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Universal scaling law for the velocity of dominoes toppling motion

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By using directed dimensional analysis and data fitting, an explicit universal scaling law for the velocity of dominoes toppling motion is formulated. The scaling law shows that domino propagational velocity is linearly proportional to the 1/2 power of domino separation and thickness, and $-1/2$ power of domino height and gravitation. The study also proved that dominoes width and mass have no influence on the domino wave traveling velocity. he scaling law obtained in this Letter is very useful to the dominoes game and will help the domino player to place the dominoes for fast speed and have a quick estimation on the speed without doing complicated multi-bodies dynamical simulation.

Keywords: dominoes, toppling motion, velocity, height, thickness, separation

INTRODUCTION

The falling of dominoes is a successive toppling of regularly spaced elements in a periodic array plotted in Figure 1. The domino effect is not only an interesting game but also an important physical phenomena, and often be used to describes some social catastrophe, such as the cascading consequences of research misconduct [1].

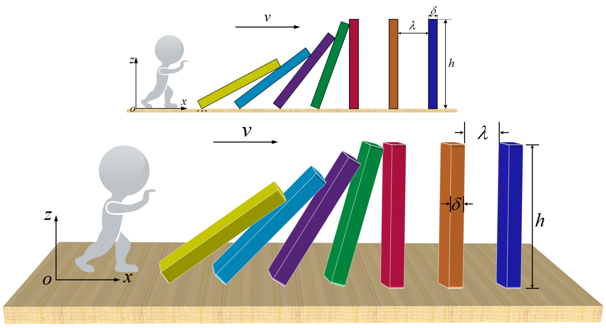

FIG. 1: The dominoes toppling motion.

The mechanics of domino falling has been studied extensively by number of leading scholars [2–14]. In 1983, McLachlan *et al.* [4] found a scaling law for the velocity v in the limiting case of dominoes with zero thickness spaced in a straight line. With these assumptions M-cLachlan *et al.* [4] the functional relation:

$$v_{\text{McLachlan}} = v(h, \lambda, g) = \sqrt{gh} f(\frac{\lambda}{h}), \qquad (1)$$

here g is gravitation acceleration, h the height of the dominoes, λ the spacing between dominoes, and $f(x)$ an undetermined function of x. Efthimiou and Johnson [6] proposed a $f(x)$ by complete elliptic integral of the first kind. Shi *et al.* [13] developed a precise numerical model with consideration of multipoint impacts between dominoes. Shi *et al.* [14] studies the toppling dynamics of a mass-varying domino system for which the mass of the domino changes at an exponential rate of its sequence number.

Szirtes and Rozsa [6] studied domino by using dimensional analysis [15] for a domino with equal thickness δ, separation λ and height h. Hence the five variables and their dimensions are listed in Table I below:

TABLE I: Dimensions of physical quantity

Variables	Symbol	Dimension
velocity	v	LT^{-1}
height	h	L
thickness	δ	L
separation	λ	L
gravitational acceleration	g	LT^{-2}

The dimensional basis used is length (L) and time (T).

Szirtes and Rozsa [6] applied dimensional analysis to find domino velocity, $v = v(h, \lambda, \delta, g)$. The problem has five variables and two dimensions (L, T), therefore there are $5 - 2 = 3$ dimensionless variables Π as follows:

$$\Pi_1 = \frac{v}{\sqrt{gh}}, \quad \Pi_2 = \frac{\lambda}{h}, \quad \Pi_3 = \frac{\delta}{h}, \qquad (2)$$

From dimensional analysis, $\Pi_1 = f(\Pi_2, \Pi_3)$, namely

$$v_{\text{Szirtes}} = \sqrt{gh} f(\frac{\lambda}{h}, \frac{\delta}{h}). \qquad (3)$$

This relation is similar to Eq.(1) except the separation and height ratio δ/h.

Although we have Eq.(1) and Eq.(3), there is no much useful information can be get from them, because the function $f(\frac{\lambda}{h}, \frac{\delta}{h})$ is still undetermined. In the following, we will try to decode the function by using directed dimensional analysis proposed by Huntley [16] and Siano [17, 18].

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DIRECTED DIMENSIONAL ANALYSIS

According to the directed dimensional analysis, we can distinct the length dimension in both x and z direction. The problem has five variables and three dimensions (L_x, L_z and T) listed in Table II below:

TABLE II: Dimensions of physical quantity

Variables	Symbol	Dimension
velocity	v	$L_x T^{-1}$
height	h	L_z
thickness	δ	L_x
separation	λ	L_x
gravitational acceleration	g	$L_z T^{-2}$

The dimensional basis used is length (L_x, L_z) and time (T).

Therefore there are $5 - 3 = 2$ dimensionless variables Π as follows:

$$\Pi_1 = v h^a \lambda^b g^c, \quad \Pi_2 = \delta h^{a_1} \lambda^{b_1} g^{c_1}, \tag{4}$$

where the exponents a, b, c and a_1, b_1, c_1 can be determined by following dimensionless conditions: $\dim(\Pi_1) = \dim(\Pi_2) = L_x^0 L_z^0 T^0$, namely

$$\begin{aligned}\dim(\Pi_1) &= L_x T^{-1} (L_z)^a (L_x)^b (L_z T^{-2})^c \\ &= L_x^{1+b} T^{-1-2c} L_z^{a+c}.\end{aligned} \tag{5}$$

From dimensionless condition, $1+b = 0$, $-1-2c = 0$ and $a + c = 0$, leads to $a = \frac{1}{2}$, $b = -1$ and $c = -\frac{1}{2}$. Hence, we have the first dimensionless variable

$$\Pi_1 = \frac{v}{\lambda}\sqrt{\frac{h}{g}}. \tag{6}$$

Similarly, we have $a_1 = 0, b_1 = -1$ and $c_1 = 0$ and the second dimensionless variable

$$\Pi_2 = \frac{\delta}{\lambda}, \tag{7}$$

From Buckingham dimensional theorem [15], the domino velocity $v = v(h, \lambda, \delta, g)$ can be replaced by $\Pi_1 = f(\Pi_2)$ as follows

$$v = \lambda\sqrt{\frac{g}{h}} f(\frac{\delta}{\lambda}). \tag{8}$$

This relation is a universal scaling law of dominoes toppling motion, where the function $f(\frac{\delta}{\lambda})$ can be determined by experiments.

Stronge [9] conducted comprehensive study with high-velocity photography on toppling of domino array, who obtained three data for domino dimensions: $h = 41.78mm$, $\delta = 7.58mm$:

To determined the function $f(x)$, let's us assume that it is a power function, ie. $f(\frac{\delta}{\lambda}) \approx C(\frac{\delta}{\lambda})^\alpha$, where the C is a constant and α is an exponent, both of them are to be confirmed with experimental data.

TABLE III: Experimental data from Stronge [9]

height	thickness	separation	velocity
h (m)	δ (m)	λ (m)	v (m/s)
0.04178	0.00758	0.0219	0.65
0.04178	0.00758	0.02949	0.80
0.04178	0.00758	0.03419	0.86

Using the data from the above table, data fitting gives $C = 0.298$ and $\alpha = 1/2$, finally, we have an explicit velocity of dominoes toppling motion as follows:

$$v = 0.298\lambda^{1/2}\sqrt{\frac{\delta g}{h}}. \tag{9}$$

This explicit scaling law for the velocity of dominoes toppling motion has never been reported in literature before, which is plotted in Fig 2.

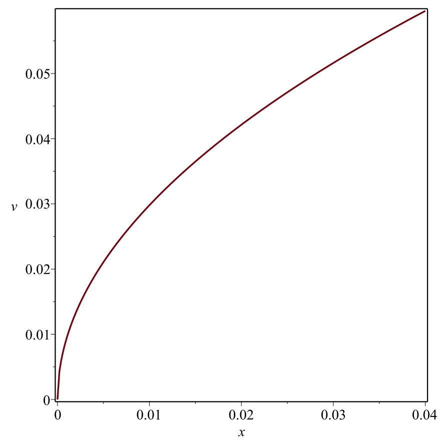

FIG. 2: Scaling law of dominoes toppling motion.

INFLUENCE OF DOMINOES WIDTH AND MASS ON THE TOPPLING VELOCITY

All previous investigation did not take into account the dominoes width [3–14]. The reason is perhaps that the dominoes width has little influence on the velocity of dominoes toppling motion, the problem is how to justify this statement.

Let's us revisit this problem by using directed dimensional analysis. To introduce the domino's width w into the formulation, we have to introduce a new dimension L_y in y direction, hence there are five variables in the problem, which are listed in Table IV below:

Therefore there are $5 - 4 = 1$ dimensionless variables Π as follows:

$$\Pi = v A^a \lambda^b g^c w^d. \tag{10}$$

TABLE IV: Dimensions of physical quantity

Variables	Symbol	Dimension
velocity	v	$L_x T^{-1}$
area	A	$L_x L_z$
width	w	L_y
separation	λ	L_x
gravitational acceleration	g	$L_z T^{-2}$

The dimensional basis used is length (L_x, L_y, L_z) and time (T).

The dimension $\dim(\Pi) = L_x^{1+a+b} T^{-1-2c} L_z^{a+c} = L_x^0 T^0 L_z^0 L_y^d$, hence, $a = \frac{1}{2}$, $b = -\frac{3}{2}$, $c = -\frac{1}{2}$ and $d = 0$.

Since the exponent of dominoes width is null, therefore, the domino's width has no influence on the velocity of dominoes toppling motion. The reason behind this is that there is no other variables has dimension in y direction.

In other words, the weight of dominoes is not a dominate issue, but the cross-section area of dominoes is a vital parameter affecting the domino velocity.

CONCLUSIONS

In conclusion, an explicit universal scaling law for the velocity of dominoes toppling motion has been formulated by using directed dimensional analysis. It is surprised to see that the domino velocity is not linearly proportional to $\sqrt{gh}$ as reported in literature (McLachlan [4] and Szirtes and Rozsa [6]). This study shown that the domino wave prorogation velocity is proportional to the 1/2 power law of domino's separation λ and thickness δ. The domino's width has no influence to the domino's velocity has also been proved. The scaling law obtained in this Letter is very useful to the dominoes game and will help the domino player to place the dominoes for fast speed and have a quick estimation on the speed without doing complicated multi-bodies dynamical simulation.

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