An explicit scaling law for the speed of falling dominoes

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(Dated: December 4, 2019)

By using directed dimensional analysis and data fitting, an explicit universal scaling law for the speed of falling dominoes is formulated. The scaling law shows that domino propagational speed is linear proportional to the 3/2 power of domino separation, and -1/2 power of domino height and thickness.

Keywords: domino, speed, height, thickness, separation

INTRODUCTION

The falling of dominoes is a successive toppling of regularly spaced elements in a periodic array plotted in Figure 1. The domino effect is not only an interesting game but also an important physical phenomena, and often be used to describes some social catastrophe, such as the cascading consequences of research misconduct [1].

FIG. 1: The falling dominoes.

The mechanics of domino falling has been studied extensively by number of leading scholars [2–14]. In 1983, McLachlan et al. [4] found a scaling law for the speed \( v \) in the limiting case of dominoes with zero thickness spaced in a straight line. With these assumptions McLachlan et al. [4] the functional relation:

\[
v^{\text{McLachlan}} = v(h, \lambda, g) = \sqrt{gh} f\left(\frac{\lambda}{h}\right), \tag{1}
\]

here \( g \) is gravitation acceleration, \( h \) the height of the dominoes, \( \lambda \) the spacing between dominoes, and \( f(x) \) an undetermined function of \( x \). Efthimiou and Johnson [6] applied dimensional analysis to find domino velocity, \( v = v(h, \lambda, \delta, g) \). The problem has five variables and two dimensions (L, T), therefore there are \( 5 - 2 = 3 \) dimensionless variables \( \Pi \) as follows:

\[
\Pi_1 = \frac{v}{\sqrt{gh}}, \quad \Pi_2 = \frac{\lambda}{h}, \quad \Pi_3 = \frac{\delta}{h}, \tag{2}
\]

From dimensional analysis, \( \Pi_1 = f(\Pi_2, \Pi_3) \), namely

\[
v^{\text{Szirtes}} = \sqrt{gh} f\left(\frac{\lambda}{h}, \frac{\delta}{h}\right). \tag{3}
\]

This relation is similar to Eq.(1) except the separation and height ratio \( \delta/h \).

Although we have Eq.(1) and Eq.(3), there is no much useful information can be get from them, because the function \( f\left(\frac{\lambda}{h}, \frac{\delta}{h}\right) \) is still undetermined. In the following, we will try to decode the function by using directed dimensional analysis proposed by Huntley [20] and Siano [21, 22].

DIRECTED DIMENSIONAL ANALYSIS I

According to the directed dimensional analysis, we can distinct the length dimension in both \( x \) and \( z \) direc-
tion. The problem has five variables and three dimensions \((L_x, L_z\) and \(T)\) listed in Table IV below:

<table>
<thead>
<tr>
<th>Variables</th>
<th>Symbol</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>(v)</td>
<td>(L_x T^{-1})</td>
</tr>
<tr>
<td>height</td>
<td>(h)</td>
<td>(L_z)</td>
</tr>
<tr>
<td>thickness</td>
<td>(\delta)</td>
<td>(L_z)</td>
</tr>
<tr>
<td>separation</td>
<td>(\lambda)</td>
<td>(L_z)</td>
</tr>
<tr>
<td>gravitational acceleration</td>
<td>(g)</td>
<td>(L_x T^{-2})</td>
</tr>
</tbody>
</table>

The dimensional basis used is length \((L_x, L_z)\) and time \((T)\).

Therefore there are \(5 - 3 = 2\) dimensionless variables \(\Pi\) as follows:

\[
\Pi_1 = \frac{v h^a \lambda^b g^c}{\delta}, \quad \Pi_2 = \frac{\delta h^a \lambda^b g^c}{\lambda},
\]

where the exponents \(a, b, c\) and \(a_1, b_1, c_1\) can be determined by following dimensionless conditions: \(\dim(\Pi_1) = \dim(\Pi_2) = L_x^0 L_z^b T^1\), namely

\[
\dim(\Pi_1) = L_x T^{-1} (L_z)^a (L_z)^{b} (L_z T^{-2})^c = L_x^a T^{-1-2c} L_z^{a+b+c}.
\]

From dimensionless condition, \(1 + b = 0, \ -1 - 2c = 0\) and \(a + c = 0\), leads to \(a = -\frac{1}{2}, b = -1\) and \(c = -\frac{1}{2}\). Hence, we have the first dimensionless variable

\[
\Pi_1 = \frac{v}{\lambda} \left(\frac{h}{g}\right)^{\frac{1}{2}}
\]

Similarly, we have \(a_1 = 0, b_1 = -1\) and \(c_1 = 0\) and the second dimensionless variable

\[
\Pi_2 = \frac{\delta}{\lambda}
\]

From Buckingham dimensional theorem [15], the domino speed \(v = v(h, \lambda, \delta, g)\) can be replaced by \(\Pi_1 = f(\Pi_2)\) as follows

\[
v = \lambda \left(\frac{h}{g}\right)^{\frac{1}{2}} f\left(\frac{\delta}{\lambda}\right).
\]

This relation has not been reported in literature, however, the function \(f\left(\frac{\delta}{\lambda}\right)\) is still undetermined.

**DIRECTED DIMENSIONAL ANALYSIS II**

In order to reduce the number of variables, let’s consider the domino’s area instead of its height and thickness separately. The problem has four variables and three dimensions \((L_x, L_z\) and \(T)\) listed in Table IV below:

Therefore there are \(4 - 3 = 1\) dimensionless variable \(\Pi\) as follows:

\[
\Pi = v A^a \lambda^b g^c.
\]

### TABLE II: Dimensions of physical quantity

<table>
<thead>
<tr>
<th>Variables</th>
<th>Symbol</th>
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</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>(v)</td>
<td>(L_x T^{-1})</td>
</tr>
<tr>
<td>height</td>
<td>(h)</td>
<td>(L_z)</td>
</tr>
<tr>
<td>thickness</td>
<td>(\delta)</td>
<td>(L_z)</td>
</tr>
<tr>
<td>separation</td>
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<td>(L_z)</td>
</tr>
<tr>
<td>gravitational acceleration</td>
<td>(g)</td>
<td>(L_x T^{-2})</td>
</tr>
</tbody>
</table>

The dimensional basis used is length \((L_x, L_z)\) and time \((T)\).

The dimension \(\dim(\Pi) = L_x^{1+a+b} T^{-1-2c} L_z^{a+c}\) = \(L_x ^0 T^0 L_z^0\), hence, \(a = \frac{1}{2}, b = -\frac{3}{2}\) and \(c = -\frac{1}{2}\).

Since we have only one dimensionless variable \(\Pi\), therefore, this \(\Pi\) can only be a constant, namely

\[
v = C \frac{\lambda^{3/2}}{\sqrt{g A}} = C \frac{\lambda^{3/2}}{\sqrt{gh\delta}}.
\]

where the area of rectangular domino \(A = h\delta\) is used. The beauty of Eq.(10) is that it is an explicit formula for the speed of falling dominoes has never been reported in literature before.

### TABLE III: Dimensions of physical quantity

<table>
<thead>
<tr>
<th>Variables</th>
<th>Symbol</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>(v)</td>
<td>(L_x T^{-1})</td>
</tr>
<tr>
<td>area</td>
<td>(A)</td>
<td>(L_x L_z)</td>
</tr>
<tr>
<td>separation</td>
<td>(\lambda)</td>
<td>(L_z)</td>
</tr>
</tbody>
</table>

The dimensional basis used is length \((L_x, L_z)\) and time \((T)\).

Stronge [9] conducted comprehensive study with high-speed photography on topping of domino array, who obtained three data for domino dimensions: \(h = 41.78\text{mm}, \ \delta = 7.58\text{mm}\):

### TABLE IV: Experimental data from Stronge [9]

<table>
<thead>
<tr>
<th>height ((h))</th>
<th>thickness ((\delta))</th>
<th>separation ((\lambda))</th>
<th>velocity ((v))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.04178)</td>
<td>(0.00758)</td>
<td>(0.0219)</td>
<td>(0.65)</td>
</tr>
<tr>
<td>(0.04178)</td>
<td>(0.00758)</td>
<td>(0.02949)</td>
<td>(0.80)</td>
</tr>
<tr>
<td>(0.04178)</td>
<td>(0.00758)</td>
<td>(0.03419)</td>
<td>(0.86)</td>
</tr>
</tbody>
</table>

Using the data from the above table, we can calculate the constant by \(C = v \frac{\lambda^{3/2}}{\sqrt{gh\delta}}\), hence, we have \(C_1 = 11.17323782\). \(C_2 = 8.800547398\) and \(C_3 = 7.578472140\), and its average \(C \approx (C_1 + C_2 + C_3)/3 \approx 9.18 \approx 2.923\pi\).

Finally, we have an explicit speed of falling dominoes as follows:

\[
v = 2.923\pi \frac{\lambda^{3/2}}{\sqrt{gh\delta}}.
\]

This explicit scaling law for the speed of falling dominoes has never been reported in literature before.
CONCLUSION

In conclusion, an explicit universal scaling law for the speed of falling dominoes has been formulated by using directed dimensional analysis. It is surprising to see that the domino speed is not linearly proportional to $\sqrt{gh}$ as reported in literature (McLachlan [4] and Szirtes and Rozsa [6]). The study also shown that the domino wave propagation speed is proportional to the $3/2$ power law of domino's separation $\lambda$, which reveals that the domino’s separation $\lambda$ is a control parameter for domino motion.

Acknowledgement: The author appreciates the financial supports from Xi’an University of Architecture and Technology and Mr Zhe Liu for the preparation of Figure 1.