

OSCILLATION CRITERIA FOR THIRD ORDER NEUTRAL GENERALIZED DIFFERENCE EQUATIONS WITH DISTRIBUTED DELAY

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Abstract

This paper aims to investigate the criteria of behaviour of certain type of third order neutral generalized difference equations with distributed delay. With the technique of generalized Riccati transformation and Philos-type method, some oscillation criteria are obtained to ensure convergence and oscillatory solution of suitable example is listed to illustrate the main result.

keywords: Generalized difference operator, Oscillation, Non-oscillation, Converge to zero, Distributed delay, Riccati transformation.

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1. INTRODUCTION

In this article, third order neutral generalized difference equation with distributed delay of the form

$$\Delta_{\ell} \left(p_1(\lambda) \Delta_{\ell} \left(p_2(\lambda) \Delta_{\ell} \left(y(\lambda) + \sum_{s=a}^b g(\lambda, s) y(\lambda + s\ell - \tau\ell) \right) \right) \right) + \sum_{s=c}^d h(\lambda, s) f(y(\lambda + s\ell - \sigma\ell)) = 0, \quad (1)$$

is considered and analyze behavior of its solutions. Here Δ_{ℓ} is the forward generalized difference operator defined by $\Delta_{\ell} y(\lambda) = y(\lambda + \ell) - y(\lambda)$, $\mathbb{N}_{\ell}(\lambda_0) = \{\lambda_0, \lambda_0 + \ell, \lambda_0 + 2\ell, \dots\}$, $\lambda_0 \in [0, \infty)$, $\ell \in (0, \infty)$ and $a, b, c, d \in \mathbb{N}(\lambda_0)$ are being assumed.

c_1 : $\{p_i(\lambda)\}$ is a positive real sequence with $\sum_{\lambda=\lambda_0}^{\infty} \frac{1}{p_i(\lambda)} = \infty$, and $P_i(\lambda) > 0$, for $i = 1, 2$;

c_2 : $\{g(\lambda, s)\}$ and $\{h(\lambda, s)\}$ are non-negative real sequence with $0 \leq \sum_{s=a}^b g(\lambda, s) \equiv g(\lambda) \leq g < 1$;

c_3 : $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $\frac{f(y)}{y} \geq L > 0$, for $y \neq 0$.

c_4 : $m_i(\lambda) = \left[\frac{\lambda - \lambda_i - j - \ell}{\ell} \right]$, $\bar{\lambda} = \lambda + j$ and $j = \lambda - \lambda_i - \left[\frac{\lambda - \lambda_i}{\ell} \right] \ell$.

Define a function $x(\lambda)$ by

$$y(\lambda) + \sum_{s=a}^b g(\lambda, s)y(\lambda + s\ell - \tau\ell) = x(\lambda) \quad (2)$$

We consider only those solution $\{y(\lambda)\}$ of equation (1) which satisfy $\sup\{|y(\lambda)| : \lambda \geq T\} > 0$ for $\lambda \in \mathbb{N}_\ell(\lambda_0)$. A solution of equation (1) is said to be non oscillatory if it is neither eventually positive or negative and oscillatory otherwise. The generalized difference operator defined for any real sequence $\{y(\lambda)\}$ by $\Delta_\ell y(\lambda) = y(\lambda + \ell) - y(\lambda) \equiv z(\lambda)$, then its inverse is defined by

$$y(\lambda) = y(\lambda_0 + j) + \sum_{r=0}^{m_0(\lambda)} z(\lambda_0 + j + r\ell). \quad (3)$$

Recently many authors obtained certain behaviors of nonlinear difference equations and their applications. See for example monographs by Agarwal [1], Gyori and Ladas [4] and Elaydi [6]. The study of third order delay difference equations has also received much attention. The oscillatory and asymptotic behaviors of solutions of the third order difference equations were studied in Schmeidal [7]. Behaviors of oscillation of the third order nonlinear delay difference equation by Riccati transformation technique were obtained by several authors like Aktas et al. [2], Elabbasy et al. [3], Saker et al. [8], Selvaraj et al. [9, 10, 11], Thandapani et al. [12].

Here we arrive condition for getting convergent oscillatory solution of equation (1) generalized Riccati transformation and a new functions. In fact, by choosing appropriate function, we shall present several oscillation criteria easily. The technique adopted in the present paper are different from the above references, and the results are extended the existing results.

This paper is structured as follows: Few standard definitions and preliminaries are discussed in section 2. Section 3 deals with new oscillation results for (1) and in Section 4, we provide suitable examples are provided to demonstrate the main findings.

2. PRELIMINARIES

In this section, some basic definitions and preliminary results are presented, which will be used in the further discussions.

Definition 2.1. [5] For $\mu \in \mathbb{N}(1)$, the generalized polynomial factorial is defined by

$$\lambda_\ell^{(\mu)} = \lambda(\lambda - \ell)(\lambda - 2\ell) \cdots (\lambda - (\mu - 1)\ell). \quad (4)$$

Lemma 2.2. [5] Let $\ell \in [0, \infty)$. then $\Delta_\ell(\lambda_\ell^{(\mu)}) = (\mu\ell)\lambda_\ell^{(\mu-1)}$

Lemma 2.3. [5] Let $u(\lambda)$ and $v(\lambda)$ be any two real valued functions. Then

$$\Delta_\ell\{u(\lambda)v(\lambda)\} = u(\lambda + \ell)\Delta_\ell v(\lambda) + v(\lambda)\Delta_\ell u(\lambda) = v(\lambda + \ell)\Delta_\ell u(\lambda) + u(\lambda)\Delta_\ell v(\lambda).$$

Lemma 2.4. Let $y(\lambda) > 0$ be a solution of (1). Then function $x(\lambda)$ satisfies exactly one of the following two properties.

- (i) $x(\lambda) > 0$, $\Delta_\ell x(\lambda) > 0$, $\Delta_\ell (p_2(\lambda)\Delta_\ell x(\lambda)) > 0$;
- (ii) $x(\lambda) > 0$, $\Delta_\ell x(\lambda) < 0$, $\Delta_\ell (p_2(\lambda)\Delta_\ell x(\lambda)) > 0$.

where $\lambda \geq \lambda_2$ for large λ_2

Proof. Let $\{y(\lambda)\} > 0$ be a solution of equation (1) for all $\lambda \geq \lambda_0$. Then from $x(\lambda)$, we have $x(\lambda) \geq y(\lambda) > 0$ for $\lambda \geq \lambda_1$, and also from (1),

$$\Delta_\ell (p_1(\lambda)\Delta_\ell (p_2(\lambda)\Delta_\ell x(\lambda))) = - \sum_{s=c}^d h(\lambda, s)f(y(\lambda + s\ell - \sigma\ell)) < 0,$$

We know that $p_1(\lambda) (\Delta_\ell (p_2(\lambda) (\Delta_\ell x(\lambda))))$ is a decreasing function on $[\lambda_1, \infty)$ and it is either positive or negative eventually. It is possible to that $p_1(\lambda) (\Delta_\ell (p_2(\lambda) (\Delta_\ell x(\lambda)))) > 0$ for $\lambda \geq \lambda_1 \geq \lambda_0$. If not, then there exists a constant $M_1 > 0$ such that

$$\Delta_\ell (p_2(\lambda)\Delta_\ell x(\lambda)) < -\frac{M_1}{p_1(\lambda)} < 0, \text{ for } \lambda \geq \lambda_1.$$

Hence by (3)

$$p_2(\lambda) (\Delta_\ell x(\lambda)) \leq p_2(\lambda_1) (\Delta_\ell x(\lambda_1)) - M_1 \sum_{r=0}^{m_1(\lambda)} \frac{1}{p_1(\lambda_1 + r\ell)}. \quad (5)$$

Letting $n \rightarrow \infty$, then using condition (c_1) , $\lim_{n \rightarrow \infty} p_2(\lambda) (\Delta_\ell x(\lambda)) = -\infty$. Then we can find a $\lambda_2 \geq \lambda_1$ also, constant $M_2 > 0$ with the condition

$$p_2(\lambda) (\Delta_\ell x(\lambda)) < -M_2, \text{ for } \lambda_0 \geq \lambda.$$

Dividing the last inequality by $p_2(\lambda)$ and summing from λ_2 to $\lambda - \ell$, we get

$$x(\lambda) < x(\lambda_2) - M_2 \sum_{r_1=0}^{m_2(\lambda)} \frac{1}{p_2(\lambda_2 + r_1\ell)}, \quad (6)$$

Letting $n \rightarrow \infty$ and using condition (c_1) , give $x(n) \rightarrow -\infty$. That is $x(n) < 0$ eventually which is contradictory with $x(n) > 0$. Therefore $\Delta_\ell (p_2(\lambda) (\Delta_\ell x(\lambda)))$ is positive, that is $\Delta_\ell (p_2(\lambda) (\Delta_\ell x(\lambda))) > 0$ holds.

It can be known from $\Delta_\ell (p_2(\lambda) (\Delta_\ell x(\lambda))) > 0$, that is monotonically increasing sign in the interval $[\lambda_2, \infty)$. Therefore either $\Delta_\ell x(\lambda) > 0$ or $\Delta_\ell x(\lambda) < 0$ for all λ , which gives only property (i) or (ii) for $\{x(\lambda)\}$. \square

Lemma 2.5. Let $\{y(\lambda)\}$ be a positive solution of equation (1), and $x(\lambda)$ satisfies (ii) of Lemma 2.4. If

$$\sum_{r_2=\lambda_3}^{\infty} \frac{1}{p_2(\bar{\lambda}_3 + r_2\ell)} \left(\sum_{r_1=\lambda_1}^{m_2(r_2)} \frac{1}{p_1(\bar{\lambda}_2 + r_1\ell)} \left(\sum_{r=0}^{m_1(r_1)} \sum_{s=c}^d h(\bar{\lambda}_1 + r\ell, s) \right) \right) = \infty. \quad (7)$$

Then the solution $y(\lambda)$ of equation (1) goes to zero when $\lambda \rightarrow \infty$.

Proof. Let $\{y(\lambda)\} > 0$ be a solution of equation (1). From (ii) of Lemma 2.4, there exist $\gamma \geq 0$ with the condition

$$0 \leq r = \lim_{\lambda \rightarrow \infty} x(\lambda).$$

Now, we shall prove that $\gamma = 0$. Let $\gamma > 0$, $\gamma + \epsilon > x(\lambda) > \gamma \forall \epsilon > 0$ where λ is enough large. Choosing $0 < \epsilon < \frac{1-g}{g}\gamma$, From (2), it follows that

$$\begin{aligned} y(\lambda) &= x(\lambda) - \sum_{s=a}^b y(\lambda + s\ell - \tau\ell)g(\lambda, s) \\ &> \gamma - \sum_{s=a}^b y(\lambda + s\ell - \tau\ell)g(\lambda, s) > \gamma - g(\gamma + \epsilon) = \frac{\gamma - g(\gamma + \epsilon)}{\gamma + \epsilon}(\gamma + \epsilon) > Mx(\lambda), \end{aligned} \quad (8)$$

where $M = \frac{\gamma - g(\gamma + \epsilon)}{\gamma + \epsilon} > 0$. Hence, from equation (1) and (c₃), we have

$$\Delta_\ell(p_1(\lambda)\Delta_\ell(p_2(\lambda)\Delta_\ell x(\lambda))) = - \sum_{s=c}^d f(y(\lambda + s\ell - \sigma\ell))h(\lambda, s) \leq - \sum_{s=c}^d Ly(\lambda + s\ell - \sigma\ell)h(\lambda, s)$$

Now using (8), we obtain

$$\Delta_\ell(\Delta_\ell(p_2(\lambda)\Delta_\ell x(\lambda))p_1(\lambda)) \leq -ML \sum_{s=c}^d h(\lambda, s)x(\lambda + s\ell - \sigma\ell).$$

Summing the last inequality from λ_1 to $\lambda - \ell$ and from (3), we get

$$-p_1(\lambda_1)\Delta_\ell(p_2(\lambda_1)\Delta_\ell x(\lambda_1)) \leq -ML \sum_{r=0}^{m_1(\lambda)} \sum_{s=c}^d h(\bar{\lambda}_1 + r\ell, s)x(\bar{\lambda}_1 + r\ell + s\ell - \sigma\ell).$$

The above equation can also be written as

$$\Delta_\ell(p_2(\lambda_1)\Delta_\ell x(\lambda_1)) \geq \frac{ML\gamma}{p_1(\lambda_1)} \sum_{r=0}^{m_1(\lambda)} \sum_{s=c}^d h(\bar{\lambda}_1 + r\ell, s).$$

Summing again from $\lambda_1 < \lambda_2$ to $\lambda - \ell$, we get

$$-\Delta_\ell x(\lambda_2) \geq \frac{ML\gamma}{p_2(\lambda_2)} \sum_{r_1=\lambda_1}^{m_2(\lambda)} \frac{1}{p_1(\bar{\lambda}_2 + r_1\ell)} \left(\sum_{r=0}^{m_1(r_1)} \sum_{s=c}^d h(\bar{\lambda}_1 + r\ell, s) \right).$$

Summing the last inequality from $\lambda_2 < \lambda_3$ to ∞ , we get

$$x(\lambda_3) \geq ML\gamma \sum_{r_2=\lambda_3}^{\infty} \frac{1}{p_2(\lambda_3 + r_2\ell)} \left(\sum_{r_1=\lambda_1}^{m_2(r_2)} \frac{1}{p_1(\lambda_2 + r_1\ell)} \left(\sum_{r=0}^{m_1(r_1)} \sum_{s=c}^d h(\bar{\lambda}_1 + r\ell, s) \right) \right).$$

This contradicts to the condition (7). Thus $\gamma = 0$. Also $0 < y(\lambda) < x(\lambda)$ gives that $\lim_{n \rightarrow \infty} x(n) = 0$.

The proof is now complete. \square

Lemma 2.6. *Let $y(\lambda) > 0$ be a solution of equation (1), and $x(\lambda)$ satisfies (i). Then*

$$\Delta_\ell x(\lambda) \geq \frac{p_1(\lambda + \ell)\Delta_\ell(p_2(\lambda + \ell)\Delta_\ell x(\lambda + \ell))P(\lambda)}{p_2(\lambda)}. \quad (9)$$

where $P(\lambda) = \sum_{r=0}^{m_1(\lambda)} \frac{1}{p_1(\lambda_1 + r\ell)}$

Proof. Let $y(\lambda) > 0$ be a solution of equation (1). Since $x(\lambda)$ satisfies (i),

$$\Delta_\ell(p_1(\lambda)\Delta_\ell(p_2(\lambda)\Delta_\ell x(\lambda))) < 0,$$

From (3) and for all $\lambda_1 \geq \lambda_0$, we have

$$\begin{aligned} \Delta_\ell x(\lambda)p_2(\lambda) &= \Delta_\ell x(\lambda)p_2(\lambda) + \sum_{r=0}^{m_1(\lambda)} \Delta_\ell(\Delta_\ell x(\bar{\lambda}_1 + r\ell)p_2(\bar{\lambda}_1 + r\ell)) \\ &\geq \sum_{r=0}^{m_1(\lambda)} \frac{p_1(\bar{\lambda}_1 + r\ell)\Delta_\ell(p_2(\bar{\lambda}_1 + r\ell)\Delta_\ell x(\bar{\lambda}_1 + r\ell))}{p_1(\bar{\lambda}_1 + r\ell)} \\ &\geq \Delta_\ell(p_2(\lambda + \ell)\Delta_\ell x(\lambda + \ell))p_1(\lambda + \ell) \sum_{r=0}^{m_1(\lambda)} \frac{1}{p_1(\bar{\lambda}_1 + r\ell)} \\ &\geq \Delta_\ell(p_2(\lambda + \ell)\Delta_\ell x(\lambda + \ell))p_1(\lambda + \ell)P(\lambda). \end{aligned}$$

Hence, we obtain

$$\Delta_\ell x(\lambda) \geq \frac{\Delta_\ell(p_2(\lambda + \ell)\Delta_\ell x(\lambda + \ell))p_1(\lambda + \ell)P(\lambda)}{p_2(\lambda)}. \quad (10)$$

this complete the proof. \square

3. MAIN RESULTS

In this section, we obtain new oscillation criteria for the equation (1) by using the generalized Riccati transformation and Philos type technique. let us define function $q, Q : \mathbb{N}_\ell \times \mathbb{N}_\ell \rightarrow \mathbb{R}$ such that

- (1) $Q(\lambda, \lambda) = 0$ for $\lambda \geq \lambda_1 \geq 0$;
- (2) $Q(\lambda, s) > 0$ for $\lambda > s \geq \lambda_1$;

- (3) $\Delta_{\ell(s)}Q(\lambda, s) = Q(\lambda, s + \ell) - Q(\lambda, s) \leq 0$ for $\lambda > s \geq \lambda_1$ and a positive real sequence $\{\rho(\lambda)\}$ with the condition

$$-q(\lambda, s)\sqrt{Q(\lambda, s)} = \Delta_{\ell(s)}Q(\lambda, s) + \frac{\Delta_{\ell}\rho(s)}{\rho(s + \ell)}Q(\lambda, s) \quad (11)$$

Theorem 3.1. Consider the condition (7) and $\{\rho(\lambda)\}$ satisfies

$$\sum_{r=0}^{m_1(\lambda)} \left(C(\bar{\lambda}_1 + r\ell) - \frac{B^2(\bar{\lambda}_1 + r\ell)}{4A(\bar{\lambda}_1 + r\ell)} \right) = \infty, \quad (12)$$

where

$$A(\lambda) = \frac{\rho(\lambda)P(\lambda)}{\rho^2(\lambda + \ell)p_2(\lambda)}, \quad B(\lambda) = \frac{\Delta_{\ell}\rho(\lambda)}{\rho(\lambda + \ell)}, \quad C(\lambda) = H(\lambda)\rho(\lambda), \quad (13)$$

and

$$H(\lambda) = L(1 - g) \sum_{s=c}^d h(\lambda, s), \quad (14)$$

then every solution of equation (1) is either oscillatory or converges to zero.

Proof. Assume that $\{y(\lambda)\}$ is a non-oscillatory solution of equation (1). Without loss of generality we may assume that $y(\lambda) > 0$, $y(\lambda + s\ell - \tau\ell) > 0$ for $\lambda \geq \lambda_1 \geq \lambda_0 \in \mathbb{N}_{\ell}$ and $\{x(\lambda)\}$ is defined as in (2). Then $\{x(n)\}$ will satisfy two case of the lemma 2.4.

Let $\{x(\lambda)\}$ satisfies property (i) of Lemma 2.4. From equation (2), we have

$$y(\lambda) \geq x(\lambda) - \sum_{s=a}^b x(\lambda + s\ell - \tau\ell)g(\lambda, s) \geq x(\lambda) \left(1 - \sum_{s=a}^b g(\lambda, s) \right) \geq x(\lambda)(1 - g). \quad (15)$$

Using condition (c_3) in equation (1),

$$\Delta_{\ell}(p_1(\lambda)\Delta_{\ell}(p_2(\lambda)\Delta_{\ell}x(\lambda))) \leq - \sum_{s=c}^d h(\lambda, s)Lx(\lambda + s\ell - \sigma\ell). \quad (16)$$

Now using equation (15) in the above inequality, we obtain

$$\begin{aligned} \Delta_{\ell}(p_1(\lambda)\Delta_{\ell}(p_2(\lambda)\Delta_{\ell}x(\lambda))) &\leq -L(1 - p) \sum_{s=c}^d h(\lambda, s)x(\lambda + s\ell - \sigma\ell) \\ &\leq -H(\lambda)x(\lambda + c\ell - \sigma\ell). \end{aligned} \quad (17)$$

Define

$$w(\lambda) = \rho(\lambda) \frac{p_1(\lambda)\Delta_{\ell}(p_2(\lambda)\Delta_{\ell}x(\lambda))}{x(\lambda)}, \quad \lambda \geq \lambda_1. \quad (18)$$

Then $w(\lambda) > 0$ for all $\lambda \geq \lambda_1$ and equations (10) and (17), yield

$$\Delta_{\ell}w(\lambda) \leq -q_1(\lambda)\rho(\lambda) + w(\lambda + \ell) \frac{\Delta_{\ell}\rho(\lambda)}{\rho(\lambda + \ell)} - w^2(\lambda + \ell) \frac{\rho(\lambda)P(\lambda)}{\rho^2(\lambda + \ell)p_2(\lambda)}. \quad (19)$$

The above equation is also expressed as

$$\Delta_{\ell}w(\lambda) \leq -C(\lambda) + w(\lambda + \ell)B(\lambda) - w^2(\lambda + \ell)A(\lambda). \quad (20)$$

Where

$$A(\lambda) = \frac{P(\lambda)\rho(\lambda)}{\rho^2(\lambda + \ell)p_2(\lambda)}, \quad B(\lambda) = \frac{\Delta_\ell \rho(\lambda)}{\rho(\lambda + \ell)}, \quad C(\lambda) = \rho(\lambda)H(\lambda).$$

Using the inequality

$$Au - Bu^{\frac{1+\beta}{\beta}} \leq \frac{\beta^\beta}{(1+\beta)^{1+\beta}} \times \frac{A^{1+\beta}}{B^\beta}. \quad (21)$$

Now, using the above inequality the equation (20) can be written as

$$C(\lambda) - \frac{B^2(\lambda)}{4A(\lambda)} \leq -\Delta_\ell w(\lambda) \quad (22)$$

Summing (22) from λ_1 to $\lambda - \ell$, gives

$$\sum_{r=0}^{m_1(\lambda)} \left(C(\bar{\lambda}_1 + r\ell) - \frac{B^2(\bar{\lambda}_1 + r\ell)}{4A(\bar{\lambda}_1 + r\ell)} \right) \leq w(\lambda_1) - w(\lambda) \leq w(\lambda_1), \quad (23)$$

From $w(\lambda) > 0$, we get contradiction to (12), and $x(\lambda)$ is oscillatory. When $x(\lambda)$ has property (ii), from (7) $\lim_{\lambda \rightarrow \infty} y(\lambda) = 0$ which gives the proof. \square

Theorem 3.2. Consider the condition (7) and let $\{\rho(\lambda)\}$ satisfies

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, s)} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} \left[Q(\lambda, \bar{s} + r\ell)C(\bar{s} + r\ell) - \frac{h^2(\lambda, \bar{s} + r\ell)}{4A(\bar{s} + r\ell)} \right] = \infty, \quad (24)$$

then every solution of equation (1) is either oscillatory or converges to zero.

Proof. Assume that $\{y(\lambda)\}$ is a non-oscillatory solution of equation (1). As theorem 3.1, we have equation (20). Now if the inequality (20) is multiplied by $Q(\lambda, s)$ then summing the resulting inequality form λ_2 to $\lambda - \ell$ for all $\lambda \geq \lambda_2 \geq \lambda_0$, we have

$$\begin{aligned} \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell)C(\bar{\lambda}_2 + r\ell) &\leq - \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell)\Delta_\ell w(\bar{\lambda}_2 + r\ell) \\ &+ \sum_{r=0}^{m_2(\lambda)} (B(\bar{\lambda}_2 + r\ell)w(\bar{\lambda}_2 + r\ell + \ell) - A(\bar{\lambda}_2 + r\ell)w^2(\bar{\lambda}_2 + r\ell + \ell)) Q(\lambda, \bar{\lambda}_2 + r\ell). \end{aligned} \quad (25)$$

By summation by parts,

$$\begin{aligned} \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell)C(\bar{\lambda}_2 + r\ell) &\leq Q(\lambda, \lambda_2)w(\lambda_2) \\ &+ \sum_{r=0}^{m_2(\lambda)} \left[Q(\lambda, \bar{\lambda}_2 + r\ell)\Delta_\ell(\bar{\lambda}_2) + Q(\lambda, \bar{\lambda}_2 + r\ell)B(\bar{\lambda}_2 + r\ell) \right] w(\bar{\lambda}_2 + r\ell + \ell) \\ &- \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell)A(\bar{\lambda}_2 + r\ell)w^2(\bar{\lambda}_2 + r\ell + \ell). \end{aligned} \quad (26)$$

The inequality (21), gives

$$\sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell)C(\bar{\lambda}_2 + r\ell) \leq Q(\lambda, \lambda_2)w(\lambda_2) + \sum_{r=0}^{m_2(\lambda)} \frac{[\Delta_{\ell}(\bar{\lambda}_2)Q(\lambda, \bar{\lambda}_2 + r\ell) + B(\bar{\lambda}_2 + r\ell)Q(\lambda, \bar{\lambda}_2 + r\ell)]^2}{4A(\bar{\lambda}_2 + r\ell)Q(\lambda, \bar{\lambda}_2 + r\ell)}. \tag{27}$$

From (11), we have

$$\sum_{r=0}^{m_2(\lambda)} \left[C(\bar{\lambda}_2 + r\ell)Q(\lambda, \bar{\lambda}_2 + r\ell) - \frac{h^2(\lambda, \bar{\lambda}_2 + r\ell)}{4A(\bar{\lambda}_2 + r\ell)} \right] \leq Q(\lambda, \lambda_2)w(\lambda_2). \tag{28}$$

$$\frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \left[C(\bar{\lambda}_2 + r\ell)Q(\lambda, \bar{\lambda}_2 + r\ell) - \frac{h^2(\lambda, \bar{\lambda}_2 + r\ell)}{4A(\bar{\lambda}_2 + r\ell)} \right] \leq w(\lambda_2). \tag{29}$$

Letting $n \rightarrow \infty$, (24) is contradictory. If $x(\lambda)$ satisfies property (ii) of Lemma 2.4, then by condition (7) we have $\lim_{\lambda \rightarrow \infty} y(\lambda) = 0$. □

Corollary 3.3. *If $(\lambda - s)_\ell^{(m)} = Q(\lambda, s)$ for all $0 \leq s \leq \lambda$, $\rho(\lambda) = 1$ and*

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda_\ell^{(m)}} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} \left[(\lambda - \bar{s} - r\ell)_\ell^{(m)} C(\bar{s} + r\ell) - \frac{\left(m\ell(\lambda - \bar{s} - r\ell)_\ell^{(m-1)} \right)^2}{4A(\bar{s} + r\ell)\sqrt{(\lambda - \bar{s} - r\ell)_\ell^{(m)}}} \right] = \infty, \tag{30}$$

for every $m \geq 1$, then each solution of (1) is oscillatory or converges to 0.

Corollary 3.4. *If $Q(\lambda, s) = \left(\log \frac{\lambda + \ell}{s + \ell} \right)^m$ for $\lambda \geq s \geq 0$, $\rho(\lambda) = 1$ and*

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{(\log(\lambda + \ell))_\ell^{(m)}} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} \left[\left(\log \frac{\lambda + \ell}{\bar{s} + r\ell + \ell} \right)^m C(\bar{s} + r\ell) - \frac{m \left(\log \frac{\lambda + \ell}{\bar{s} + r\ell + \ell} \right)^{\frac{m-2}{2}}}{4(\bar{s} + (r + 1)\ell)A(\bar{s} + r\ell)} \right] = \infty, \tag{31}$$

for every $m \geq 1$, then each solution of (1) is oscillatory or converges to 0.

Theorem 3.5. *Assume that conditions (7) holds. Also let*

$$0 < \inf_{s \geq \lambda_0} \left[\liminf_{\lambda \rightarrow \infty} \frac{Q(\lambda, s)}{Q(\lambda, \lambda_0)} \right] \leq \infty, \tag{32}$$

and

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_0)} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} \frac{h^2(\lambda, \bar{s} + r\ell)}{A(\bar{s} + r\ell)} < \infty, \tag{33}$$

holds. If there is a sequence $\{\Phi(n)\}$ such that

$$\sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} A(\bar{s} + r\ell)\Phi_+^2(\bar{s} + r\ell + \ell) = \infty, \tag{34}$$

and

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, s)} \sum_{r=0}^{\lambda - \bar{s} - \ell} \left[Q(\lambda, \bar{s} + r\ell) C(\bar{s} + r\ell) - \frac{h^2(\lambda, \bar{s} + r\ell)}{4A(\bar{s} + r\ell)} \right] \geq \Phi(s), \quad (35)$$

where

$$\Phi_+(\bar{s} + r\ell + \ell) = \max\{\Phi(\bar{s} + r\ell + \ell), 0\}, \quad (36)$$

Let $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $Q(\lambda, s)$ respectively are defined in (13) and (11). Then each solution of equation of equation (1) is either oscillatory or $y(\lambda) \rightarrow 0$.

Proof. Let $y(\lambda)$ be a non oscillatory solution of equation (1) as in Theorem 3.2, when $x(\lambda)$ has a property (i), from (26) and rearranging terms

$$\begin{aligned} w(\lambda_2) &\geq \limsup_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \left[C(\bar{\lambda}_2 + r\ell) Q(\lambda, \bar{\lambda}_2 + r\ell) - \frac{h^2(\lambda, \bar{\lambda}_2 + r\ell)}{4A(\bar{\lambda}_2 + r\ell)} \right] \\ &+ \liminf_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \left[\frac{Q(\lambda, \bar{\lambda}_2 + r\ell)}{2\sqrt{A(\bar{\lambda}_2 + r\ell)}} + \sqrt{A(\bar{\lambda}_2 + r\ell) Q(\lambda, \bar{\lambda}_2 + r\ell) w(\bar{\lambda}_2 + r\ell + \ell)} \right]^2. \end{aligned}$$

for $\lambda \geq \lambda_2$. It follows from (35) that

$$\begin{aligned} w(\lambda_2) &\geq \Phi(\lambda_2) \\ &+ \liminf_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \left[\frac{Q(\lambda, \bar{\lambda}_2 + r\ell)}{2\sqrt{A(\bar{\lambda}_2 + r\ell)}} + \sqrt{A(\bar{\lambda}_2 + r\ell) Q(\lambda, \bar{\lambda}_2 + r\ell) w(\bar{\lambda}_2 + r\ell + \ell)} \right]^2. \end{aligned} \quad (37)$$

which means that,

$$w(\lambda_2) \geq \Phi(\lambda_2) \quad (38)$$

and then

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \left[\frac{Q(\lambda, \bar{\lambda}_2 + r\ell)}{2\sqrt{A(\bar{\lambda}_2 + r\ell)}} + \sqrt{A(\bar{\lambda}_2 + r\ell) Q(\lambda, \bar{\lambda}_2 + r\ell) w(\bar{\lambda}_2 + r\ell + \ell)} \right]^2 < \infty. \quad (39)$$

Therefore

$$\begin{aligned} &\liminf_{\lambda \rightarrow \infty} \left[\frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \lambda_2 + r\ell) A(\lambda_2 + r\ell) w^2(\lambda_2 + r\ell + \ell) \right. \\ &+ \sum_{r=0}^{m_2(\lambda)} q(\lambda, \lambda_2 + r\ell) \sqrt{Q(\lambda, \lambda_2 + r\ell) w(\lambda_2 + r\ell + \ell)} \frac{1}{Q(\lambda, \lambda_2)} \\ &\left. + \frac{1}{4Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \frac{q^2(\lambda, \lambda_2 + r\ell)}{\sqrt{A^2(\lambda_2 + r\ell)}} \right] < \infty. \end{aligned} \quad (40)$$

Then

$$\liminf_{\lambda \rightarrow \infty} \left[\sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell) A(\bar{\lambda}_2 + r\ell) w^2(\bar{\lambda}_2 + r\ell + \ell) \frac{1}{Q(\lambda, \lambda_2)} + \sum_{r=0}^{m_2(\lambda)} q(\lambda, \bar{\lambda}_2 + r\ell) \sqrt{Q(\lambda, \bar{\lambda}_2 + r\ell)} w(\bar{\lambda}_2 + r\ell + \ell) \frac{1}{Q(\lambda, \lambda_2)} \right] < \infty. \quad (41)$$

The above inequality can be expressed as

$$\liminf_{\lambda \rightarrow \infty} [U(\lambda) + V(\lambda)] < \infty \quad \text{for } \lambda \geq \lambda_2, \quad (42)$$

where

$$U(\lambda) = \sum_{r=0}^{m_2(\lambda)} Q(\lambda, \bar{\lambda}_2 + r\ell) A(\bar{\lambda}_2 + r\ell) w^2(\bar{\lambda}_2 + r\ell + \ell) \frac{1}{Q(\lambda, \lambda_2)}$$

$$V(\lambda) = \sum_{r=0}^{m_2(\lambda)} q(\lambda, \bar{\lambda}_2 + r\ell) \sqrt{Q(\lambda, \bar{\lambda}_2 + r\ell)} w(\bar{\lambda}_2 + r\ell + \ell) \frac{1}{Q(\lambda, \lambda_2)}.$$

Here we assert

$$\sum_{r=0}^{m_2(\lambda)} w^2(\bar{\lambda}_2 + r\ell + \ell) A(\bar{\lambda}_2 + r\ell) < \infty. \quad (43)$$

Suppose to the contrary that

$$\sum_{r=0}^{m_2(\lambda)} A(\bar{\lambda}_2 + r\ell) w^2(\bar{\lambda}_2 + r\ell + \ell) = \infty. \quad (44)$$

From equation (32), we have

$$\inf_{s \geq \lambda_0} \left[\liminf_{\lambda \rightarrow \infty} \frac{Q(\lambda, s)}{Q(\lambda, \lambda_0)} \right] > \mu \quad (45)$$

for $\mu > 0$, then $\frac{Q(\lambda, s)}{Q(\lambda, \lambda_0)} > \mu$ for $\lambda \geq \lambda_2 \geq \lambda_1$. There we can find a positive constant $M_3 > 0$ with the condition

$$\sum_{r=0}^{m_2(\lambda)} w^2(\bar{\lambda}_2 + r\ell + \ell) A(\bar{\lambda}_2 + r\ell) \geq \frac{M_3}{\mu}. \quad (46)$$

Thus for $\lambda \geq \lambda_3$ and using equation (45), gives

$$\begin{aligned}
 U(\lambda) &= \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_2(\lambda)} \Delta_\ell \left(\sum_{r_1=0}^{m_2(\lambda)} A(\bar{\lambda}_2 + r_1 \ell) w^2(\bar{\lambda}_2 + r_1 \ell + \ell) \right) Q(\lambda, \bar{\lambda}_2 + r \ell) \\
 &\quad + A(\bar{\lambda}_2 + r \ell) w^2(\bar{\lambda}_2 + r \ell + \ell) \\
 &= -\frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_3(\lambda)} \left(\sum_{r_1=0}^{m_2(r)} A(\bar{\lambda}_2 + r_1 \ell) w^2(\bar{\lambda}_2 + r_1 \ell + \ell) \right) \Delta_{\ell(\lambda_3)} Q(\lambda, \bar{\lambda}_3 + r \ell) \\
 &\quad - \frac{A(\lambda_2) w^2(\lambda_2 + \ell)}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_3(\lambda)} Q(\lambda, \bar{\lambda}_3 + r \ell) + A(\bar{\lambda}_2 + r \ell) w^2(\bar{\lambda}_2 + r \ell + \ell) \\
 &\geq \frac{1}{Q(\lambda, \lambda_2)} \sum_{r=0}^{m_3(\lambda)} \left(\sum_{r_1=0}^{m_2(r)} A(\bar{\lambda}_2 + r_1 \ell) w^2(\bar{\lambda}_2 + r_1 \ell + \ell) \right) (-\Delta_{\ell(\lambda_3)} Q(\lambda, \bar{\lambda}_3 + r \ell)) \\
 &\geq \frac{M_3}{\mu Q(\lambda, \lambda_2)} \sum_{r=0}^{m_3(\lambda)} (-\Delta_{\ell(\lambda_3)} Q(\lambda, \bar{\lambda}_3 + r \ell)) \geq \frac{M_3 Q(\lambda, \lambda_3)}{\mu Q(\lambda, \lambda_2)} \geq M_3.
 \end{aligned}$$

Since M_3 is arbitrary,

$$\lim_{\lambda \rightarrow \infty} U(\lambda) = \infty. \quad (47)$$

Next, consider a sequence $\{\lambda_n\}$ with the condition

$$\lim_{n \rightarrow \infty} [U(\lambda_n) + V(\lambda_n)] = \liminf_{\lambda \rightarrow \infty} [Q(\lambda) + V(\lambda)]. \quad (48)$$

It follows from (42) that we can find a number M_4 such that

$$U(\lambda_n) + V(\lambda_n) \leq M_4 \quad \forall n = 0, 1, 2, \dots. \quad (49)$$

In view of (47), we conclude that

$$\lim_{n \rightarrow \infty} V(\lambda_n) = -\infty \quad (50)$$

By (49), for a large enough n , we have

$$1 + \frac{V(\lambda_n)}{U(\lambda_n)} \leq \frac{M_2}{U(\lambda_n)} < \frac{1}{2}. \quad (51)$$

In view of (50), this implies that

$$\lim_{n \rightarrow \infty} \frac{V^2(\lambda_n)}{U(\lambda_n)} = \infty \quad (52)$$

On the other hand, by Schwarz's inequality, we have

$$\begin{aligned} V^2(\lambda_n) &= \left(\frac{1}{Q(\lambda_n, \lambda_2)} \sum_{r=0}^{m_2(\lambda_n)} q(\lambda_n, \bar{\lambda}_2 + r\ell) \sqrt{Q(\lambda_n, \bar{\lambda}_2 + r\ell)} w(\bar{\lambda}_2 + r\ell + \ell) \right)^2 \\ &\leq \left(\frac{1}{Q(\lambda_n, \lambda_2)} \sum_{r=0}^{m_2(\lambda_n)} Q(\lambda_n, \bar{\lambda}_2 + r\ell) A(\bar{\lambda}_2 + r\ell) w^2(\bar{\lambda}_2 + r\ell + \ell) \right) \\ &\quad \left(\frac{1}{Q(\lambda_n, \lambda_2)} \sum_{r=0}^{m_2(\lambda_n)} \frac{q^2(\lambda_n, \bar{\lambda}_2 + r\ell)}{A(\bar{\lambda}_2 + r\ell)} \right) \\ &\leq U(\lambda_n) \left(\frac{1}{Q(\lambda_n, \lambda_2)} \sum_{r=0}^{m_2(\lambda_n)} \frac{q^2(\lambda_n, \bar{\lambda}_2 + r\ell)}{A(\bar{\lambda}_2 + r\ell)} \right). \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{V^2(\lambda_n)}{U(\lambda_n)} &\leq \frac{1}{Q(\lambda_n, \lambda_2)} \sum_{r=0}^{m_2(\lambda_n)} \frac{q^2(\lambda_n, \bar{\lambda}_2 + r\ell)}{A(\bar{\lambda}_2 + r\ell)}. \\ &\leq \frac{1}{\mu Q(\lambda_n, \lambda_0)} \sum_{r=0}^{m_2(\lambda_n)} \frac{q^2(\lambda_n, \bar{\lambda}_2 + r\ell)}{A(\bar{\lambda}_2 + r\ell)}. \end{aligned}$$

It follows from (52) that

$$\lim_{n \rightarrow \infty} \frac{1}{Q(\lambda_n, \lambda_0)} \sum_{r=0}^{m_2(\lambda_n)} \frac{q^2(\lambda_n, \bar{\lambda}_2 + r\ell)}{A(\bar{\lambda}_2 + r\ell)} = \infty. \quad (53)$$

which contradicts (33). Then, (43) holds. Hence, by (38)

$$\sum_{r=0}^{m_2(\lambda)} A(\bar{\lambda}_2 + r\ell) \Phi_+^2(\bar{\lambda}_2 + r\ell + \ell) \leq \sum_{r=0}^{m_2(\lambda)} A(\bar{\lambda}_2 + r\ell) w^2(\bar{\lambda}_2 + r\ell + \ell) < \infty. \quad (54)$$

which contradicts (34) and completes the proof. If $x(\lambda)$ satisfies property (ii) of Lemma 2.4, by condition (7) we have $\lim_{\lambda \rightarrow \infty} y(\lambda) = 0$. \square

Theorem 3.6. Assume that all hypothesis of Theorem 3.5 are satisfied except condition (33). Also let

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, \lambda_0)} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} Q(\lambda, \bar{s} + r\ell) C(\bar{s} + r\ell) < \infty, \quad (55)$$

and

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{Q(\lambda, s)} \sum_{r=0}^{\frac{\lambda - \bar{s} - \ell}{\ell}} \left[Q(\lambda, \bar{s} + r\ell) C(\bar{s} + r\ell) - \frac{q^2(\lambda, \bar{s} + r\ell)}{4A(\bar{s} + r\ell)} \right] \geq \Phi(s), \quad (56)$$

then every solution of (1) is convergent to zero or oscillatory.

Proof. The proof is similar to that of Theorem 3.5 and hence the details are omitted. □

Corollary 3.7. Let $m \geq 1$ be a constant, $\rho(\lambda) = 1$, suppose that

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda^{(m)}} \sum_{r=0}^{\lambda - \bar{s} - \ell} \frac{\left(m\ell(\lambda - \bar{s} - r\ell)_\ell^{(m-1)}\right)^2}{\sqrt{(\lambda - \bar{s} - r\ell)_\ell^{(m)}} A(\bar{\lambda}_2 + r\ell)} < \infty, \tag{57}$$

If there is a sequence $\{\Phi(\lambda)\}$ satisfying (34) and

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{(\lambda - s)_\ell^{(m)}} \sum_{r=0}^{\lambda - \bar{s} - \ell} \left[(\lambda - \bar{s} - r\ell)_\ell^{(m)} C(\bar{s} + r\ell) - \frac{\left(m\ell(\lambda - \bar{s} - r\ell)_\ell^{(m-1)}\right)^2}{4A(\bar{s} + r\ell)\sqrt{(\lambda - \bar{s} - r\ell)_\ell^{(m)}}} \right] \geq \Phi(s), \tag{58}$$

then every solution of equation (1) is oscillatory or converges to zero.

Corollary 3.8. Let $m \geq 1$ be a constant, and

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{\lambda^{(m)}} \sum_{r=0}^{m_2(\lambda)} m\ell(\lambda - \bar{\lambda}_2 - r\ell)_\ell^{\left(\frac{m}{2}-1\right)} C(\bar{\lambda}_2 + r\ell) < \infty, \tag{59}$$

If there is a sequence $\{\Phi(\lambda)\}$ satisfying (34) and

$$\liminf_{\lambda \rightarrow \infty} \frac{1}{(\lambda - s)_\ell^{(m)}} \sum_{r=0}^{\lambda - \bar{s} - \ell} \left[(\lambda - \bar{s} - r\ell)_\ell^{(m)} C(\bar{s} + r\ell) - \frac{\left(m\ell(\lambda - \bar{s} - r\ell)_\ell^{(m-1)}\right)^2}{4A(\bar{s} + r\ell)\sqrt{(\lambda - \bar{s} - r\ell)_\ell^{(m)}}} \right] \geq \Phi(s), \tag{60}$$

then every solution of equation (1) will oscillate or converge to 0.

4. EXAMPLES

Example 4.1. Consider the third order neutral generalized difference equation with distributed delay

$$\Delta_\ell^3 \left(y(\lambda) + \sum_{s=1}^2 \frac{1}{4s} y(\lambda + s\ell - \ell) \right) + \sum_{s=1}^2 9 \left(\lambda + \frac{2}{s} \right) x(\lambda + s\ell - \ell) = 0. \tag{61}$$

Here $p_1(\lambda) = p_2(\lambda) = 1$, $g(\lambda, s) = \frac{1}{4s}$, $h(\lambda, s) = 9 \left(\lambda + \frac{2}{s} \right)$, $\tau = \sigma = 1$ and $\rho(n) = 1$. Then, $P(\lambda) = \frac{\lambda - \lambda_1}{\ell}$ and $H(\lambda) = \frac{45L}{8}(2\lambda + 3)$. which implies

$$\limsup_{\lambda \rightarrow \infty} \sum_{r=0}^{m_1(\lambda)} \frac{45L}{8} (2(\bar{\lambda}_1 + r\ell) + 3) = \infty \quad \text{for } 0 \leq L \leq 1.$$

It follows from Theorem 3.1 that every solution of equation (61) is oscillatory. In fact $\{y(\lambda)\} = \{(-1)^{\lfloor \frac{\lambda}{\ell} \rfloor}\}$ is one such oscillatory solution of equation (61).

Example 4.2. Consider the third order neutral generalized difference equation with distributed delay

$$\Delta_\ell^2 \left(\lambda \Delta_\ell \left(y(\lambda) + \frac{1}{2} y(\lambda - 2\ell) + \frac{1}{4} y(\lambda - \ell) \right) \right) + y(\lambda) 10(\lambda + \ell) = 0. \tag{62}$$

Here $p_1(\lambda) = \lambda$, $p_2(\lambda) = 1$, $g(\lambda, s) = \frac{1}{2s}$, $h(\lambda, s) = 10s(\lambda + \ell)$, $\tau = 3$, $\sigma = 1$, $a = c = d = 1$ and $b = 2$. By choosing $\rho(\lambda) = 1$ and $Q(\lambda, s) = (\lambda - s)_\ell^{(2)}$. By Theorem 3.2, $P(\lambda) = \frac{\lambda - \lambda_1}{\ell}$, $q(\lambda, s) = \frac{2\ell\sqrt{\lambda - s}}{\sqrt{\lambda - s + \ell}}$, $H(\lambda) = \frac{5}{2}(\lambda + \ell)$, $A(\lambda) = \frac{\lambda - \lambda_1}{\lambda\ell}$, $B(\lambda) = 0$ and $C(\lambda) = \frac{5(\lambda + \ell)}{2}$ and

$$\limsup_{\lambda \rightarrow \infty} \sum_{r=0}^{\frac{\lambda - s - \ell}{\ell}} \left[Q(\lambda, s + r\ell)C(s + r\ell) - \frac{h^2(\lambda, s + r\ell)}{4A(s + r\ell)} \right] \frac{1}{Q(\lambda, s)} = \infty,$$

Hence by Theorem 3.2, every solution of equation (62) is oscillatory. In fact $\{y(\lambda)\} = \{(-1)^{\lfloor \frac{\lambda}{\ell} \rfloor}\}$ is one such oscillatory solution of equation (62).

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