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Baryon Physics and Tight Coupling Approximation in Boltzmann Code

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Boltzmann codes are essential tools for studying cosmology. Most of the codes are based on a seminal work by Ma and Bertschinger [1]. We found that the formalism employed in those codes has at least three possible problems which need to be understood. i) The equation of motion for baryons are gauge incompatible, ii) they break the Bianchi identity, and iii) it is not clear from the equations of motion which physical system is considered. In this work we revisit the baryon physics and address all the above mentioned issues, based solely on conservation of stress-energy tensor, resulting in taking into account a correction that is usually neglected and that is numerically small. We also study the tight coupling approximation up to the second order without choosing any gauge. We implement the improved baryon equations in a Boltzmann code and investigate the change in the estimate of cosmological parameters by performing an MCMC analysis. While in this paper we study the ΛCDM model only, our baryon equations can be easily implemented in other models and various modified gravity theories.

I. INTRODUCTION

Studies in modern cosmology heavily rely on cosmological linear perturbation theory. Perturbations in the gravitational fields and matter fields in the early universe give rise to the current distributions of radiation, baryons, dark matter and dark energy, that we observe today. To understand the evolution of these linear perturbations, Einstein equations are solved numerically at the linear order in perturbations around a homogeneous and isotropic background. These numerical solvers are often referred to as Boltzmann solvers or Boltzmann codes. There are several open-source linear Boltzmann solvers available namely, CLASS [2], CAMB [3], CMBEASY [4], CMBFAST [5], etc. Among them CAMB and CLASS are maintained frequently. These codes provide us with a platform to test any theory against observations.

To understand the nature of the dark sector of our universe there are several future experiments planned, such as EUCLID [6], DESI [7] and LSST [8]. All these experiments focus on higher precision for the estimate of cosmological parameters. In order to take full advantages of these experiments the Boltzmann codes need to be precise enough and any inconsistencies present in the implementation of the codes need to be fixed. Otherwise, theoretical predictions cannot be matched with the observations.

The first calculation of cosmological perturbation theory was performed by Lifshitz [9]. Later Bardeen [10] and Kodama & Sasaki [11] fixed the gauge issues in the scalar sector. CMBFAST [5] introduced a new line-of-sight integration method to compute anisotropies, and their code was made publicly available. This could reduce the time for computation up to two times. There were two other Boltzmann codes available before, one developed by Sugiyama [12, 13], based on gauge invariant formalism, and the other developed by White in the synchronous gauge [14–16]. CMBFAST is also based on the synchronous gauge. CMBEASY and CAMB are basically formulated based on CMBFAST. The Boltzmann equations in CMBFAST is taken from COSMIC [17], which is based on the seminal work by Ma and Bertschinger [1]. CLASS is also based on it, implemented in Newtonian and synchronous gauges.

However, in [1] there appear at least three possible problems in the evolution equation for the baryon fluid.

- First, there is a gauge incompatibility, that is, the equations break general covariance. In particular, the equations of motion in the Newtonian gauge and those in the synchronous gauge are not related to each other by a gauge transformation. This results in different physical outcomes for different gauge choices. The consequence is that it is not clear which gauge one should choose from the beginning to study baryons. This should not happen in a covariant theory, as a gauge choice merely represents a choice of coordinates and does not affect physical results.

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- Second, it breaks the Bianchi identity. This aspect also leads to inconsistencies. For example, breaking Bianchi identity implies that solving all componets of the Einstein equation would lead in general to a solution which is not consistent with the conservation equation for the matter fields.

- Third, in the limit of no interaction between the baryon fluid and the photon gas, we have an equation of motion for the baryons in which the squared sound speed c_s^2 is present. It is difficult to understand the nature of this term with c_s^2, as no known covariant action for matter would lead to such a term in the dynamical equation of motion.

As mentioned above these equations are taken for code implementation in almost all existing Boltzmann solvers. We feel that in this era of precision cosmology, the Boltzmann solvers should have these issues fixed. These issues cause artificial deviations from general relativity. These problems appear also in [15]. We find that there are terms missing at the order of c_s^2 in the baryon evolution equation. In this work, we derive the correct equations of motion for baryons from an action. The resulting equations are devoid of the above mentioned issues. We shall see that our new terms do not modify strongly the final results (and this is a bit reassuring), nonetheless, we believe our corrections are to be made for Boltzmann solvers, especially at early times, when the sound speed of the baryon fluid (though small) cannot be neglected completely.

We shall then implement our corrections in the tight coupling approximation scheme. In fact, in the regime before recombination, photons and baryons coupled to form a stiff baryon-photon fluid. Since, in this case, the equations for this photon-baryon fluid are numerically stiff, in-order to study this regime, both CLASS and CAMB make use of the so called tight coupling approximations (tca). These were first developed by Peebles and Yu [18]. In [1], the authors have developed this approximation up to the first order and they have further assumed $\tau_c \propto a^2$and $c_s^2 \propto a^{-1}$. The full first order tca is implemented in [3] with an additional approximation, $\tau_c \propto a^2$. For CMBEASY tca is calculated for the Newtonian gauge in which they considered terms beyond the first order [4]. CLASS developed both first order and second order tca without assuming any further assumptions like in COSMIC and CAMB. Several other authors have worked on the tca. For example, the tca up to the second order was implemented for the calculation in the synchronous gauge in [19]. An extension of the tca to the second order cosmological perturbation was developed in [20]. The approximation depends on baryon equations. Hence, also the approximation methods used to solve baryon-photon system need to be corrected.

In order to fix the covariance issues, we find it useful to understand the baryon physics, starting from a newly written Lagrangian, which up to a field redefinition is equivalent to the Schutz-Sorkin Lagrangian [21], which was studied also in [22, 23] in the context of perfect fluids, with general equations of state. We then consider the baryon fluid as an ideal gas. This model of a gas in fact describes non relativistic particles with a non-zero speed of propagation, i.e. $c_s^2 \neq 0$. This system then allows us to find covariant equations of motion for the perturbation variables. (We also give an alternative derivation of the same set of equations based on the conservation of the stress-energy tensor.) We find that there is an extra term of the order of c_s^2 in both the evolution of energy density and velocity perturbations for the baryon fluid. This fix solves all the three problems mentioned above. We then use these new baryon equations of motion in order to derive the tight coupling approximation equations up to the second order. We implement these corrections for the baryon evolution, in the CLASS code. Finally, we make a parameter estimation for the ΛCDM model with these new corrections, using Monte Python [24, 25] Monte Carlo sampler. We find that the new equations, solved by CLASS, give some deviations from the previous results, but for the parameter estimation of the ΛCDM model such deviations are inside the statistical error bars. (On the other hand, we do not know whether the deviation remains within the statistical error bars in other models and various modified gravity theories.)

For the clarity of the reader, we would like to summarize the main difference between Ma & Bertschinger's paper [1] and our paper. Ma & Bertschinger's starting equations, i.e. Eqs. (29)-(30) of [1], are equivalent to ours but are not ready for implementation to Boltzmann codes. On the other hand, equations that are ready for implementation are different between Ma & Bertschinger and us. While Ma & Bertschinger removed some terms in the equations, we did not. As a result, Ma & Bertschinger's equations (66) and (67) (that are used in the Boltzmann codes) are not gauge compatible and do not satisfy the conservation equation. In the following section below, we also argue that some of the terms that Ma & Bertschinger removed are not a priori small enough to be ignored. We can justify their treatment only a posteriori. However, the a-posteriori justification is possible only if we have the correct equations ready for implementation. This is what we shall do in this paper. As a result, for the first time, we were able to justify the use of Ma & Bertschinger's equations (66) and (67) in the Boltzman code. Also, we were able to improve the tca scheme in the Boltzman code, following the same approximation made by Ma & Bertschinger.

The fact that the equations of motion implemented in Boltzmann codes for the baryon in Ma & Bertschinger do not satisfy stress-energy momentum conservation (and general covariance), we believe is due to an approximation we want to discuss here. For Ma & Bertschinger, baryons being non-relativistic means: 1) $T/\mu_g \ll 1$ (or equivalently $c_s^2 \ll 1$), and 2) only terms in $k^2\, c_s^2$ should be kept (while neglecting terms only in c_s^2) . If we agree with this approximation scheme, we are forced to conclude that whenever there is a combination $c_s^2 k^2 \times$ (perturbation field), in general such a term cannot be ignored a priori, as it may give non-negligible contributions at sufficiently high k. A non-trivial consequence of this fact is the following. In synchronous gauge, in Boltzmann codes, the metric perturbation field $\dot{h}$, is replaced on using the perturbed $G^0{}_0$ Einstein equation. Such an equation replaces everywhere $\dot{h}$ with other terms including a term proportional to $k^2\eta$. On the other hand, one of the terms which are neglected in the baryon equations of motion by Ma & Bertschinger is actually $c_s^2\dot{h}$. But by doing this, one will end up neglecting a term, as we shall see later on, proportional to $k^2 c_s^2 \eta$. In order to avoid this potential problem, for us instead, in this paper, baryons being non-relativistic only means $T \ll \mu_g$ (i.e. $c_s^2 \ll 1$), without neglecting any term up to first order in c_s^2. This is the reason why we call our equation for baryon as exact (up to c_s^2 order). Since we are not ignoring any term, our equation obeys general covariance and gauge compatibility (once again, up to c_s^2). On the other hand, Ma & Bertschinger baryon equations will then break general covariance at first order in c_s^2.

While in the present paper we restrict our consideration to the ΛCDM model, our baryon equations can be easily implemented in other models and various modified gravity theories.

This paper is organized as follows. In section II we briefly expose the key problems in the current Boltzmann codes. Then in section III we discuss a monoatomic ideal gas with a non-zero speed of propagation in the non-relativistic limit. In section IV, we study a baryon fluid from a Lagrangian and derive the equation of motion in the non-relativistic limit. Then in section V , the tight coupling approximation up to the second order is discussed to overcome the stiffness problem for the new equations of motion of the coupled baryon-photon system. A brief code implementation is discussed in section VI. Subsequently, we present the results of cosmological parameters after doing MCMC analysis in section VII. Finally we give our conclusion in section VIII.

II. COVARIANCE PROBLEMS IN CURRENT BOLTZMANN CODES

In this section we make a short outline of the problems related to the equations of motion for the baryon sector used in [1] and in modern Boltzmann solvers. All these problems can be related to the breaking of general covariance present inside the equations of motion. Since in this section we only want to briefly show the key points of this study, we consider here, for simplicity, only the scalar perturbations of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which can be written as follows

$$ds^2 = -\left(1+2\alpha\right)a^2\,d\tau^2 + 2a\partial_i\chi\,d\tau\,dx^i + a^2\left[\left(1+2\zeta\right)\delta_{ij} + \frac{2\partial_i\partial_j E}{a^2}\right]dx^i dx^j\,. \tag{1}$$

In the above, we have not fixed any gauge yet.

Then, on choosing the Newtonian gauge, i.e. on setting $\chi = 0$ and $E = 0$, the equations of motion for the baryon fluid as given in [1] (their Eq. (67)) – which are also used in current Boltzmann codes – read in the Fourier space as follows

$$\dot{\delta}_b = -\theta_b - 3\dot{\zeta}, \tag{2}$$

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + k^2\alpha + c_s^2 k^2\delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b}\,a n_e\sigma_T\left(\theta_\gamma - \theta_b\right), \tag{3}$$

where $\delta_b \equiv \left(\rho_b - \bar{\rho}_b\right)/\bar{\rho}_b$ is the baryon density perturbation, θ_b is the scalar-part of the baryon velocity perturbation defined as $\left(\bar{\rho}_b + \bar{P}_b\right)\theta_b \equiv ik^j\delta T_b^0{}_j$, θ_γ is the scalar-part of the photon velocity perturbation, and the term $a n_e\sigma_T\left(\theta_\gamma - \theta_b\right)$ represents the momentum transfer into the photon gas. A dot here represents a derivative with respect to τ, the conformal time. The equations (2) and (3) are written in the Newtonian gauge but can be rewritten in terms of the gauge invariant variables (which reduce to the corresponding perturbation variables in the Newtonian gauge when $\chi = E = 0$)

$$\delta_b^{\rm NG} = \delta_b + \frac{\dot{\bar{\rho}}_b}{a\bar{\rho}_b}\chi - \frac{\dot{\bar{\rho}}_b}{\bar{\rho}_b}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right), \tag{4}$$

$$\theta_b^{\rm NG} = \theta_b + \frac{k^2}{a}\chi - k^2\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right), \tag{5}$$

$$\theta_\gamma^{\rm NG} = \theta_\gamma + \frac{k^2}{a}\chi - k^2\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right), \tag{6}$$

$$\zeta^{\rm NG} = \zeta + \frac{\dot{a}}{a^2}\chi - \frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right), \tag{7}$$

$$\alpha^{\rm NG} = \alpha + \frac{1}{a}\dot{\chi} - \frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - \frac{\partial^2}{\partial\tau^2}\left(\frac{E}{a^2}\right), \tag{8}$$

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as

$$\dot{\delta}_b^{\rm NG} = -\theta_b^{\rm NG} - 3\dot{\zeta}^{\rm NG}, \tag{9}$$

$$\dot{\theta}_b^{\rm NG} = -\frac{\dot{a}}{a}\theta_b^{\rm NG} + k^2\alpha^{\rm NG} + c_s^2 k^2 \delta_b^{\rm NG} + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T \left(\theta_\gamma^{\rm NG} - \theta_b^{\rm NG}\right), \tag{10}$$

Since the general covariance is supposed to hold (as we are discussing General Relativity in the presence of standard matter), we are now able to rewrite the previous evolution equations in any other gauge, in particular in the synchronous gauge, for which $\alpha = 0$ and $\chi = 0$. Then the dynamical equations (9) and (10) reduce to[1]

$$\dot{\delta}_b = -\theta_b + k^2 \frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - 3\dot{\zeta}\,, \tag{11}$$

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2 \left[\delta_b + 3\frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)\right] + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T \left(\theta_\gamma - \theta_b\right), \tag{12}$$

where now the fields are all evaluated in the synchronous gauge. Notice that the interaction term proportional to σ_T is gauge invariant since $\theta_\gamma - \theta_b = \theta_\gamma^{\rm NG} - \theta_b^{\rm NG}$.

Now the problem is evident. In fact, the above differential equation for the velocity field, Eq. (12), is different from the one written in [1] (precisely their Eq. (66)) which is also supposed to hold in the synchronous gauge. More precisely, the term proportional to $k^2 c_s^2$ makes the baryon velocity equation incompatible between the two gauges. To look at this same problem from another point of view, we can start from writing the dynamical baryon equations of motion in the synchronous gauge, as given in Eq. (66) of [1], and then transform them to the Newtonian gauge. However, on doing so, the resulting baryon-velocity equation of motion turns out to be once again different from the one shown in Eq. (3) (or Eq. (67) in [1]).

Therefore, up to now, solving Boltzmann equations in the two gauges leads to solving two intrinsically different equations of motion, so that the two gauges give rise to two physically different solutions. Then one may wonder which of the two should be considered. As we will see later on the answer is: none of them, as new equations need to be introduced.

First of all, it is obvious from Eq. (3) that setting c_s^2 to vanish would make the baryon-velocity equation compatible among the two gauges. Therefore, in order to fix the problem, we should somehow introduce further counter terms proportional to c_s^2 to make this equation gauge-compatible. We will see how to perform this in the next section.

An immediate consequence of this breaking of covariance is the fact that $T^{\mu\nu}$ is not conserved any longer. To show this simple fact, on using standard results from the literature [11] (see also appendix B), we find that for a general perfect fluid[2], the equations of motion $T^\mu{}_{\nu;\mu} = 0$ can be shown to lead to (see Section IV for more details):

$$\dot{\delta} = 3\,\frac{\dot{a}}{a}\left(\frac{p}{\rho} - c_s^2\right)\delta + \left(1 + \frac{p}{\rho}\right)(3\dot{\phi} - \theta)\,, \tag{13}$$

$$\dot{\theta} = k^2\psi - \frac{\dot{a}}{a}\,\left(1 - 3\,c_s^2\right)\theta + \frac{k^2 c_s^2\,\rho}{\rho + p}\,\delta\,. \tag{14}$$

The equations of state will fix p in terms of ρ. On comparing these equations with the ones written in Eqs. (2) and (3), we notice we cannot match the equations of motion, even taking any correction/redefinition for c_s^2. Therefore the equations of motion are breaking general covariance and lead to non-conservation of the stress-energy tensor. One may object that baryons interact with photons, and the previous example does not apply. However, the reader knows well that the total stress-energy tensor needs to be conserved, otherwise Einstein equations $G_{\mu\nu} = 8\pi G\, T^{\rm tot}_{\mu\nu}$ would not hold. Therefore on adding the two contributions from baryons and photons, all the relative interactions cancel out, so that the equations of motion will reduce to the sum of the equations of motion for the free fields. Therefore, the previous example can be actually used, and, as a consequence, Eqs. (2) and (3) would still break energy-conservation law.

One should now be convinced that Eqs. (2) and (3) break covariance, but may think this is due merely to an approximation. Let us then consider *which* approximation has been considered. We have already stated above that on top of considering baryons to be non-relativistic, a small-scale limit has been taken, namely $k \gg aH = \dot{a}/a \equiv \mathcal{H}$,

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[1] The relation between fields defined in this paper and in [1] are $2E/a^2 = -1/k^2\,(h + 6\eta)$ and $\zeta = -\eta$. For the synchronous gauge we add the gauge choice $\alpha = 0$, $\chi = 0$, which is actually incomplete. For a complete gauge fixing we need to choose also the following two initial conditions at the time $\tau = \tau_{\rm ini}$: $\theta_c(\tau_{\rm ini}) = 0$, $\delta_\gamma(\tau_{\rm ini}) + \frac{2}{3}h(\tau_{\rm ini}) = 0$, where θ_c represents the field θ for the cold dark matter fluid, and δ_γ is the photon density perturbation. For Newtonian gauge, the authors in [1] use the following field redefinition, $\alpha = \psi$, $\zeta = -\phi$, together with the complete gauge fixing, $E = 0$, $\chi = 0$.

[2] As well explained in standard textbooks of GR [26], a perfect fluid is a system which needs to be defined by specifying two equations of state, for example $p = p(\rho, s)$ and $T = T(\rho, s)$ (e.g. for dust, we have $p = 0 = T$). Therefore for a perfect fluid $\delta p = c_s^2 \delta\rho + (\partial p/\partial s)_\rho\, \delta s$, where $c_s^2 \equiv (\partial p/\partial \rho)_s$ as established by Laplace for an ideal gas. See appendix A for more details.

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but we will show it here explicitly. As we shall see later on, the correct equations of motion for the baryon fluid, in synchronous gauge (see footnote 1 for the redefinition of field variables), can be written as (the subscript b indicates baryon)

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\,\theta_b + c_s^2\,k^2\,\delta_b + R\,a n_e \sigma_T(\theta_\gamma - \theta_b) + 3\,c_s^2\,\frac{\dot{a}}{a}\,\theta_b\,, \tag{15}$$

$$\dot{\delta}_b = -\theta_b - \frac{1}{2}\,\dot{h} - \frac{6}{5}\,c_s^2\,\frac{\dot{a}}{a}\,\delta_b - \frac{3}{5}\,c_s^2\left(\theta_b + \frac{1}{2}\,\dot{h}\right), \tag{16}$$

whereas the non-covariant equations of motion read $\dot{\delta}_b = -\theta - \frac{\dot{h}}{2}$ and $\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2 \delta_b$. As stated in the previous section, in Boltzmann codes, $\dot{h}$ is actually replaced by using the Hamiltonian constraint, which includes a term $c_s^2 k^2 \eta$, and so we cannot remove the $c_s^2 \dot{h}$ term a priori, even following Ma & Bertschinger approximation choice for baryons.

Since we only want to know the approximation done and its meaning let us switch off the coupling with the photons and take the time derivative of $\dot{\delta}_b$ and represent this expression as $\ddot{\delta}_b = \dot{P}$ where P corresponds to the rhs of Eq. (16). Then we can also write $\ddot{\delta}_b + \mathcal{H}\,\dot{\delta}_b = \dot{P} + \mathcal{H}\,P$. Substituting in the rhs of such equation both $\dot{\theta}_b$ and $\dot{\delta}_b$ by the above given equations, and replacing $\dot{h}$ and $\ddot{h}$ (also this latter term introduces terms proportional to $k^2\eta$) by the Einstein equations we can write the second order baryon equation in the form[3]

$$\ddot{\delta}_b + \mathcal{H}\dot{\delta}_b = -\left[c_s^2 k^2 + 1/25\,(120\,\mathcal{H}^2 + 30\,\dot{\mathcal{H}})c_s^2 + 3\,\mathcal{H}\dot{c_s^2}\right]\delta_b \tag{17}$$

$$- \frac{3\,\mathcal{H} c_s^2 \theta}{5} + \frac{12}{5}\,c_s^2 k^2 \eta + \frac{12}{5}\,(3\,c_s^2 + 5)\,\pi G_N a^2 \sum_i \delta p_i$$

$$+ 4(1 + 3\,c_s^2)\pi G_N a^2 \sum_i \delta\rho_i\,. \tag{18}$$

On comparing this last equation with the one given in [27] for baryon sector, namely

$$\ddot{\delta}_b + \mathcal{H}\dot{\delta}_b + c_s^2 k^2 \delta_b = 4\pi G_N a^2 \sum_i (\delta\rho_i + 3\delta p_i)\,, \tag{19}$$

we see all the additional terms which have been neglected. As expected a term proportional to $c_s^2 k^2 \eta$ has appeared and, as we shall show in Fig. 1, it cannot be neglected a priori during radiation domination era, up to dust domination.

Nonetheless, the missing terms correspond to the realization and the definition of the approximation itself. This approximation consists of taking $k \gg aH$, $\delta_b \gg \eta$ and $k^2\delta_b \gg aH\theta$. However, during radiation domination (valid at least for the initial conditions taken in [1] at $z \geq 10^6$) at which the speed of propagation for the baryons cannot be neglected (as $c_s^2 \propto a^{-1}$ approximately), on considering $H \approx H_0\sqrt{\Omega_{r0}}(1+z)^2$, where $H_0 \approx h\mathrm{Mpc}^{-1}/(2997.9)$, we find that k is constrained as to be $k \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)\,h\mathrm{Mpc}^{-1}$. Furthermore, even considering the high-k limit, in the equations we should at least keep the term proportional to $c_s^2 k^2 \eta$. However, since the term is usually neglected then it implies that at any time and scale one should check that $\eta \ll \delta_b$. If so, this approximation should be extended in other equations, whenever both η and δ_b. Finally the approximation consists of taking $k^2\delta_b \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)\,h\mathrm{Mpc}^{-1}\,\theta_b$. However, several authors, e.g. [1], consider a range for k given by $0.01\,\mathrm{Mpc}^{-1} \leq k \leq 10\,\mathrm{Mpc}^{-1}$, so that inside this set the approximation scheme in general fails for large redshifts. We will show in this section, also numerically, for all redshifts, the validity of the assertions made here.

We have another strong reason why one should not ignore the terms in c_s^2 that instead we have kept. Let us explain this more in detail. When we study the tca scheme, on using the approximated baryon equations of motion of [1], one is bound to miss some relevant terms (relevant in the sense of [1] point of view), e.g. $c_s^2 k^2 \psi$ (in Newtonian gauge), and $c_s^2 k^2 \delta_\gamma$ (in any gauge), which are meant, a priori, to contribute at sufficiently high k Eq. (132). Removing these terms (as they do not appear in previous Boltzmann codes) could lead to self inconsistencies inside the regime of validity of Ma & Bertschinger approximation.

If we had to follow baryon approximation [1] at any cost, then only after making all manipulations with covariantly complete equations of motion, we can a posteriori select the $c_s^2 k^2$ terms, if present. Ignoring any of these terms would lead to further approximate $\delta_b \gg \eta$ or $\delta_b \gg \delta_\gamma$ at all times, which is not true in general. To show this more transparently we show in Fig. 1 a plot of the perturbations η, δ_b, δ_γ, $a^2H^2\delta_b/k^2$, ψ (this latter one being relevant

[3] The extra terms in the c_s^2 is due to the fact that, in Eq. (15) and Eq. (16) has c_s^2 terms, which are not neglected a priori. On using Einstein equations in synchronous gauge one get these extra terms. Especially, the term $c_s^2 k^2 \eta$, which can never be neglected since $\delta_b < \eta$ in the relevent redshift as we show the Fig. 1

for tca in Newtonian gauge), and $aH\theta_b/k^2$ in the high-k regime, namely for $k = 0.1$ Mpc^{-1} at all times for default values of parameters. We can see once for all that keeping only the $c_s^2 k^2 \delta_b$ term is not a good approximation during radiation domination and up to dust domination.

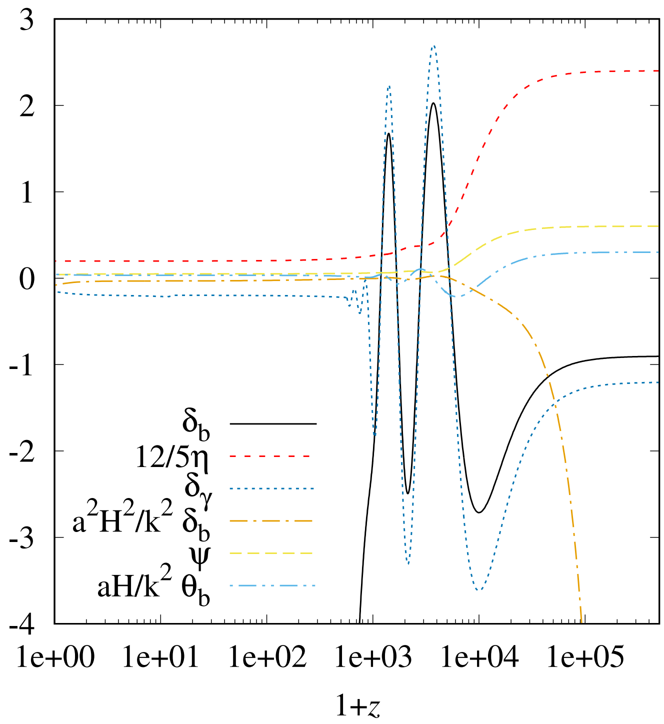

Figure 1. Evolution of perturbations out of which we can test the motivation of our study in the high-k regime (fixing $k = 0.1$ Mpc^{-1}) at all times for default values of parameters. During radiation domination (when c_s^2 should not be neglected) and up to $z > 10^4$, η typically dominates over δ_b, so that $c_s^2 k^2 \eta > c_s^2 k^2 \delta_b$, and this term cannot be neglected. Instead δ_g dominates over δ_b up to dust domination, so that in this range of redshift $c_s^2 k^2 \delta_g > c_s^2 k^2 \delta_b$. So such a term cannot be neglected in tca schemes. Finally even for this high-k mode, the subhorizon approximation breaks down during radiation domination as it is clear that for $z > 5 \times 10^4$, we have $c^2 k^2 \delta_b \ll c_s^2 a^2 H^2 \delta_b$. Since c_s^2 cannot be neglected in this redshift range, also this term should be included into the equations of motion for the perturbations. All these new terms are terms which are imposed by conservation of stress-energy tensor.

Furthermore, the results presented in Fig. 1 are calculated for ΛCDM. If we were studying the phenomenology of modified theories of gravity, especially early-time modified gravity theories (which in general are implemented as not affecting the matter equations of motion which come from conservation of stress-energy tensor), we could expect non-trivial behaviour from the metric field perturbations h, η (in synchronous gauge) or ϕ, ψ (in Newtonian gauge) which could change the results of Fig. 1. A priori, we would not be sure that the gravitational perturbation field can be really neglected. This may result in missing relevant information in studying the cosmology of modified theories of gravity. On the other hand keeping equations which obeys general covariance ensures we do not miss any information within the theory.

As we shall see later on, in Section IV, introducing back the gauge-compatibility and the general covariance ends up with considering a new set of equations of motion for the baryon fluid, which will give different results from the equations found in [1]. We will also see that on making the equations of motion explicitly covariant will not make the numerical code unstable, or slow. Therefore, we do not have a clear reason why the corrections which are to be introduced in the next section should not be implemented in today's Boltzmann solvers. It is true that the corrections, as we will show, are not large enough to change the final results beyond the error bar, but in the equations of motion, we see that these corrections (of order of $k^2 c_s^2$) are of the similar order as second-order tca quantities. Therefore, implementing tca correctly for the aim of reaching precision cosmology should also lead to consider exact and covariant baryon equations of motion.

We will also build tight coupling approximation schemes (as used e.g. in CLASS) for the new equations of motion as to address the stiffness of the ODEs of the numerical code during early time cosmological eras.

We point out here that there is another problem related to the gauge-incompatibility of the equations of motion. In fact, since the equations of motion are not gauge-compatible, we should conclude that the general covariance has

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been broken. In turn, this behavior leads to the fact that the equations of motion will not close in general. That is, the Bianchi identities will not hold any longer for the system of the perturbed Einstein equations. If Bianchi identities do not hold any longer, then, in general, picking up a subset of equations will lead to a solution which does not solve the other remaining equations. This implies that in general there is no solution to the full set of equations.

To understand how to solve all the previous problems for the baryon equations of motion in the presence of a non-vanishing c_s^2, in the next section, we will study the ideal gas Lagrangian which describes a non-relativistic physical system with non-zero pressure, which is then capable of describing non-relativistic baryons with $1 \gg c_s^2 = (\partial p/\partial \rho)_s \neq 0$.

Finally the new equations of motions introduced here should be considered as the basic baryon equations of motion. Therefore any other additional physical phenomenon which has to do with baryon physics, such as reionization, higher order tca approximation schemes, should be considered as starting from the most basic level, i.e. from the equations of motion that we introduce in this paper.

III. IDEAL GAS

In order to understand the reason why the equations of motion in the baryon sector break general covariance it is sufficient to study the limit in which the interaction with the photon gas can be switched off. This can be done by setting $\sigma_T \to 0$. In this limit the baryon fluid becomes a self-gravitating fluid whose equations of motion in Newtonian gauge read as

$$\dot{\delta}_b = -\theta_b - 3\dot{\zeta}, \tag{20}$$

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + k^2\alpha + c_s^2 k^2 \delta_b \,. \tag{21}$$

The problems is that in this limit the fluid does reduce either to a dust fluid or to any other perfect fluid equations of motions. The free equations describe an unknown system, which cannot be described in terms of a perfect fluid in General Relativity. That is the reason why these same equations break general covariance. In this same case, we should also expect that these equations cannot come from a Lagrangian and therefore they cannot be found by any conservation law, i.e. they cannot come from considering $T^{\mu\nu}{}_{;\nu} = 0$.

Therefore in order to reintroduce general covariance we impose that the free Lagrangian for the baryon reduces to the Lagrangian of an ideal gas, because we are interested in a non-zero sound speed of propagation, i.e. $c_s^2 \neq 0$. In fact, a perfect fluid for baryons may be modeled in two possible ways, with or without temperature. The model without temperature is described by a dust fluid. But this model would lead to $c_s^2 = 0$, and this is not the model that we are looking for. Let us consider then the case of a baryon fluid with a tiny, but non-zero temperature, as only in this case the baryon fluid will possess $c_s^2 \neq 0$. The model for baryons considered here consists of an ideal gas, whose fluid particles are considered to be non-relativistic. To this fluid gas a collision term with photons will be then added, as we do in the case of a dust-like baryon-gas without temperature.

A monoatomic ideal gas is defined by the following two equations of state:

$$p = n\,T\,, \qquad \rho = \mu_g\, n + \frac{3}{2}\, n\, T\,, \tag{22}$$

where p, n, ρ and T are the pressure, number density, energy density and temperature of the fluid respectively. Furthermore, μ_g is a constant and represents the mass of the fluid particle. Since the fluid is assumed to be non-relativistic, these equations of state hold only for $T \ll \mu_g$.

The first law of thermodynamics for a general perfect fluid can be written as

$$d\rho = \mu\, dn + n\, T\, ds\,, \tag{23}$$

where the enthalpy per particle μ is defined as

$$\mu = \frac{\rho + p}{n}\,, \tag{24}$$

and s represents the entropy per particle. Therefore, on considering $\rho = \rho(n, s)$, we find

$$\mu = \left(\frac{\partial \rho}{\partial n}\right)_s, \tag{25}$$

$$T = \frac{1}{n}\left(\frac{\partial \rho}{\partial s}\right)_n. \tag{26}$$

On combining the two equations of state Eqs. (22), it is easy to show that

$$p = \frac{\rho T}{\mu_g + 3T/2}\,, \tag{27}$$

which shows that an ideal gas does *not* represent a fluid with a barotropic equation of state because $p \neq p(\rho)$; instead we have $p = p(\rho, T)$. Furthermore, this equation shows that for an ideal gas $p/\rho = \mathcal{O}(T/\mu_g) \ll 1$, so that at the zero-th order in T/μ_g this fluid can be well approximated by a dust fluid. However, whenever in the history of the universe, on cosmological scales, the speed of propagation for the baryon fluid cannot be neglected, then we have[4]

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\dot p}{\dot \rho} \approx \frac{\dot\rho T}{\mu_g \dot\rho} + \frac{\rho \dot T}{\mu_g \dot\rho} \approx \frac{T}{\mu_g} - \frac{a\dot T}{\mu_g(3\dot a)} = \frac{T}{\mu_g}\left(1 - \frac{1}{3}\,\frac{a}{T}\,\frac{\dot T}{\dot a}\right) = \frac{T}{\mu_g}\left(1 - \frac{1}{3}\,\frac{d\ln T}{d\ln a}\right), \tag{28}$$

confirming the Eq. (68) in [1] at the first approximation. Hereafter, by an overdot we represent the derivative with respect to the conformal time τ.

In particular this shows that we cannot in general neglect the pressure of such a fluid, and we have

$$p \approx \frac{T}{\mu_g}\,\rho\,. \tag{29}$$

In the presence of a collision term with photons, there is an exchange of entropy with the photon fluid, which when combined with the first law of thermodynamics gives

$$\dot T = -2\,\frac{\dot a}{a}\,T + \frac{8}{3}\,\frac{\rho_\gamma}{\rho}\,\frac{\mu_g}{m_e}\,a n_e \sigma_T (T_\gamma - T)\,, \tag{30}$$

as shown in [1].

IV. BARYON EQUATION OF MOTION

In order to study the dynamics of the perturbations of an ideal gas, which is meant to represent a more realistic and covariant model for non-relativistic baryons at early times, on a cosmological background, we introduce here an action for perfect fluids which is able to completely describe the scalar modes of a non-barotropic perfect fluid (a class to which an ideal gas belongs). The action can be written as follows

$$S_m = -\int d^4x\,\sqrt{-g}[\rho(n,s) + J^\mu\,(\partial_\mu\phi + \vartheta\,\partial_\mu s)]\,, \tag{31}$$

where the fundamental variables are the vector J^α, the metric $g_{\mu\nu}$, and the scalars ϕ, ϑ, s, whereas

$$n \equiv \sqrt{-J^\mu J^\nu g_{\mu\nu}}\,, \tag{32}$$

and here, since the fluid is non-barotropic we have $\rho = \rho(n, s)$. Notice here that the minus sign in Eq. (32) is needed, as J^μ represents a timelike vector.

This action written in these variables, to the best of our knowledge, has not been introduced before. However, on redefining the vector variable J^α in terms of a vector density, as in $J^\alpha = \bar J^\alpha/\sqrt{-g}$, the action reduces to the Schutz-Sorkin action of [21] (see also, e.g. [22, 23]). The point of introducing the Lagrangian defined in Eq. (31) is that it allows us to study general FLRW cosmology with curvature terms in terms of fields whose interpretation and dynamics is at the same time simpler and clearer.

The equation of motion for the field ϕ gives:

$$\nabla_\mu J^\mu = 0\,, \tag{33}$$

which is related to the conservation of number of particles. In fact, on defining u^μ so that

$$J^\mu = n\,u^\mu\,, \tag{34}$$

[4] It should be noted that for a general perfect fluid, the notion of the sound speed is purely thermodynamical, so that once the equations of state are imposed, its expression is independent of the background. For an ideal gas, we find $c_s^2 = 10T/(6\mu_g + 15T)$. Furthermore, since p can be written in terms of two other thermodynamical variables, we have that at linear order $\delta p = (\partial p/\partial\rho)_s\,\delta\rho + (\partial p/\partial s)_\rho\,\delta s$, which can be rewritten as $\delta p = c_s^2\,\delta\rho + 4\mu_g p/(6\mu_g + 15T)\,\delta s$. In particular we find that, in general, $c_s^2 \neq \delta p/\delta\rho$.

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then Eq. (32) leads to the following constraint on the u^μ:

$$u^\mu u_\mu = -1\,. \tag{35}$$

Therefore the time-like vector u^α represents the 4-velocity of the fluid, and J^α is such a 4-velocity vector multiplied by the 4-scalar n, the gas number density. This property, in particular, implies that on a general FLRW background, $J^i = 0$.

The equation of motion for the field ϑ leads instead to the conservation of entropy, namely

$$u^\mu \nabla_\mu s = u^\mu \partial_\mu s = 0\,. \tag{36}$$

The equation of motion for the field s leads to $u^\mu \partial_\mu \vartheta = T$ (because $(\partial\rho/\partial s)_n = nT$, from the first law of thermodynamics), whereas the equations of motion for J^μ relates J^μ (or u^μ) to the other fields ϕ, ϑ, s.

We can decompose the scalar contributions from the matter field action, at linear order in perturbation theory about a general FLRW background, as follows

$$s = s_0 + \delta s\, Y\,, \tag{37}$$

$$\phi = -\int^\tau d\eta\, a(\eta)\, \overline{\rho}_{,n} + \delta\phi\, Y\,, \tag{38}$$

$$\vartheta = \int^\tau d\eta\, a^4\, \overline{\rho}_{,s}/N_0 + \delta\vartheta\, Y\,, \tag{39}$$

$$J^0 = \frac{N_0}{a^4}(1 + W_0\, Y)\,, \tag{40}$$

$$J^i = \frac{W}{a^2}\,\gamma^{ij} Y_{|j}\,, \tag{41}$$

where N_0 represents the total number of fluid particles and we have also defined

$$ds^2 = -(1+2\alpha Y)\, a^2\, d\tau^2 + 2a\chi\, Y_{|i}\, d\tau\, dx^i + [a^2(1+2\zeta Y)\gamma_{ij} + 2E\, Y_{|ij}]\, dx^i dx^j\,. \tag{42}$$

Here, $a = a(\tau)$ is the scale factor, γ_{ij} is the metric of a 3-dimensional constant-curvature space, the time-independent function Y is determined by the property $\gamma^{ij} Y_{|ij} = -k^2 Y$, and the subscript $|i$ represents the spatial covariant derivative compatible with the 3D metric γ_{ij}. All the coefficients (δs etc.) are functions of time only.

On a FLRW background, Eq. (36) leads to $s = \text{constant} = s_0$. As a consequence, the perturbation of entropy per particle δs becomes gauge invariant, and corresponds to a non-adiabatic mode. On perturbing Eq. (36) at the first order, we find

$$\delta u^\mu \partial_\mu s_0 + u^0 \partial_0 \delta s = 0\,, \tag{43}$$

which, in the real space, implies that

$$\frac{\partial}{\partial\tau}\,\delta s = 0\,. \tag{44}$$

This condition of adabaticity, i.e. being constant the entropy per baryon is a direct consequence of the equations of state and conservation of stress-energy tensor, and it was also correctly considered in Ma & Bertschinger (see statemet before their Eq. (96)). Only interactions with photon may affect this, as these may also affect exchange of heat the systems. However quantum mechanically, the fluids only exchange relative-momentum (which is not zero in general, even without interactions) at tree level, ad in particular, do not generate momentum. So we do not expect generation of entropy either. Actually we can (and have to) choose the initial conditions for the entropy perturbations (at the end of inflation) so that we have an adiabatic fluid, namely $\delta s(\vec{x}) = 0$, having assumed that, at the end of inflation, no non-adiabatic mode was produced or present. In this case, since we impose δs to vanish, then, because of this boundary condition, we find $\delta p = c_s^2 \delta\rho + (\partial p/\partial s)_\rho\, \delta s = c_s^2 \delta\rho$.

Then we further find it convenient to define new perturbation variables v, δ, $\delta\vartheta_v$, and θ so that

$$\delta\phi = \rho_{,n}\, v - \vartheta(\tau)\, \delta s\,, \tag{45}$$

$$\delta\vartheta = \delta\vartheta_v - \frac{\rho_{,s}}{n}\, v\,, \tag{46}$$

$$W_0 = \frac{\rho}{n\rho_{,n}}\,\delta - \alpha - \frac{\rho_{,s}}{n\rho_{,n}}\,\delta s\,, \tag{47}$$

$$v = -\frac{a}{k^2}\,\theta\,, \tag{48}$$

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where Eq. (47) has been found on considering the definition of the variable δ, namely $\delta \equiv \delta\rho/\bar{\rho}$. As a result, the two main equations of motion coming from the matter Lagrangian can be written as

$$\dot{\theta} = -\frac{\dot{a}}{a}\,\theta + k^2\alpha + \frac{\rho}{\rho+p}\,c_s^2 k^2\left(\delta + 3\,\frac{\dot{a}}{a}\,\frac{\rho+p}{\rho}\,\frac{\theta}{k^2}\right), \tag{49}$$

$$\frac{\partial}{\partial\tau}\left(\frac{\rho}{\rho+p}\,\delta\right) = -\theta - 3\dot{\zeta} + k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - \frac{k^2}{a}\,\chi\,, \tag{50}$$

where we have not fixed any gauge yet. Of course, the same equations can be derived from conservation of the stress-energy tensor, namely ${T^\mu}_{\nu;\mu} = 0$.

It can be noticed that, in the first equation, a gauge-invariant combination associated with δ is present, namely the comoving matter energy density perturbation

$$\delta_v = \delta + 3\,\frac{\dot{a}}{a}\,\frac{\rho+p}{\rho}\,\frac{\theta}{k^2}\,, \tag{51}$$

whereas, in the second equation, another gauge-invariant combination associated again with δ appears, namely the flat-gauge energy density perturbation, or

$$\delta_{\rm FG} = \delta + \frac{3\,(\rho+p)}{\rho}\,\zeta\,. \tag{52}$$

In fact, the second equation can also be rewritten as

$$\dot{\delta} = -\frac{\rho+p}{\rho}\left[\theta + 3\dot{\zeta} + \frac{k^2}{a}\,\chi - k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)\right] - 3\frac{\dot{a}}{a}\left[c_s^2 - \frac{p}{\rho}\right]\delta\,. \tag{53}$$

where $c_s^2 = \dot{p}/\dot{\rho} = n\rho_{,nn}/\rho_{,n}$ is the speed of propagation for the fluid. Eq. (49) and Eq. (53) are the equations of motion for the fluid that we will consider from now on. (In appendix B we give an alternative derivation of (49) and (53), based on the conservation of the stress-energy tensor.)

In the above derivation (and also in the derivation presented in appendix B), the Einstein equation is not used. Therefore, for a system with multiple fluids, Eq. (49) and Eq. (53) hold for each perfect fluid component which is separately conserved and adiabatic, provided that ρ, p and perturbation variables are replaced with the corresponding quantities for the component.

A. Expansion in T

Up to now, Eqs. (49) and (53) still hold for a general adiabatic perfect fluid, i.e. not only for an ideal gas. From now on, we will instead restrict our consideration to the case of an ideal gas with a non-zero collision term with a photon gas. We will fix the equations of motion by making an expansion in $T/\mu_g \ll 1$, as the baryon particles are supposed to be non-relativistic. The dynamical equation for T, Eq. (30), reads

$$\frac{1}{aH}\,\frac{d}{d\tau}\left(\frac{T}{\mu_g}\right) = -2\,\frac{T}{\mu_g} + \frac{8}{3}\,\frac{\rho_\gamma}{\rho}\,\frac{n_e\sigma_T}{H}\,\frac{T_\gamma - T}{m_e}\,,$$

where we have the Hubble expansion rate as $H = \dot{a}/a^2$, so that, we will need to assume also that

$$\frac{\rho_\gamma}{\rho}\,\frac{n_e\sigma_T}{H}\,\frac{T_\gamma - T}{m_e} \ll 1\,. \tag{54}$$

Besides we have Eqs. (22) which imply

$$c_s^2 = \left(\frac{\partial p}{\partial\rho}\right)_s = \frac{10\,T}{3\,(5T + 2\mu_g)} \approx \frac{5}{3}\,\frac{T}{\mu_g}\,. \tag{55}$$

The Boltzmann equations, on introducing also the interaction term, can be written as

$$\dot{\theta} = -\frac{\dot{a}}{a}\,\theta + k^2\alpha + \frac{\rho}{\rho+p}\,c_s^2 k^2\left(\delta + 3\,\frac{\dot{a}}{a}\,\frac{\rho+p}{\rho}\,\frac{\theta}{k^2}\right) + \frac{4\rho_\gamma}{3\rho}\,a n_e\sigma_T(\theta_\gamma - \theta), \tag{56}$$

$$\dot{\delta} = -\frac{\rho+p}{\rho}\left[\theta + 3\dot{\zeta} + \frac{k^2}{a}\,\chi - k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)\right] - 3\,\frac{\dot{a}}{a}\left[c_s^2 - \frac{p}{\rho}\right]\delta\,, \tag{57}$$

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then at the lowest order in T/μ_g, we find

$$\dot{\theta} = -\frac{\dot{a}}{a}\,\theta + k^2\alpha\,, \tag{58}$$

$$\dot{\delta} = -\theta - 3\dot{\zeta} + k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - \frac{k^2}{a}\,\chi\,, \tag{59}$$

which represents the eoms of a cold dust component with no interactions.

At the first order in $c_s^2 \simeq T/\mu_g$, these equations of motion can be rewritten so that they look as similar as possible to the ones present in [1], as follows:

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\,\theta_b + k^2\alpha + c_s^2\,k^2\left(\delta_b + 3\,\frac{\dot{a}}{a}\,\frac{\theta_b}{k^2}\right) + R\,an_e\sigma_T(\theta_\gamma - \theta_b)\,, \tag{60}$$

$$\dot{\delta}_b = -\frac{6}{5}\,c_s^2\,\frac{\dot{a}}{a}\,\delta_b - \left(1 + \frac{3}{5}\,c_s^2\right)\left[\theta_b + 3\dot{\zeta} - k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) + \frac{k^2}{a}\,\chi\right], \tag{61}$$

where $R \equiv \frac{4}{3}\,\rho_\gamma/\rho_b$, c_s^2 can be found by solving Eq. (28), and the subscript b and γ denotes baryon and photon respectively. Both of these equations of motion for the baryon perturbation variables are gauge invariant (up to the first order in c_s^2). In case higher precision is needed, then one can further write down the equations of motion at any order in T/μ_g. In this work, we will only consider the first order approximation corrections to the dust fluid case, and we will apply them in a consistent and covariant way to a well-known Boltzmann code, CLASS.

Actually, the above two equations are indeed different from the baryon equations of motion given in [1]: they have to be, as the latter ones are not covariant. In this work, we claim that, on introducing these "new" covariant equations, we can solve all the three problems we have already stated in the introduction. The solution here merely comes from the fact that our baryon equations of motion have been derived directly from a covariant action, and, on top of that, we are expanding them in terms of $c_s^2 = (\partial p/\partial\rho)_s$, which is a scalar.

We believe that these are the equations of motions which need to be implemented in any Boltzmann code, otherwise baryon physics will be described out of general relativity.

V. TIGHT COUPLING APPROXIMATION

In 1970, Peebles & Yu [18] introduced a technique to solve the cosmological evolution of a tightly coupled photon-baryon fluid. The interaction time scale of photons and baryons is given by $\tau_c \equiv (an_e\sigma_T)^{-1}$, where σ_T is the Thomson scattering amplitude. This time scale of the interaction is shorter than both sub-Horizon and super-Horizon scales, on which most of the modes of our interest are evolved. At the time when photons and baryons are tightly coupled together, the dynamical equations of motion become stiff, so that standard numerical integrators become invalid. They solved this system perturbatively in τ_c for terms which are considerably small in the limit $\tau_c \to 0$. These perturbative solutions are implemented numerically in the Boltzmann code. Here we recalculate the tight coupling approximation equations using the gauge invariant equation of motion of baryons derived in the previous section.

The first order tight coupling approximation is implemented in [1] by making two additional assumptions/approximations, namely $\tau_c \propto a^2$ and $c_s^2 \propto a^{-1}$. CAMB [3] also have implemented the first order approximation, assuming only $\tau_c \propto a^2$. Here we discuss tight coupling approximation up to the second order with corrected equations of baryons.

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We have the following set of equations for the photon fluid, without fixing any gauge[5]:

$$\dot{\delta}_\gamma = -\frac{4}{3}\,\theta_\gamma - \frac{4}{3}\,\frac{k^2}{a}\,\chi + \frac{4}{3}\,\partial_\tau\left(\frac{k^2 E}{a^2}\right) - 4\dot{\zeta}, \tag{62}$$

$$\dot{\theta}_\gamma = \frac{k^2}{4}\,\delta_\gamma - k^2\sigma_\gamma + k^2\alpha - \frac{1}{\tau_c}\,(\theta_\gamma - \theta_b)\,, \tag{63}$$

$$2\dot{\sigma}_\gamma = \frac{8}{15}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right] - \frac{3}{5}\,k\,F_{\gamma 3} - \frac{9}{5\tau_c}\,\sigma_\gamma + \frac{1}{10\tau_c}\,(G_{\gamma 0} + G_{\gamma 2})\,, \tag{64}$$

$$\dot{F}_{\gamma l} = \frac{k}{2l+1}\,\left[l F_{\gamma(l-1)} - (l+1)\,F_{\gamma(l+1)}\right] - \frac{1}{\tau_c} F_{\gamma l}\,, \qquad l \geq 3 \tag{65}$$

$$\begin{aligned}\dot{G}_{\gamma l} &= \frac{k}{2l+1}\,\left[l G_{\gamma(l-1)} - (l+1)\,G_{\gamma(l+1)}\right] \\ &\quad + \frac{1}{\tau_c}\left[-G_{\gamma l} + \frac{1}{2}\,(F_{\gamma 2} + G_{\gamma 0} + G_{\gamma 2})\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right]\,,\end{aligned} \tag{66}$$

where $F_{\gamma 2} = 2\sigma_\gamma$, $F_{\gamma l}$ is higher multi poles of the photon Boltzmann hierarchical equations and $G_{\gamma l}$ is multi poles of Boltzmann hierarchical equations for the difference in the photon linear polarization components [1, 28].

We can rewrite the two equations for the speed of photons and baryon, given respectively by (63) and (60), as

$$\tau_c\left[\dot{\theta}_\gamma - \frac{k^2}{4}\,\delta_\gamma + k^2\sigma_\gamma - k^2\alpha\right] + \Theta_{\gamma b} = 0\,, \tag{67}$$

$$\tau_c\left[-\dot{\theta}_b - \frac{\dot{a}}{a}\,\theta_b + k^2\alpha + c_s^2\,k^2\left(\delta_b + 3\,\frac{\dot{a}}{a}\,\frac{\theta_b}{k^2}\right)\right] + R\,\Theta_{\gamma b} = 0\,, \tag{68}$$

where[6]

$$\Theta_{\gamma b} = \theta_\gamma - \theta_b\,. \tag{69}$$

Then adding both these two equations we obtain

$$\tau_c\left[\dot{\Theta}_{\gamma b} - \mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma + k^2\sigma_\gamma\right] + (1+R)\,\Theta_{\gamma b} = 0\,, \tag{70}$$

where

$$\mathcal{H} \equiv \frac{\dot{a}}{a}\,. \tag{71}$$

The above equation determines the evolution of $\Theta_{\gamma b}$, which is often referred to as the slip parameter. The equation (70) involves the shear of photons. The shear equation (64) for the photon can be rewritten as,

$$\sigma_\gamma = \frac{\tau_c}{9}\left\{\frac{8}{3}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right] - 3\,k\,F_{\gamma 3} - 10\dot{\sigma}_\gamma\right\} + \frac{1}{18}\,(G_{\gamma 0} + G_{\gamma 2})\,. \tag{72}$$

We can consider linear combination of Eqs. (67) and (68) in order to eliminate $\Theta_{\gamma b}$, so that we find:

$$\dot{\theta}_b + R\dot{\theta}_\gamma = Rk^2\left(\frac{1}{4}\,\delta_\gamma - \sigma_\gamma\right) + (1+R)\,k^2\alpha - \mathcal{H}\left(1 - 3c_s^2\right)\,\theta_b + c_s^2\,k^2\delta_b\,. \tag{73}$$

From the above equation we obtain the equation for $\dot{\theta}_\gamma$ as

$$\dot{\theta}_\gamma = -\frac{\dot{\theta}_b}{R} + k^2\left(\frac{1}{4}\,\delta_\gamma - \sigma_\gamma\right) + \frac{1+R}{R}\,k^2\alpha - \frac{\mathcal{H}}{R}\left(1 - 3c_s^2\right)\,\theta_b + \frac{c_s^2\,k^2}{R}\delta_b\,. \tag{74}$$

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5 In the case of non-flat 3D slices, the equations of motion need to be changed. For example, in Eq. (63), the shear field gets an extra factor, $\sigma_\gamma \to s_2^2\,\sigma_\gamma$, where, following the CLASS-code notation, $s_2^2 \equiv 1 - 3K/k^2$.

6 We have neglected the contribution from the baryon pressure from the term $R = (\rho_\gamma + p_\gamma)/(\rho_b + p_b)$, because the c_s^2 correction term, typically of order of $\Theta_{\gamma b}^2$ will affect tight coupling approximation only at higher orders (e.g. the cubic order).

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Since $\dot{\Theta}_{\gamma b} = \dot{\theta}_\gamma - \dot{\theta}_b$, we can rewrite Eq. (73) as

$$\dot{\theta}_b + R(\dot{\Theta}_{\gamma b} + \dot{\theta}_b) = Rk^2 \left(\frac{1}{4}\, \delta_\gamma - \sigma_\gamma \right) + (1+R)\, k^2 \alpha - \mathcal{H} \left(1 - 3c_s^2\right)\, \theta_b + c_s^2\, k^2 \delta_b \,, \tag{75}$$

or

$$\dot{\theta}_b = -\frac{1}{1+R} \left\{ \mathcal{H}\, (1 - 3c_s^2)\, \theta_b - (1+R)\, k^2 \alpha - Rk^2 \left(\frac{1}{4}\, \delta_\gamma - \sigma_\gamma \right) - c_s^2\, k^2 \delta_b + R\dot{\Theta}_{\gamma b} \right\} , \tag{76}$$

$$\begin{aligned} \dot{\theta}_\gamma &= \dot{\Theta}_{\gamma b} - \frac{1}{1+R} \left\{ \mathcal{H}\, (1 - 3c_s^2)\, \theta_b - (1+R)\, k^2 \alpha - Rk^2 \left(\frac{1}{4}\, \delta_\gamma - \sigma_\gamma \right) - c_s^2\, k^2 \delta_b + R\dot{\Theta}_{\gamma b} \right\} \\ &= -\frac{1}{1+R} \left\{ \mathcal{H}\, (1 - 3c_s^2)\, \theta_b - (1+R)\, k^2 \alpha - Rk^2 \left(\frac{1}{4}\, \delta_\gamma - \sigma_\gamma \right) - c_s^2\, k^2 \delta_b - \dot{\Theta}_{\gamma b} \right\} . \end{aligned} \tag{77}$$

All these equations are exact. In what follows, we will mainly use Eqs. (69), (70), (72) and (76) in order to find approximate solutions for $\Theta_{\gamma b}$ and σ_γ.

A. Terms in G

In the equation of motion for the shear, Eq. (66), there appear terms in $G_{\gamma 0}$ and $G_{\gamma 2}$. Let us first see the perturbative solution of these two terms. Let us consider equation for $G_{\gamma l}$ with $l = 1, 3$,

$$\tau_c \dot{G}_{\gamma 1} = \frac{k\tau_c}{3}\, [G_{\gamma 0} - 2G_{\gamma 2}] - G_{\gamma 1} \,, \tag{78}$$

$$\tau_c \dot{G}_{\gamma 3} = \frac{k\tau_c}{7}\, [3G_{\gamma 2} - 4G_{\gamma 4}] - G_{\gamma 3} \,, \tag{79}$$

with the assumption, to be confirmed later on, that $G_{\gamma 0} = \mathcal{O}(\tau_c)$ and $G_{\gamma 2} = \mathcal{O}(\tau_c)$, and that $G_{\gamma 4}$ is even more suppressed. In this case, let us look for solutions of the kind

$$G_{\gamma 1} = G_{\gamma 1}^{(0)} + \tau_c G_{\gamma 1}^{(1)} + \tau_c^2 G_{\gamma 1}^{(2)} \,. \tag{80}$$

Then we find

$$\tau_c[\dot{G}_{\gamma 1}^{(0)} + \tau_c \dot{G}_{\gamma 1}^{(1)} + \dot{\tau}_c G_{\gamma 1}^{(1)} + \tau_c^2 \dot{G}_{\gamma 1}^{(2)} + 2\tau_c \dot{\tau}_c G_{\gamma 1}^{(2)}] = \mathcal{O}(\tau_c^2) - (G_{\gamma 1}^{(0)} + \tau_c G_{\gamma 1}^{(1)} + \tau_c^2 G_{\gamma 1}^{(2)}) \,, \tag{81}$$

where we have assumed that $\dot{\tau}_c/(a\tau_c) \simeq H$, which is valid, as long as tight coupling approximation is at work. This last equation leads to the lowest order to

$$G_{\gamma 1}^{(0)} = 0 \,. \tag{82}$$

Then

$$\tau_c[\tau_c \dot{G}_{\gamma 1}^{(1)} + \dot{\tau}_c G_{\gamma 1}^{(1)} + \tau_c^2 \dot{G}_{\gamma 1}^{(2)} + 2\tau_c \dot{\tau}_c G_{\gamma 1}^{(2)}] = \mathcal{O}(\tau_c^2) - (\tau_c G_{\gamma 1}^{(1)} + \tau_c^2 G_{\gamma 1}^{(2)}) \,, \tag{83}$$

or, to the lowest order now

$$G_{\gamma 1}^{(1)} = 0 \,, \tag{84}$$

and

$$\tau_c[\tau_c^2 \dot{G}_{\gamma 1}^{(2)} + 2\tau_c \dot{\tau}_c G_{\gamma 1}^{(2)}] = \mathcal{O}(\tau_c^2) - \tau_c^2 G_{\gamma 1}^{(2)} \,,$$

so that

$$G_{\gamma 1}^{(2)} \neq 0 \,, \tag{85}$$

and

$$G_{\gamma 1} = \mathcal{O}(\tau_c^2) \,. \tag{86}$$

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A similar argument leads to

$$G_{\gamma 3} = \mathcal{O}(\tau_c^2)\,. \tag{87}$$

Now we need to verify that indeed $G_{\gamma 0} = \mathcal{O}(\tau_c)$ and $G_{\gamma 2} = \mathcal{O}(\tau_c)$. In fact, we have

$$\begin{aligned} \tau_c \dot{G}_{\gamma 0} &= k\tau_c\,[-G_{\gamma 1}] - G_{\gamma 0} + \frac{1}{2}\,(2\sigma_\gamma + G_{\gamma 0} + G_{\gamma 2})\,, \\ &\approx -\frac{1}{2}\,G_{\gamma 0} + \frac{1}{2}\,(2\sigma_\gamma + G_{\gamma 2})\,, \end{aligned} \tag{88}$$

$$\begin{aligned} \tau_c \dot{G}_{\gamma 2} &= \frac{k\tau_c}{5}\,[2G_{\gamma 1} - 3G_{\gamma 3}] - G_{\gamma 2} + \frac{1}{10}\,(2\sigma_\gamma + G_{\gamma 0} + G_{\gamma 2})\,, \\ &\approx -\frac{9}{10}\,G_{\gamma 2} + \frac{1}{10}\,(2\sigma_\gamma + G_{\gamma 0})\,, \end{aligned} \tag{89}$$

so that on looking for solutions of the kind $G_{\gamma 0} = G^{(1)}_{\gamma 0} + \tau_c G^{(2)}_{\gamma 0}$, and $G_{\gamma 2} = G^{(1)}_{\gamma 2} + \tau_c G^{(2)}_{\gamma 2}$, we find

$$\tau_c\left[\dot{G}^{(1)}_{\gamma 0} + \tau_c\left(\dot{G}^{(2)}_{\gamma 0} + \frac{\dot{\tau}_c}{\tau_c}\,G^{(2)}_{\gamma 0}\right)\right] = -\frac{1}{2}\,(G^{(1)}_{\gamma 0} + \tau_c G^{(2)}_{\gamma 0}) + \frac{1}{2}\,(2\sigma_\gamma + G^{(1)}_{\gamma 2} + \tau_c G^{(2)}_{\gamma 2})\,, \tag{90}$$

$$\tau_c\left[\dot{G}^{(1)}_{\gamma 2} + \tau_c\left(\dot{G}^{(2)}_{\gamma 2} + \frac{\dot{\tau}_c}{\tau_c}\,G^{(2)}_{\gamma 2}\right)\right] = -\frac{9}{10}\,(G^{(1)}_{\gamma 2} + \tau_c G^{(2)}_{\gamma 2}) + \frac{1}{10}\,(2\sigma_\gamma + G^{(1)}_{\gamma 0} + \tau_c G^{(2)}_{\gamma 0})\,, \tag{91}$$

so that, at the lowest order we find

$$-\frac{1}{2}\,G^{(1)}_{\gamma 0} + \frac{1}{2}\,(2\sigma_\gamma + G^{(1)}_{\gamma 2}) = 0\,, \tag{92}$$

$$-\frac{9}{10}\,G^{(1)}_{\gamma 2} + \frac{1}{10}\,(2\sigma_\gamma + G^{(1)}_{\gamma 0}) = 0\,, \tag{93}$$

or

$$G^{(1)}_{\gamma 2} = \frac{1}{2}\,\sigma_\gamma\,, \tag{94}$$

$$G^{(1)}_{\gamma 0} = \frac{5}{2}\,\sigma_\gamma\,. \tag{95}$$

Here we have assumed for the moment that $\sigma_\gamma = \mathcal{O}(\tau_c)$. We will check later on that this assumption is consistent. Then at the next order

$$\tau_c[\dot{G}^{(1)}_{\gamma 0}] = -\frac{1}{2}\,(\tau_c G^{(2)}_{\gamma 0}) + \frac{1}{2}\,(\tau_c G^{(2)}_{\gamma 2})\,, \tag{96}$$

$$\tau_c[\dot{G}^{(1)}_{\gamma 2}] = -\frac{9}{10}\,(\tau_c G^{(2)}_{\gamma 2}) + \frac{1}{10}\,(\tau_c G^{(2)}_{\gamma 0})\,, \tag{97}$$

or

$$\frac{5}{2}\,\dot{\sigma}_\gamma = -\frac{1}{2}\,G^{(2)}_{\gamma 0} + \frac{1}{2}\,G^{(2)}_{\gamma 2}\,, \tag{98}$$

$$\frac{1}{2}\,\dot{\sigma}_\gamma = -\frac{9}{10}\,G^{(2)}_{\gamma 2} + \frac{1}{10}\,G^{(2)}_{\gamma 0}\,, \tag{99}$$

which leads to

$$G^{(2)}_{\gamma 0} = -\frac{25}{4}\,\dot{\sigma}_\gamma\,, \tag{100}$$

$$G^{(2)}_{\gamma 2} = -\frac{5}{4}\,\dot{\sigma}_\gamma\,, \tag{101}$$

and to

$$G_{\gamma 0} = \frac{5}{2}\,\sigma_\gamma - \frac{25}{4}\,\tau_c\dot{\sigma}_\gamma\,, \tag{102}$$

$$G_{\gamma 2} = \frac{1}{2}\,\sigma_\gamma - \frac{5}{4}\,\tau_c\dot{\sigma}_\gamma\,. \tag{103}$$

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For general $l \geq 3$, we have

$$\tau_c \dot{G}_{\gamma l} = \frac{k\tau_c}{2l+1}\left[l G_{\gamma,l-1} - (l+1)\, G_{\gamma,l+1}\right] - G_{\gamma l}\,, \tag{104}$$

and, since there are no source terms, we will assume that each $(l+1)$-th term is suppressed by τ_c with respect to the l-th term, that is

$$G_{\gamma l} = \beta_l\, \tau_c G_{\gamma,l-1}\,, \tag{105}$$

so that we find

$$\beta_l\, \tau_c^2 \left[\dot{G}_{\gamma,l-1} + \frac{\dot{\tau}_c}{\tau_c}\, G_{\gamma,l-1}\right] = \frac{k\tau_c}{2l+1}\left[l G_{\gamma,l-1} - (l+1)\, \beta_{l+1}\, \tau_c G_{\gamma,l}\right] - \beta_l\, \tau_c G_{\gamma,l-1}\,, \tag{106}$$

or

$$\beta_l\, \tau_c^2 \left[\dot{G}_{\gamma,l-1} + \frac{\dot{\tau}_c}{\tau_c}\, G_{\gamma,l-1}\right] = \frac{k\tau_c}{2l+1}\left[l G_{\gamma,l-1} - (l+1)\, \beta_{l+1}\beta_l\, \tau_c^2\, G_{\gamma,l-1}\right] - \beta_l\, \tau_c G_{\gamma,l-1}\,, \tag{107}$$

which leads, at leading order, to

$$0 = \frac{k\tau_c}{2l+1}\, l G_{\gamma,l-1} - \beta_l\, \tau_c G_{\gamma,l-1}\,, \tag{108}$$

or

$$\beta_l = \frac{kl}{2l+1}\,, \tag{109}$$

so that we find

$$G_{\gamma l} = \frac{l}{2l+1}\, k\tau_c\, G_{\gamma,l-1}\,, \qquad \text{for } l \geq 3. \tag{110}$$

And since we obtain the same eoms for the terms $F_{\gamma l}$ for $l \geq 3$, then we also have

$$F_{\gamma l} = \frac{l}{2l+1}\, k\tau_c\, F_{\gamma,l-1}\,, \qquad \text{for } l \geq 3. \tag{111}$$

All Eqs. (102), (103), (110), and (111) agree with the results given in [2].

B. Shear solution

Now we need to look for an approximate solution for the shear. Using the solutions for $G_{\gamma 0,2}$ and $F_{\gamma 3}$, we can rewrite Eq. (64) as

$$2\tau_c \dot{\sigma}_\gamma = \frac{8}{15}\, \tau_c \left[\theta_\gamma + \frac{k^2}{a}\, \chi - k^2\, \partial_\tau \left(\frac{E}{a^2}\right)\right] - \frac{3}{5}\, k\, \tau_c\, F_{\gamma 3} - \frac{9}{5}\, \sigma_\gamma + \frac{1}{10}\left(\frac{5}{2}\, \sigma_\gamma - \frac{25}{4}\, \tau_c \dot{\sigma}_\gamma + \frac{1}{2}\, \sigma_\gamma - \frac{5}{4}\, \tau_c \dot{\sigma}_\gamma\right), \tag{112}$$

which leads to

$$2\tau_c \dot{\sigma}_\gamma = \frac{8}{15}\, \tau_c \left[\theta_\gamma + \frac{k^2}{a}\, \chi - k^2\, \partial_\tau \left(\frac{E}{a^2}\right)\right] - \frac{3}{5}\, k\, \tau_c\, F_{\gamma 3} - \frac{9}{5}\, \sigma_\gamma + \frac{1}{10}\left(3\, \sigma_\gamma - \frac{15}{2}\, \tau_c \dot{\sigma}_\gamma\right), \tag{113}$$

and

$$\begin{aligned}\left(2 + \frac{3}{4}\right) \tau_c \dot{\sigma}_\gamma &= \frac{8}{15}\, \tau_c \left[\theta_\gamma + \frac{k^2}{a}\, \chi - k^2\, \partial_\tau \left(\frac{E}{a^2}\right)\right] - \frac{18}{35}\, k^2\, \tau_c^2\, \sigma_\gamma - \frac{3}{2}\, \sigma_\gamma\,, \\ &= \frac{8}{15}\, \tau_c \left[\theta_\gamma + \frac{k^2}{a}\, \chi - k^2\, \partial_\tau \left(\frac{E}{a^2}\right)\right] - \frac{3}{2}\, \sigma_\gamma + \mathcal{O}(\tau_c^3)\,. \end{aligned} \tag{114}$$

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Now, let us assume that we have a solution of the form

$$\sigma_\gamma = \tau_c\sigma_\gamma^{(1)} + \tau_c^2\sigma_\gamma^{(2)}\,, \tag{115}$$

then we obtain

$$\frac{11}{4}\,\tau_c^2\left[\dot\sigma_\gamma^{(1)} + \frac{\dot\tau_c}{\tau_c}\,\sigma_\gamma^{(1)} + \tau_c\left(\dot\sigma_\gamma^{(2)} + 2\frac{\dot\tau_c}{\tau_c}\sigma_\gamma^{(2)}\right)\right] = \frac{8}{15}\,\tau_c\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right] - \frac{3}{2}\,\left[\tau_c\sigma_\gamma^{(1)} + \tau_c^2\sigma_\gamma^{(2)}\right], \tag{116}$$

which at the lowest order leads to

$$\sigma_\gamma^{(1)} = \frac{16}{45}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right], \tag{117}$$

so that, at the next order we have

$$\frac{11}{4}\,\tau_c^2\left[\dot\sigma_\gamma^{(1)} + \frac{\dot\tau_c}{\tau_c}\,\sigma_\gamma^{(1)}\right] = -\frac{3}{2}\,\tau_c^2\sigma_\gamma^{(2)}\,, \tag{118}$$

or

$$\sigma_\gamma^{(2)} = -\frac{11}{6}\,\left[\dot\sigma_\gamma^{(1)} + \frac{\dot\tau_c}{\tau_c}\,\sigma_\gamma^{(1)}\right] = -\frac{88}{135}\,\frac{d}{d\tau}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right] - \left(\frac{11}{6}\,\frac{\dot\tau_c}{\tau_c}\right)\frac{16}{45}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right]. \tag{119}$$

Hence we find the approximate solution as

$$\sigma_\gamma = \frac{16}{45}\,\tau_c\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right]\left(1 - \frac{11}{6}\,\dot\tau_c\right) - \frac{88}{135}\,\tau_c^2\,\frac{d}{d\tau}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right]. \tag{120}$$

This approximate solution agrees with the one found in [2].

C. Slip equation

From now on, because we will make use of the new equations of motion for the baryon fluid, our results will start differing from the ones given in [2]. To find an approximate solution for the slip parameter, up to the second order in τ_c, let us start with Eq. (70):

$$\tau_c\left[\dot\Theta_{\gamma b} - \mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma + k^2\left(\tau_c\sigma_\gamma^{(1)} + \tau_c^2\sigma_\gamma^{(2)}\right)\right] + (1+R)\,\Theta_{\gamma b} = 0\,, \tag{121}$$

and let us search for a solution for $\Theta_{\gamma b}$. Then we have, up to the second order

$$\tau_c\left[\dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma + k^2\left(\tau_c\sigma_\gamma^{(1)} + \tau_c^2\sigma_\gamma^{(2)}\right)\right] = -(1+R)\left[\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}\right]. \tag{122}$$

At the lowest order, we find

$$\tau_c\left[-\mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma\right] = -(1+R)\,\Theta_{\gamma b}^{(1)}, \tag{123}$$

or

$$\Theta_{\gamma b}^{(1)} = -\frac{\tau_c}{1+R}\left[-\mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma\right]. \tag{124}$$

At the second order we find

$$\tau_c\left[\dot\Theta_{\gamma b}^{(1)} + k^2\tau_c\sigma_\gamma^{(1)}\right] = -(1+R)\,\Theta_{\gamma b}^{(2)}, \tag{125}$$

or[7]

$$\Theta_{\gamma b}^{(2)} = -\frac{\tau_c}{1+R}\left[\dot\Theta_{\gamma b}^{(1)} + k^2\tau_c\sigma_\gamma^{(1)}\right]. \tag{126}$$

[7] If the background spatial curvature is present, then, as already mentioned in footnote 5, we need to replace $\sigma_\gamma^{(1)} \to s_2^2\,\sigma_\gamma^{(1)}$.

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The solution is then found as

$$\dot{\Theta}_{\gamma b} = \dot{\Theta}^{(1)}_{\gamma b} + \dot{\Theta}^{(2)}_{\gamma b}\,. \tag{127}$$

In the following, we will rewrite the above solution for the slip parameter in such a way that we can easily implement them in the new CLASS code.

D. First order contribution

We first manipulate the first order solutions as follows

$$\begin{aligned}
\dot{\Theta}_{\gamma b} = \dot{\Theta}^{(1)}_{\gamma b} &= \dot{\Theta}^{(1)}_{\gamma b} - \frac{\dot{\tau}_c}{\tau_c}\,\Theta^{(1)}_{\gamma b} + \frac{\dot{\tau}_c}{\tau_c}\,\Theta^{(1)}_{\gamma b} \\
&= \dot{\Theta}^{(1)}_{\gamma b} - \frac{\dot{\tau}_c}{\tau_c}\,\Theta^{(1)}_{\gamma b} + \frac{\dot{\tau}_c}{\tau_c}\,\Theta_{\gamma b} \\
&= \dot{\Theta}^{(1)}_{\gamma b} - \frac{\dot{\tau}_c}{\tau_c}\,\Theta^{(1)}_{\gamma b} + \frac{\beta_1\mathcal{H}}{1+R}\,\Theta^{(1)}_{\gamma b} - \frac{\beta_1\mathcal{H}}{1+R}\,\Theta^{(1)}_{\gamma b} + \frac{\dot{\tau}_c}{\tau_c}\,\Theta_{\gamma b} \\
&= \dot{\Theta}^{(1)}_{\gamma b} - \frac{\dot{\tau}_c}{\tau_c}\,\Theta^{(1)}_{\gamma b} + \frac{\beta_1\mathcal{H}}{1+R}\,\Theta^{(1)}_{\gamma b} - \frac{\beta_1\mathcal{H}}{1+R}\,\Theta_{\gamma b} + \frac{\dot{\tau}_c}{\tau_c}\,\Theta_{\gamma b}\,,
\end{aligned} \tag{128}$$

In this equation, we replace the quantity $\dot{\theta}_b$ (which appears inside $\dot{\Theta}^{(1)}_{\gamma b}$) with its solution for Eq. (68), $\dot{R} = -\mathcal{H}R$, $\dot{c}_s^2 = \bar{c}_s^2 - \mathcal{H}c_s^2$, and the quantity $\Theta^{(1)}_{\gamma b}$ with Eq. (124). It should be noticed that Eq. (68) comes in the form

$$\dot{\theta}_b = \cdots + R\,\frac{\Theta_{\gamma b}}{\tau_c}\,. \tag{129}$$

Therefore, when we replace $\dot{\theta}_b$, we will end up with an overall coefficient of $\Theta_{\gamma b}$ which partly depends on $\dot{\tau}_c/\tau_c$, and partly on another coefficient which in turn, depends on the free function β_1. Then we choose β_1 so that the result looks as close as possible to the result given in [2] (Eq. (2.19)), namely

$$\beta_1 = 2 + R - 3c_s^2 R\,. \tag{130}$$

Finally, Eq. (128) can be rewritten as

$$\dot{\Theta}_{\gamma b} = \left(\frac{\dot{\tau}_c}{\tau_c} - \frac{2\mathcal{H}}{R+1}\right)\Theta_{\gamma b} + \mathcal{T}_1 + \mathcal{O}(\tau_c^2)\,, \tag{131}$$

where

$$\begin{aligned}
\mathcal{T}_1 \equiv{}& \frac{\tau_c}{R+1}\left[\left(\mathcal{H}^2 + \dot{\mathcal{H}}\right)\theta_b + \left(\frac{\dot{\delta}_\gamma}{4} - c_s^2\dot{\delta}_b + \mathcal{H}\alpha + \frac{\mathcal{H}\delta_\gamma}{2} - \bar{c}_s^2\delta_b\right)k^2\right] \\
&- \frac{3\tau_c}{R+1}\left[\frac{\left\{3\mathcal{H}^2c_s^4 + [\dot{\mathcal{H}}\,(R+1) - \mathcal{H}^2]c_s^2 + \mathcal{H}\bar{c}_s^2\,(R+1)\right\}\theta_b}{R+1} + \mathcal{H}c_s^2k^2\left(\alpha + \frac{1}{4}\,\frac{R\delta_\gamma}{R+1} + \frac{c_s^2\delta_b}{R+1}\right)\right].
\end{aligned} \tag{132}$$

Comparing our results to the ones in [2], the new parts of this equation consist of the following two parts: 1) in the α-term in the first line (corresponding to the fact that a gauge has not been chosen yet) and; 2) in the entire second line which is due to gauge-choice plus the corrections to the baryon dynamics.

Also in this last equation, even if we consider the high-k limit, we should not neglect the new terms proportional to either α or δ_γ.

E. Second order contribution

As we already know, at the second order we find

$$\dot{\Theta}_{\gamma b} = \dot{\Theta}^{(1)}_{\gamma b} + \dot{\Theta}^{(2)}_{\gamma b}\,, \tag{133}$$

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where

$$\Theta_{\gamma b}^{(1)} = -\frac{\tau_c}{1+R}\left[-\mathcal{H}\,\theta_b + c_s^2\,k^2\left(\delta_b + 3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right) - \frac{k^2}{4}\,\delta_\gamma\right], \tag{134}$$

$$\Theta_{\gamma b}^{(2)} = -\frac{\tau_c}{1+R}\left[\dot\Theta_{\gamma b}^{(1)} + k^2\tau_c\sigma_\gamma^{(1)}\right], \tag{135}$$

$$\sigma_\gamma^{(1)} = \frac{16}{45}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right]. \tag{136}$$

Therefore we can substitute all the terms inside Eq. (133). This will lead to substituting $\ddot\Theta_{\gamma b}^{(1)}$, $\dot\Theta_{\gamma b}^{(1)}$, $\dot\sigma_\gamma^{(1)}$, $\sigma_\gamma^{(1)}$, $\ddot R = R\,(\mathcal{H}^2 - \dot{\mathcal{H}})$, $\dot R = -\mathcal{H}R$, $c_s^{2\prime} = \bar c_s^2 - \mathcal{H}c_s^2$. We then obtain a quite complicated expression which corresponds to the needed answer for the slip parameter valid up to the second order in τ_c. However, once more, we try to write it as close as possible to the expression written in [2]. Then we write

$$\begin{aligned}
\dot\Theta_{\gamma b} &= \dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) + \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) \\
&= \dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) + \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,\Theta_{\gamma b}\,.
\end{aligned} \tag{137}$$

In this last line we should think that the quantities $\Theta_{\gamma b}^{(i)}$ are given explicitly, whereas $\Theta_{\gamma b}$ is left as it is. Then we still add new contributions

$$\begin{aligned}
\dot\Theta_{\gamma b} &= \left[\dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)})\right] + \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,\Theta_{\gamma b} \\
&+ \frac{\beta_2}{R+1}\left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\mathcal{H}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) - \frac{\beta_2}{R+1}\left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\mathcal{H}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) \\
&+ \beta_3\,\tau_c\,\Theta_{\gamma b}^{(1)} - \beta_3\,\tau_c\,\Theta_{\gamma b}^{(1)} \\
&= \left[\dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)})\right] + \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,\Theta_{\gamma b} \\
&- \frac{\beta_2}{R+1}\left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\mathcal{H}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) + \frac{\beta_2}{R+1}\left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\mathcal{H}\,\Theta_{\gamma b} \\
&+ \beta_3\,\tau_c\,\Theta_{\gamma b}^{(1)} - \beta_3\,\tau_c\,\Theta_{\gamma b}\,,
\end{aligned} \tag{138}$$

where the functions $\beta_{2,3}$ are supposed to be of order $\mathcal{O}(\tau_c^0)$, and in the very last line the equality holds up to the second order in τ_c. Therefore we can rewrite

$$\dot\Theta_{\gamma b} = \mathcal{A} + \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\left(\frac{\dot\tau_c}{\tau_c} + \frac{\beta_2\mathcal{H}}{R+1}\right)\Theta_{\gamma b} - \beta_3\,\tau_c\,\Theta_{\gamma b}\,, \tag{139}$$

where

$$\mathcal{A} \equiv \left[\dot\Theta_{\gamma b}^{(1)} + \dot\Theta_{\gamma b}^{(2)} - \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\frac{\dot\tau_c}{\tau_c}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)})\right] - \frac{\beta_2}{R+1}\left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\mathcal{H}\,(\Theta_{\gamma b}^{(1)} + \Theta_{\gamma b}^{(2)}) + \beta_3\,\tau_c\,\Theta_{\gamma b}^{(1)}\,. \tag{140}$$

Furthermore, we can easily find the decomposition of

$$\mathcal{A} = \mathcal{A}^{(1)} + \mathcal{A}^{(2)}\,, \tag{141}$$

into the linear and quadratic contributions.

Let us then focus on $\mathcal{A}^{(2)}$. This term will contain terms of the kind $\ddot\theta_\gamma$, $\dot\theta_\gamma$, θ_γ, and we replace them respectively by $\ddot\theta_b$, $\dot\theta_b$, θ_b as their difference appears at the cubic order. Once these terms are replaced, we can, in turn, replace the expressions of $\dot\theta_b$ and $\ddot\theta_b$ by using the zero-th order approximation found by using Eq. (76), which can be written as

$$\dot\theta_b \approx \dot\theta_b^{(0)} \equiv -\frac{1}{1+R}\left[\mathcal{H}\,(1-3c_s^2)\,\theta_b - (1+R)\,k^2\alpha - \frac{Rk^2}{4}\,\delta_\gamma - c_s^2\,k^2\delta_b\right], \tag{142}$$

$$\ddot\theta_b \approx \ddot\theta_b^{(0)}\,. \tag{143}$$

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This substitution is allowed because it is performed inside the quadratic term $\mathcal{A}^{(2)}$. It can be checked that now in $\dot\Theta_{\gamma b}$ there is no more explicit term containing θ_γ or any of its derivatives. Also no more explicit dependence on $\ddot\theta_b$. In the end, after all these substitutions, we have $\mathcal{A}^{(2)} \to \bar{\mathcal{A}}^{(2)}$. However, in the linear contribution we have a non-zero contribution from the term $\dot\theta_b$. For such a term we can replace the exact solution coming from solving Eq. (68), as in

$$\dot\theta_b = -\mathcal{H}\,(1-3c_s^2)\,\theta_b + k^2\alpha + c_s^2\,k^2\delta_b + \frac{R}{\tau_c}\,\Theta_{\gamma b}\,. \tag{144}$$

This term will modify the coefficient of the term $\Theta_{\gamma b}$. Then we can choose the variables $\beta_{2,3}$ so that the linear result looks as similar as possible to the one written in the Eq. (2.20) of [2] . Namely we choose

$$\beta_2 = -2 - R + 3Rc_s^2\,, \tag{145}$$

$$\beta_3 = \frac{2RH^2}{(R+1)^2}(1-3c_s^2)\,, \tag{146}$$

so that in this case the term proportional to $\Theta_{\gamma b}$ reduces to

$$\dot\Theta_{\gamma b} \ni \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\left(\frac{\dot\tau_c}{\tau_c} - \frac{2\mathcal{H}}{R+1}\right)\Theta_{\gamma b}\,. \tag{147}$$

Besides, the linear term (excluding the terms explicitly dependent on $\Theta_{\gamma b}$) exactly coincides with the quantity $\mathcal{T}_1$. We can finally add and subtract a quadratic order term $2\mathcal{H}\tau_c\mathcal{T}_1/(1+R)$ to end up with

$$\dot\Theta_{\gamma b} = \left(1 - \frac{2\mathcal{H}\tau_c}{R+1}\right)\left[\left(\frac{\dot\tau_c}{\tau_c} - \frac{2\mathcal{H}}{R+1}\right)\Theta_{\gamma b} + \mathcal{T}_1\right] + \mathcal{T}_2\,, \tag{148}$$

where

$$\mathcal{T}_2 = \bar{\mathcal{A}}^{(2)} + \frac{2\mathcal{H}\tau_c}{R+1}\,\mathcal{T}_1\,. \tag{149}$$

Here, the expressions for $\beta_{2,3}$ have been replaced into $\bar{\mathcal{A}}^{(2)}$. Now, we are only left with the implementation of these results into the CLASS code.

VI. CODE IMPLEMENTATION

We choose the CLASS Boltzmann solver to implement the corrected baryon and tight coupling approximation equations. We will also make our notation as close as possible to the one used in [1, 2]. So far we have not fixed any gauge, but in the code, we choose the synchronous gauge for the tca approximation scheme. It is straightforward to implement our approximation schemes in any other gauge.

Since we have chosen synchronous gauge, $\alpha = 0 = \chi$. Furthermore, we will make the following field redefinitions

$$-\partial_\tau\left(\frac{E}{a^2}\right) \equiv \alpha_{\rm CL}, \tag{150}$$

$$\zeta = -\eta_{\rm CL}\,, \tag{151}$$

$$k^2\,\frac{E}{a^2} = -\frac{1}{2}\,(h_{\rm CL} + 6\,\eta_{\rm CL})\,, \tag{152}$$

$$\alpha_{\rm CL} = \frac{1}{2k^2}\,(\dot h_{\rm CL} + 6\,\dot\eta_{\rm CL})\,, \tag{153}$$

CLASS has implemented five different tight coupling approximation schemes. In the light of the new baryon equations, we also perform all these approximations except for the one named "second_order_CSR". As mentioned above for the Ma&B-linear-approximation scheme, we make the approximations $\tau_c \propto a^2$ and $c_s^2 \propto a^{-1}$, for the slip parameter at first order. As for the first order CAMB, we only consider the approximation $c_s^2 \propto a^{-1}$. For both first order and second order class schemes we do not make any approximation for τ_c or c_s^2.

The parameter estimation using Monte Python was carried out in a super-computer, XC-40, having 64 nodes, each node having 64 cores. In total 4096 cores were available. We chose to run three independent series[8] of 1024 chains,

[8] One for each author.

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Table I. Best fit values of cosmological parameters given by MCMC analysis: old vs new. The upper and lower limits are at 95% confidence level.

Parameters	Best fit: new (old)
$100\ \omega_b$	$2.236^{+0.041}_{-0.038}\ (2.238^{+0.04}_{-0.04})$
$\omega_{\rm cdm}$	$0.1176^{+0.0021}_{-0.0025}\ (0.117^{+0.0027}_{-0.0019})$
$100\theta_s$	$1.042^{+0.001}_{-0.001}\ (1.042^{+0.001}_{-0.001})$
$\ln 10^{10} A_s$	$3.086^{+0.046}_{-0.051}\ (3.085^{+0.047}_{-0.05})$
n_s	$0.969^{+0.0102}_{-0.0074}\ (0.9726^{+0.0067}_{-0.0111})$
$\tau_{\rm reio}$	$0.07879^{+0.02461}_{-0.02672}\ (0.08009^{+0.02301}_{-0.02841})$
Ω_Λ	$0.6989^{+0.0144}_{-0.0124}\ (0.7022^{+0.0111}_{-0.0157})$
$Y_{\rm He}$	$0.2478^{+0.0002}_{-0.0001}\ (0.2478^{+0.0002}_{-0.0001})$
H_0	$68.35^{+1.13}_{-0.98}\ (68.58^{+0.92}_{-1.19})$
σ_8	$0.8209^{+0.0174}_{-0.0198}\ (0.8194^{+0.0188}_{-0.0184})$
$\Omega_{\rm m}$	$0.301^{+0.0124}_{-0.0144}\ (0.2977^{+0.0158}_{-0.0111})$

each chains using 4 cores in parallel. For each chain, we have performed 13000 steps. We have run all the four tight coupling approximation schemes we have implemented, and found that they are all compatible with each other. On running the code, we have also found that all the tca approximation schemes we have implemented do not make the code any slower or stiffer[9]. It is also noted that the acceptance rate for MCMC analysis is increased by about 0.95% for covariantly corrected code with respect to the old code. The likelihood of the covariantly corrected code get slightly improved to $-\ln\mathcal{L}_{\rm min} = 5984.11$ with respect to the old covariance-breaking code for which the likelihood is $-\ln\mathcal{L}_{\rm min} = 5984.45$.

VII. RESULTS

Here we present our results of running the Monte Carlo sampler for the cosmological parameter estimation. For analysis we used the following data sets: Planck 2015 (high l, low l, and lensing), JLA, BAO BOSS DR12, BAO SMALLZ 2014 and Hubble Space Telescope. We compare the estimation for the cosmological parameters between the old baryon equations (which were non-covariant) and the new covariant ones. We have run the Monte Python sampler for all the four different tight coupling approximation schemes we have implemented in CLASS namely, first_order_Ma, first_order_CAMB, first_order_CLASS, and second_order_CLASS. Below, in Table I, we show the results for the second order tca scheme given by the new covariant equations of motion and the results according to old covariance-breaking baryon equations of motion for the same second order tca scheme. Also we give a combined plots of old and new code for the second order tca scheme in Fig. 2. Here, we only show the numerical results for this scheme, as this is the one whose code underwent the largest number of modifications.

We find that the new results for the cosmological parameters numerically agree with the previous results within one percent, and this fact is reassuring. Nonetheless, our corrections to the equations of motion give a contribution which is not completely negligible, and we believe this improvement can give a useful contribution in the context of "precision cosmology."

The amount of changes in the estimation of the parameters is similar among the different tca schemes we have implemented. In fact, the magnitude of such changes we have obtained is of the same size of the difference in the results that the original non-covariant code was giving for the different tca schemes. This means that in order to address the needed precision in the context of the newest (and future) cosmological probes, we should use Boltzmann solvers with the corrected covariant baryon dynamical equations.

VIII. CONCLUSION

In this paper we have pointed out that there are at least three conceptual issues in the seminal work by Ma and Bertschinger [1] due to missing terms that are usually neglected and that are numerically small: i) the baryon equations are not gauge compatible; ii) these equations violate the Bianchi identity; iii) the origin of the term with c_s^2 is not clear.

[9] The code can be found in `http://www2.yukawa.kyoto-u.ac.jp/~antonio.defelice/new_baryon.zip`

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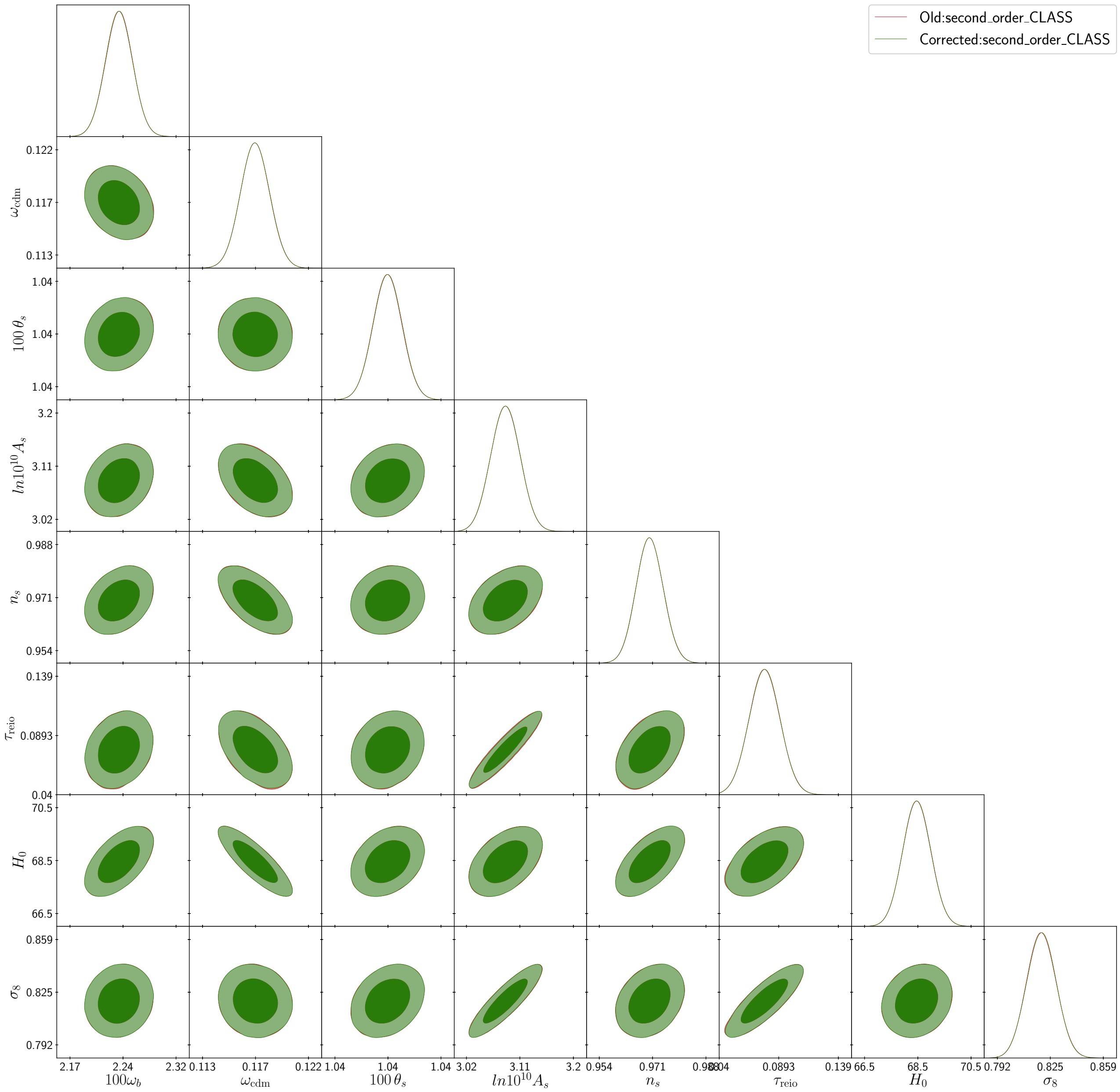

Figure 2. Combined results given by the old, original CLASS code with covariance-breaking equations of motion for the baryon and the new code with covariant equations of motion for the baryon fluid. The percent level differences shown in Table I are invisible in this figure but will be important for future surveys.

To address all these issues we have imposed the free Lagrangian of baryons to be described by the covariant action of a non-relativistic ideal gas. We find that this model for the baryon fluid, which describes a non-relativistic system of particles with a non-zero temperature and c_s^2, leads to covariant equations of motion for the perturbations, which do not violate the Bianchi identities. We have also derived the same equations from the conservation of stress-energy tensor, without relying on the Lagrangian.

Since the new covariant equations of motion for baryons represent one of the main result of our paper, we will rewrite them here both in synchronous and Newtonian gauge, in the notation of [1], as follows.

Sinchronous gauge —

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$$\dot{\theta}_b = -\frac{\dot{a}}{a}\,\theta_b + c_s^2\,k^2\,\delta_b + R\,an_e\sigma_T(\theta_\gamma - \theta_b) \tag{154a}$$

$$+\,3\,c_s^2\,\frac{\dot{a}}{a}\,\theta_b\,, \tag{154b}$$

$$\dot{\delta}_b = -\theta_b - \frac{1}{2}\,\dot{h} \tag{155a}$$

$$-\,\frac{6}{5}\,c_s^2\,\frac{\dot{a}}{a}\,\delta_b - \frac{3}{5}\,c_s^2\left(\theta_b + \frac{1}{2}\,\dot{h}\right). \tag{155b}$$

Conformal Newtonian gauge —

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\,\theta_b + k^2\psi + c_s^2\,k^2\,\delta_b + R\,an_e\sigma_T(\theta_\gamma - \theta_b) \tag{156a}$$

$$+\,3\,c_s^2\,\frac{\dot{a}}{a}\,\theta_b\,, \tag{156b}$$

$$\dot{\delta}_b = -\theta_b + 3\dot{\phi} \tag{157a}$$

$$-\,\frac{6}{5}\,c_s^2\,\frac{\dot{a}}{a}\,\delta_b - \frac{3}{5}\,c_s^2\,(\theta_b - 3\dot{\phi}), \tag{157b}$$

where the new contributions (with respect to [1]) come from the lines (154b), (155b), (156b) and (157b).

Furthermore, with our new equations we can keep terms which were before completely omitted in the previous equations of motion. For example on taking the time derivative of Eq. (155), which can be written schematically as $\dot{\delta}_b = P$, we find $\ddot{\delta}_b + \mathcal{H}\dot{\delta}_b = \dot{P} + \mathcal{H}\,P$. On replacing into this new equation the value of $\dot{\theta}_b$ with its own equation of motion, Eq. (154), together with the Einstein equations (to remove the fields $\ddot{h}$ and $\dot{h}$), we get the standard term $k^2c_s^2\delta_b$ but also other terms, e.g. one proportional to $k^2\,c_s^2\,\eta$ (η being the curvature perturbation in synchronous gauge), which cannot be neglected a priori but was absent in previous treatments of baryon physics. In this same equation, also other terms have been in the past neglected, e.g. terms proportional to $a^2H^2c_s^2\delta_b$, and from Eq. (154b) a term proportional to $aHc_s^2\theta_b$. Therefore the approximations made in the past correspond to setting $k \gg aH$, $\delta_b \gg \eta$, and $k^2\delta_b \gg aH\theta_b$. However, during radiation domination (valid at least for the initial conditions taken in [1] at $z \geq 10^6$), on approximating $H = H_0\sqrt{\Omega_{r0}}(1+z)^2$, where $H_0 \approx h\mathrm{Mpc}^{-1}/(2997.9)$, we find that k is constrained as to be $k \gg (1+z)\sqrt{\Omega_{r0}}/(3\times10^3)\,h\mathrm{Mpc}^{-1}$. Furthermore, $\eta \ll \delta_b$ (at any time and scale), and $k^2\delta_b \gg (1+z)\sqrt{\Omega_{r0}}/(3\times10^3)\,h\mathrm{Mpc}^{-1}\,\theta_b$. However, several authors, e.g. [1], consider a range for k given by $0.01\,\mathrm{Mpc}^{-1} \leq k \leq 10\,\mathrm{Mpc}^{-1}$, so that inside this set the previous approximations in general fail for large redshifts. One may wonder then why our numerical results are still similar to the previous ones. The reason is that c_s^2 is anyhow a small quantity which gives only a small correction to the baryon-dust fluid (i.e. a fluid with c_s^2 identically equal to 0). Nonetheless, we claim that only our approximation can be trusted, as the c_s^2 contributions we have introduced restore general covariance and are the only terms consistent with cosmological linear perturbation theory. We believe that the percent-level correction for the parameter estimation that we have found is expected to be important for future surveys.

With the covariant action so introduced we claim we have fixed all the three issues of [1] stated above. In fact, the new equations of motion for the baryon fluid possess additional terms of order c_s^2 which make the system of differential equations gauge compatible and hence obeying the general covariance.

In order to understand the cosmological evolution at all relevant redshifts, we need to study the solution of the equations of motion before recombination. In this regime, photons and baryons are tightly coupled, leading in general to a stiff regime during which it is difficult to solve the equations of motion numerically. In order to overcome this problem, we have adapted several approximate schemes, already introduced in the past, to our new dynamical equations of motion. We have then found the solutions of the equations of motion for the perturbations up to the second order in the tight coupling approximation using the new corrected baryon dynamics. In fact, we have implemented four different tight coupling approximation schemes without choosing any gauge, so that our code can then be immediately used in any gauge for modern Boltzmann solvers.

We have therefore used a Monte Carlo sampler in order to re-estimate the values of the cosmological parameters, after having incorporated our covariant corrections into the baryon dynamical equations of motion. We have found

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that there are some parameters, e.g. $\Omega_{\rm m}$, whose best fit values deviate from the previous code analysis by one percent. Moreover, we do not a priori know whether the deviation remains as small as one percent in other models and various modified gravity theories. In the age of precision cosmology, we therefore believe that these changes are to be considered. On top of that, we have found that both the covariantly corrected baryon equations of motion and the baryon-photon fluid tight coupling approximation schemes do not make the code (implemented in CLASS) slower of stiffer than the previous code. Hence, we do not find any reason why the modifications presented in this paper to the code should not be included permanently into modern Boltzmann codes, as to confront any gravity theory with the present and future observations in the era of precision cosmology.

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Appendix A: Speed of propagation of matter fields in the presence of several fluids

In the following we want to consider the case of N different fluids. For simplicity we will consider the case of several perfect fluids. For each of the fluid we need to give equations of state, namely

$$p_i = p_i(\rho_i, s_i)\,, \tag{A1}$$
$$T_i = T_i(\rho_i, s_i)\,, \tag{A2}$$

which together with the first principle of thermodynamics, namely

$$d\rho_i = \mu_i dn_i + n_i T_i ds_i\,, \tag{A3}$$

where $\mu_i \equiv (\rho_i{+}p_i)/n_i$, is enough to completely specify the thermodynamics of the i-th fluid, provided the integrability condition holds, namely

$$\left(\frac{\partial\mu_i}{\partial s_i}\right)_{n_i} = \left(\frac{\partial(n_i T_i)}{\partial n_i}\right)_{s_i}. \tag{A4}$$

For any of the fluid we have then automatically the speed of propagation,

$$c_{s,i}^2 \equiv \left(\frac{\partial p_i}{\partial\rho_i}\right)_{s_i}. \tag{A5}$$

So far the discussion holds in *any* environment, in particular in cosmology. In the latter, the equations of state hold at any order in perturbation theory, so that for relativistic degrees of freedom $p = \rho/3$, not only at the level of the background, but at any order in perturbation theory. And in particular, at first order of perturbation theory we have

$$\delta p_i = \overline{\left(\frac{\partial p_i}{\partial\rho_i}\right)_{s_i}}\,\delta\rho_i + \overline{\left(\frac{\partial p_i}{\partial s_i}\right)_{\rho_i}}\,\delta s_i = \overline{c_{s,i}^2}\,\delta\rho_i + \overline{\left(\frac{\partial p_i}{\partial s_i}\right)_{\rho_i}}\,\delta s_i\,, \tag{A6}$$

where the overline tells we need to consider the quantities evaluated at background level. Therefore, for fluids, whether or not in the presence of other fluids, the expression of the speed of propagation does not change. In particular a dust fluid (for which $p = 0 = T$), will have $c_{s,\rm dust}^2 = 0$, whereas for a radiation fluid $c_{s,\rm rad}^2 = 1/3$. The thermodynamical description of the fluid, well described in [26], is powerful in particular for not-polytropic fluids, such as ideal gas, for which p is not only a function of ρ. The speed of propagation in any fluid does correspond to the adiabatic speed. It should be noticed that this holds for fluids. In the case of a general action for a scalar field, e.g. quintessence, this formalism cannot be applied and the speed of propagation needs to be studied for the particular action at hand.

We want to add here a discussion on the equations of state for baryons in particular. Once can choose several (but all equivalent to each other) equations of state for the baryon fluid. In particular, once can choose $p = p(T, \rho)$, which is absolutely fine, because a thermodynamical equation of state can be written in terms of any two thermodynamical

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degrees of freedom, for example $p = p(n, s)$ or $p = p(\rho, s)$ as discussed e.g. at p. 564 of [26]. Then, it is obvious that the pressure perturbation should be of the form

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_T \delta\rho + \left(\frac{\partial p}{\partial T}\right)_\rho \delta T. \tag{A7}$$

Equivalently, we can consider as discussed in this paper $p = p(\rho, s)$. These two descriptions are actually completely equivalent. In fact, since the thermodynamical degrees of freedom are two, we can also write $T = T(\rho, s)$ and hence $\delta T = \left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \delta s$. Then, one can show that

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s = \left[\left(\frac{\partial p}{\partial \rho}\right)_T + \left(\frac{\partial p}{\partial T}\right)_\rho \left(\frac{\partial T}{\partial \rho}\right)_s\right] \delta\rho + \left(\frac{\partial p}{\partial T}\right)_\rho \left(\frac{\partial T}{\partial s}\right)_\rho \delta s = c_s^2\, \delta\rho\,, \tag{A8}$$

for adiabaticity. This adabaticity condition is also correctly considered in Ma & Bertschinger, being stated before their Eq. (96).

Appendix B: Baryon covariant equations of motion from conservation law

In this appendix we show that the equations of motion for non-barotropic perfect fluid can be derived from the conservation law $\nabla_\mu T^{\mu\nu} = 0$.

Let us suppose a perfect fluid in the conventional form

$$T^\mu{}_\nu = (\rho + p)\, u^\mu\, u_\nu + p\, \delta^\mu{}_\nu, \tag{B1}$$

where u^μ is the four velocity of the perfect fluid. We have the usual constraint on the fluid velocity

$$u_\mu u^\mu = -1. \tag{B2}$$

The linear perturbations are defined as

$$\rho = \bar\rho + \delta\rho\,, \tag{B3}$$
$$p = \bar p + \delta p\,, \tag{B4}$$
$$u_0 = -a\,(1 + \delta u)\;, \tag{B5}$$
$$u_i = \partial_i v_s\,, \tag{B6}$$
$$g_{00} = -a^2\,(1 + 2\alpha)\;, \tag{B7}$$
$$g_{0i} = a\partial_i \chi\,, \tag{B8}$$
$$g_{ij} = a^2 \left[(1 + 2\zeta)\,\delta_{ij} + \frac{2\partial_i\partial_j E}{a^2}\right]. \tag{B9}$$

where δu (which is determined by the condition $u_\mu u^\mu = -1$) and v_s are the scalar perturbations in the fluid velocity. Since p can be written in terms of two other thermodynamical variables, pressure perturbation $\delta p = (\partial p/\partial\rho)_s\, \delta\rho + (\partial p/\partial\rho)_\rho\, \delta s$, where $(\partial p/\partial\rho)_s \equiv c_s^2$. Considering adiabatic initial condition, we can choose $\delta s = 0$. Hence $\delta p = (\partial p/\partial\rho)_s\, \delta\rho = c_s^2 \delta\rho$. Notice that we have not chosen any gauge. Now we have the component of energy momentum tensor as

$$T^0{}_0 = -\bar\rho\,(1 + \delta)\;, \tag{B10}$$
$$T^0{}_i = (\bar\rho + \bar p)\,\frac{\partial_i v_s}{a}\,, \tag{B11}$$
$$T^i{}_j = (\bar p + \delta p)\,\delta^i{}_j\,, \tag{B12}$$

where $\delta \equiv \delta\rho/\rho$. Here, for simplicity, we do not consider anisotropic shear perturbations. Now, to find the equations of motion the calculation is straight forward. From the conservation law

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\beta}\, T^{\beta\nu} + \Gamma^\nu{}_{\mu\beta}\, T^{\mu\beta} = 0, \tag{B13}$$

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we find the equations of motion for linear perturbation in the Fourier space

$$\dot{\delta} = -\frac{\rho + p}{\rho}\left[\theta + 3\dot{\zeta} + \frac{k^2\chi}{a} - k^2\frac{\partial}{\partial t}\left(\frac{E}{a^2}\right)\right] - 3\frac{\dot{a}}{a}\left(c_s^2 - \frac{p}{\rho}\right)\delta\,, \tag{B14}$$

$$\dot{\theta} = -\frac{\dot{a}}{a}\theta + k^2\alpha + c_s^2\left(3\frac{\dot{a}}{a}\theta + k^2\frac{\rho}{\rho+p}\delta\right)\,, \tag{B15}$$

where we have redefined $v_s = -\theta a/k^2$ and $c_s^2 = (\partial p/\partial\rho)_s$ evaluated at the background level. The above two equations of motions exactly match with Eq. (53) and Eq. (49) respectively, which are resulting equations from the variation of the action (31). This ensures that the action discussed in the section IV is a well defined covariant action for a perfect fluid.

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