Some integral inequalities for h-Godunova-Levin preinvexity

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November 15, 2019

Abstract. In this study, we define new classes of convexity called of h-Godunova-Levin and h-Godunova-Levin preinvexity, through which some new inequalities of Hermite-Hadamard type were established. These new classes are the generalisation of several known convexity including the s-convex, P-function, Godunova-Levin. Further, the properties of h-Godunova-Levin function were also discussed. Meanwhile, the applications of h-Godunova-Levin Preinvex function are given.

Keywords: Hermite-Hadamard inequality; h-convexity; h-Godunova-Levin function; h-Godunova-Levin preinvex function.

1   Introduction

Recently, the theory of convexity has become a broad area of study since it is related to the theory of inequalities. Many such inequalities are frequently reported in the literature as a result of applications of convexity in both pure and applied sciences (see [1, 4, 13, 14]). Among all, an interesting inequality for convex function is of Hermite-Hadamard type, which can be defined as follows:

Let $\psi : S \to \mathbb{R}$ be a convex function on S and $u_1, u_2 \in S, u_1 < u_2$, then we have

$$\psi\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(x) dx \leq \frac{\psi(u_1) + \psi(u_2)}{2}. \tag{1}$$

If ψ is a concave function, the two inequalities can be held in the reversed direction. These inequalities have been extensively improved and generalized. For example, see [1, 2, 3, 5] and [21].

Definition 1. [6] A positive function $\psi : S \to \mathbb{R}$ is said to be a Godunova-Levin, if

$$\psi(\delta u_1 + (1 - \delta)u_2) \leq \frac{\psi(u_1)}{\delta} + \frac{\psi(u_2)}{1 - \delta}, \forall u_1, u_2 \in S, \delta \in (0, 1).$$

Several other properties related to this class of functions are given in [7, 8, 9]. For example, both the positive monotone and positive convex functions belong to this classes.

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This concept has been recently extended to s-Godunova-Levin type of convexity by Dragomir [11]. Further, studies were conducted on s-Godunova-Levin type of convexity and can be found in the literature [12]. Another important class of convex function is h-convexity which was introduced by Varošanec[10], through which several generalizations and extensions were made.

Definition 2. [10] Let $\psi, \omega : S \to \mathbb{R}$ be two functions, such that $u_1, u_2 \in S$, the inequality $(\psi(u_1) - \psi(u_2))(\omega(u_1) - \omega(u_2)) \geq 0$ is called similarly ordered for ψ and ω on S.

Now, the following preliminaries on invexity analysis are necessary since they can be frequently used throughout this study.

Definition 3. [20] A set S is said to be invex set with respect to $\zeta(.,.)$, if for every $u_1, u_2 \in S$ and $\delta \in [0, 1]$

$$u_1 + \delta\zeta(u_2, u_1) \in S.$$

Definition 4. [19] A function ψ on the invex set S is said to be preinvex with respect to ζ, if

$$\psi(u_1 + \delta\zeta(u_2, u_1)) \leq (1 - \delta)\psi(u_1) + \delta\psi(u_2); \forall u_1, u_2 \in S; \delta \in [0, 1]. \tag{2}$$

Usually, the preinvex functions can be convexity if $\zeta(u_2, u_1) = u_2 - u_1$ holds in (2). Other properties of preinvex functions are given in [15, 16].

We arrange this paper as follows. Section 2 introduces the new clases of h-Godunova-Levin, denoted by $SGX(\frac{1}{h}, t)$ and $SGV(\frac{1}{h}, t)$, together whith their properties. This class of function unifies different classes of convexity: s-Godunova-Levin, P-functions, s-convexity and Godunova-Levin. In Section 3, we proved new Hermite-Hadamard inequalities via h-Godunova-Levin preinvexity. Section 4 introduces a new definition of h-Godunova-Levin preinvexity, which can be the generalization of preinvexity. This Section also presents new Hermite-Hadamard type inequalities for h-Godunova-Levin preinvexity. Section 5 gives some applications to special means.

2   h-Godunova-Levin and Their properties

This section introduces the notion of h-Godunova-Levin function together with their properties. This class of function can be denoted by $SGX(\frac{1}{h}, t)$ and $SGV(\frac{1}{h}, t)$ for h-GodunovaLevin convex and h-Godunova-Levin concave, respectively.

Definition 5. *Suppose $h : (0, 1) \to \mathbb{R}$. A non-negative function $\psi : S \to \mathbb{R}$ is said to be h-Godunova-Levin, or that ψ belongs to the class $SGX(\frac{1}{h}, S)$, forall $u_1, u_2 \in S$ and $\delta \in (0, 1)$, we have*

$$\psi(\delta u_1 + (1 - \delta)u_2) \leq \frac{\psi(u_1)}{h(\delta)} + \frac{\psi(u_2)}{h(1 - \delta)}.$$

Remark 1. *If $h(\delta) = \delta$, $h(\delta) = \frac{1}{\delta}$, $h(\delta) = \delta^s$, $h(\delta) = 1$, $h(\delta) = \frac{1}{\delta^s}$ in (4), the definition of h-Godunova-Levin function can be clearly reduced to different types of convexity such as; Godunova-Levin function, classical convex, s-Godunova-Levin function, P-function and s-convex function. This indicates that h-Godunova-Levin functions is the generalaization of these different classes.*

Proposition 1. *Suppose that h_1, h_2 are two positive functions defined on the interval S satisfying the property*

$$\frac{1}{h_1(\delta)} \leq \frac{1}{h_2(\delta)}, \quad \delta \in (0,1).$$

If $\psi \in SGX(\frac{1}{h_1}, S)$, then $\psi \in SGX(\frac{1}{h_2}, S)$. If $\psi \in SGV(\frac{1}{h_1}, S)$, then $\psi \in SGV(\frac{1}{h_2}, S)$, where $h_1(t) \neq 0$ and $h_2(t) \neq 0$.

Proof. If $\psi \in SGX(\frac{1}{h_1}, S)$, then for any $u_1, u_2 \in S$ and $\delta \in (0,1)$ we get

$$\psi(\delta u_1 + (1-\delta)u_2) \leq \frac{1}{h_1(\delta)}\psi(u_1) + \frac{1}{h_1(1-\delta)}\psi(u_2)$$

$$\leq \frac{1}{h_2(\delta)}\psi(u_1) + \frac{1}{h_2(1-\delta)}\psi(u_2),$$

i.e. $\psi \in SGX(\frac{1}{h_2}, S)$. $\square$

Proposition 2. *If $\psi, \omega \in SGX(\frac{1}{h}, S)$ and $\lambda > 0$, then $\psi + \omega, \lambda\psi \in SGX(\frac{1}{h}, S)$. if $\psi, \omega \in SGV(\frac{1}{h}, S)$ and $\lambda > 0$, then $\psi + \omega, \lambda\psi \in SGV(\frac{1}{h}, S)$.*

Proof. The proof is clear from the defintion of the classes h-Godunova-Levin convex and h-Godunova-Levin concave, $SGX(\frac{1}{h}, S)$ and $SGV(\frac{1}{h}, S)$. $\square$

Proposition 3. *Suppose that ψ and ω are two h-Godunova-Levin functions and satisfying the property given in Definition 2. Then the product of these two functions is given as:*

$$\psi(\delta u_1 + (1-\delta)u_2)\omega(\delta u_1 + (1-\delta)u_2) \leq \left[\frac{1}{h(1-\delta)}\psi(u_1)\omega(u_1) + \frac{1}{h(\delta)}\psi(u_2)\omega(u_2)\right]\left[\frac{1}{h(1-\delta)} + \frac{1}{h(\delta)}\right].$$

Proof. Given that ψ and ω are h-Godunova-Levin functions, we have

$$\psi(\delta u_1 + (1-\delta)u_2)\omega(\delta u_1 + (1-\delta)u_2) \leq \left(\frac{\psi(u_1)}{h(1-\delta)} + \frac{\psi(u_2)}{h(\delta)}\right) + \left(\frac{\omega(u_1)}{h(1-\delta)} + \frac{\omega(u_2)}{\omega(\delta)}\right)$$

$$= \frac{1}{(h(1-\delta))^2}\psi(u_1)\omega(u_1) + \frac{1}{h(1-\delta)h(t)}[\psi(u_1)\omega(u_2)$$

$$+ \psi(u_2)\omega(u_1)] + \frac{1}{(h(\delta))^2}\psi(u_2)\omega(u_2)$$

$$= \left[\frac{1}{h(1-\delta)}\psi(u_1)\omega(u_1) + \frac{1}{h(\delta)}\psi(u_2)\omega(u_2)\right]\left[\frac{1}{h(1-\delta)} + \frac{1}{h(\delta)}\right].$$

$\square$

Proposition 4. *Suppose that $\psi : S_1 \to [0,\infty)$, $\omega : S_2 \to [0,\infty)$ are two functions such that $\omega(S_2) \subseteq S_1$. If the function ω is convex(concave) and the function ψ is increasing(decreasing), $\psi \in SGX(\frac{1}{h}, S_1)$, then the composition $\psi \circ \omega$ belongs to $SGX(\frac{1}{h}, S_2)$. Meanwhile, if the function ω is convex(concave) and the function ψ is decreasing (increasing), $\psi \in SGV(\frac{1}{h}, S_1)$, then the composition $\psi \circ \omega$ belongs to $SGV(\frac{1}{h}, S_2)$.*

Proof. Suppose that ω is a convex function, ψ is increasing and $\psi \in SGX(\frac{1}{h}, S_1)$. Then we have

$$(\psi \circ \omega)((\delta u_1 + (1 - \vartheta)u_2)) \leq \psi(\delta\omega(u_1) + (1 - t)\omega(u_2)) \leq \frac{1}{h(1 - \delta)}(\psi \circ \omega)(u_1) + \frac{1}{h(\delta)}(\psi \circ \omega)(u_2),$$

for all $u_1, u_2 \in S_2$ and $\delta \in (0, 1)$. $\qquad\square$

3 New Hermite-Hadamard inequality for h-Godunova-Levin convex function.

The following generalization of the Hermite-Hadamard inequalities for h-Godunova-Levin convex function can be proved in this section.

Theorem 1. *Let* $\psi \in SGX(\frac{1}{h}, S)$, $u_1, u_2 \in S$, *with* $u_1 < u_2$ *and* $\psi \in L_1([u_1, u_2])$ *where* $h : (0, 1) \to \mathbb{R}$ *is a positive function and* $h(\delta) \neq 0$, *we have*

$$\frac{h\left(\frac{1}{2}\right)}{2}\psi\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\psi(x)dx \leq [\psi(u_1) + \psi(u_2)]\int_0^1\frac{1}{h(\delta)}d\delta. \qquad (3)$$

Proof. Since ψ is h-Godunova-Levin, we have

$$\psi(\delta u_1 + (1 - \delta)u_2) \leq \frac{\psi(u_1)}{h(\delta)} + \frac{\psi(u_2)}{h(1 - \delta)}. \qquad (4)$$

Consider $v_1 = \alpha u_1 + (1 - \alpha)u_2$, $v_2 = (1 - \alpha)u_1 + \alpha u_2$ and $\delta = \frac{1}{2}$ in (4), we obtain

$$\psi\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{h(\frac{1}{2})}\psi(\alpha u_1 + (1 - \alpha)u_2) + \frac{1}{h(\frac{1}{2})}\psi((1 - \alpha)u_1 + \alpha u_2)$$

$$\leq \frac{1}{h(\frac{1}{2})}[\psi(\alpha u_1 + (1 - \alpha)u_2) + \psi((1 - \alpha)u_1 + \alpha u_2)]. \qquad (5)$$

Thus, after integrating (5), we get the following

$$\psi\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{h(\frac{1}{2})}\left[\int_0^1\psi(\alpha u_1 + (1 - \alpha)u_2)d\alpha + \int_0^1\psi((1 - \alpha)u_1 + \alpha u_2)d\alpha\right]$$

$$\leq \frac{2}{h(\frac{1}{2})(u_2 - u_1)}\int_{u_1}^{u_2}\psi(x)dx.$$

This ends the proof of the first inequality. Now, taking $v_1 = u_1$ and $v_2 = u_2$ in (4) and integrating the result over the interval $[0, 1]$ with respect to δ, we obtain

$$\frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\psi(x)dx \leq [\psi(u_1) + \psi(u_2)]\int_0^1\frac{1}{h(\delta)}d\delta.$$

This complete the proof of the second inequality (3).

$\qquad\square$

Remark 2. *In Theorem 1, choosing $h(\delta) = \delta^s$, we obtain the Hermite-Hadamard inequalities for s-convexity in the second sense, Theorem 2.1. in [17]. If we choose $h(\delta) = 1$, Theorem 1 can be reduced to the result for P-function [21]. Taking $h(\delta) = \frac{1}{\delta}$ then the theorem reduces the result for classical Hermite-Hadamard inequalities given in inequality (1).*

4 Hermite-Hadamard inequalities for h-Godunova-Levin preinvex function

The definition of h-Godunova-Levin preinvex is introuced in this section. The inequalities of Hermite-Hadamard type for functions whose first derivatives absolute values are h-Godunova-Levin preinvex are also presented here.

Definition 6. *A function $\psi : S \to \mathbb{R}$ is said to be h-Godunova-Levin preinvex function with respect to ζ, if forall $u_1, u_2 \in S, \delta \in (0,1)$, then*

$$\psi(u_1 + \delta\zeta(u_2, u_1)) \leq \frac{\psi(u_1)}{h(1-\delta)} + \frac{\psi(u_2)}{h(\delta)}$$

holds.

The following lemma can be used to prove the generalization of the Hermite-Hadamard inequalities for h-Godunova-Levin preinvexity.

Lemma 1. [18] Suppose that $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ is a differentiable function, where $u_1, u_1 + \zeta(u_2, u_1) \in S$ with $u_1 < u_1 + \zeta(u_2, u_1)$. If $\psi' \in L1[u_1, u_1 + \zeta(u_2, u_1)]$, we have

$$\frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1+\zeta(u_2,u_1)} \psi(x)dx - \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} = \frac{\zeta(u_2, u_1)}{2}\left[\int_0^1 (1 - 2t)\psi'(u_1 + \delta\zeta(u_2, u_1))d\delta\right].$$

Theorem 2. *Suppose that $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ is a differentiable mapping on S°, $u_1, \zeta(u_2, u_1) \in S^\circ$, with $u_1 < u_1 + \zeta(u_2, u_1)$. If $|\psi'|$ is a h-Godunova-Levin preinvex on $[u_1, u_1 + \zeta(u_2, u_1)]$, then we get the following inequality:*

$$\left|\frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)}\int_{u_1}^{u_1+\zeta(u_2,u_1)} \psi(x)dx\right| \leq \frac{\zeta(u_2, u_1)}{2}[|\psi'(u_1)| + |\psi'(u_2)|]$$

$$\times \int_0^1 |1 - 2\delta|\left[\frac{1}{h(\delta)} + \frac{1}{h(1-\delta)}\right]d\delta. \qquad (6)$$

Proof. We use Lemma 1 to prove inequality (2) as follows

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x)dx \right| = \left| \frac{\zeta(u_2, u_1)}{2} \int_0^1 (1 - 2\delta)\psi'(u_1 + \delta\zeta(u_2, u_1))d\delta \right|$$

$$\leq \frac{\zeta(u_2, u_1)}{2} \int_0^1 |1 - 2\delta||\psi'(u_1 + \delta\zeta(u_2, u_1))d\delta|$$

$$\leq \frac{\zeta(u_2, u_1)}{2} \int_0^1 |1 - 2\delta| \left| \frac{\psi'(u_1)}{h(\delta)} + \frac{\psi'(u_2)}{h(1 - \delta)} \right| d\delta$$

$$\leq \frac{\zeta(u_2, u_1)}{2} [|\psi'(u_1)| + |\psi'(u_2)|]$$

$$\times \int_0^1 |1 - 2\delta| \left[\frac{1}{h(\delta)} + \frac{1}{h(1 - \delta)} \right] d\delta.$$

□

Corollary 1. *Since* $\int_0^1 \frac{1}{h(\delta)}d\delta = \int_0^1 \frac{1}{h(1 - \delta)}d\delta$, *then from Theorem 2, we have*

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x)dx \right| \leq \frac{\zeta(u_2, u_1)}{2} [|\psi'(u_1)| + |\psi'(u_2)|]$$

$$\times \int_0^1 \frac{|1 - 2\delta|}{h(\delta)}d\delta. \qquad (7)$$

Taking $\zeta(u_2, u_1) = u_2 - u_1$ *in (7), we obtain the following inequality:*

$$\left| \frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(x)dx \right| \leq \frac{u_2 - u_1}{2} [|\psi'(u_1)| + |\psi'(u_2)|] \int_0^1 \frac{|1 - 2\delta|}{h(\delta)}d\delta.$$

Theorem 3. *Suppose that* $\psi : S = [u_1, u_1 + \zeta(u_2, u_1)] \to (0, \infty)$ *is a differentiable function on* S°, $u_1, \zeta(u_2, u_1) \in S^\circ$, *with* $u_1 < u_1 + \zeta(u_2, u_1)$. *If* $|\psi'|$ *is a h-Godunova-Levin preinvex on* $[u_1, u_1 + \zeta(u_2, u_1)]$, *we obtain*

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x)dx \right| \leq \frac{\zeta(u_2, u_1)}{(p + 1)^{\frac{1}{p}}} (|\psi'(u_1)|^q$$

$$+ |\psi'(u_2)|^q)^{\frac{1}{q}} \int_0^1 \frac{1}{h(\delta)}d\delta.$$

Proof. Applying Lemma 1, we have:

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x)dx \right| = \left| \frac{\zeta(u_2, u_1)}{2} \int_0^1 (1 - 2\delta)\psi'(u_1 + \delta\zeta(u_2, u_1))d\delta \right|$$

$$\leq \frac{\zeta(u_2, u_1)}{2} \int_0^1 |1 - 2\delta||\psi'(u_1 + \delta\zeta(u_2, u_1))d\delta|.$$

We use Hölder's integral inequality as follows:

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x) dx \right| \le \frac{\zeta(u_2, u_1)}{2} \left(\int_0^1 |1 - 2\delta|^p \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 |\psi'(u_1 + \delta\zeta(u_2, u_1)) d\delta|^q \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Now, since $|\psi'|^p$ is a h-Godunova-Levin preinvex, we obtain

$$\int_0^1 |\psi'(u_1 + \delta\zeta(u_2, u_1))|^q d\delta \le \int_0^1 \left(\frac{|\psi'(u_1)|^q}{h(\delta)} + \frac{|\psi'(u_2)|^q}{h(1 - \delta)} \right) d\delta$$
$$\le 2 \int_0^1 \frac{1}{h(\delta)} d\delta (|\psi'(u_1)|^q + |\psi'(u_2)|^q).$$

Using the basic calculus, we have $\int_0^1 |1 - 2\delta|^p d\delta = \frac{1}{p+1}$, this copmplete the proof of the Theorem 3. $\square$

Corollary 2. *Choosing* $\zeta(u_2, u_1) = u_2 - u_1$ *in Theoreme 3 reduces inequality (8) to the following:*

$$\left| \frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(x) dx \right| \le \frac{u_2 - u_1}{(p + 1)^{\frac{1}{p}}} (|\psi'(u_1)|^{\frac{1}{q}} + |\psi'(u_2)|^{\frac{1}{q}})^{\frac{1}{q}} \int_0^1 \frac{1}{h(\delta)} d\delta.$$

Theorem 4. *With the assumptions of Theorem 3, we get the following:*

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x) dx \right| \le \frac{\zeta(u_2, u_1)}{4} \left(\int_0^1 \frac{|1 - 2\delta|}{h(\delta)} d\delta \right)^{\frac{1}{q}}$$
$$\times (|\psi'(u_1)|^q + |\psi'(u_2)|^q).$$

Proof. We use Lemma 1 to show that

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x) dx \right| = \left| \frac{\zeta(u_2, u_1)}{2} \int_0^1 (1 - 2\delta)\psi'(u_1 + \delta\zeta(u_2, u_1)) d\delta \right|$$
$$\le \frac{\zeta(u_2, u_1)}{2} \int_0^1 |1 - 2\delta| |\psi'(u_1 + \delta\zeta(u_2, u_1)) d\delta|.$$

Applying power-mean inequality, we get

$$\left| \frac{\psi(u_1) + \psi(u_1 + \zeta(u_2, u_1))}{2} - \frac{1}{\zeta(u_2, u_1)} \int_{u_1}^{u_1 + \zeta(u_2, u_1)} \psi(x) dx \right| \le \frac{\zeta(u_2, u_1)}{2} \left(\int_0^1 |1 - 2\delta| d\delta \right)^{1 - \frac{1}{q}}$$
$$\times \left(\int_0^1 |1 - 2\delta| |\psi'(u_1 + \delta\zeta(u_2, u_1))|^q d\delta \right)^{\frac{1}{q}}.$$

Since $|\psi'|^q$ is a h-Godunova-Levin preinvex, we obtain

$$\int_0^1 |1-2\delta||\psi'(u_1 + \delta\zeta(u_2,u_1))|^q d\delta \leq \int_0^1 \left(\frac{|1-2\delta|}{h(\delta)}|\psi'(u_1)|^q + \frac{|1-2\delta|}{h(1-\delta)}|\psi'(u_2)|^q\right)d\delta$$

$$\leq \int_0^1 \frac{|1-2\delta|}{h(\delta)}d\delta(|\psi'(u_1)|^q + |\psi'(u_2)|^q).$$

Applying the basic calculus, we have $\int_0^1 |1-2\delta|d\delta = \frac{1}{2}$. $\qquad\square$

Corollary 3. *Taking $\zeta(u_2,u_1) = u_2 - u_1$ gives $h(\delta) = 1$ and $q = 1$*

$$\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_1}\psi(x)dx\right| \leq \frac{u_2 - u_1}{8}(|\psi'(u_1)| + |\psi'(u_2)|),$$

which is similar to Theorem 2.2 repoted by Dragomir and Agarwal [14].

5 Applications to special means

We finally use Hermite-Hadamard inequalities for h-GodunovaLevin preinvex function to form the inequalities for special means. Thus, the means of two positive numbers u_1, u_2 and $u_1 \neq u_2$ can be considered as follows:

1. The arithmetic mean:
   $A = A(u_1, u_2) = \dfrac{u_1 + u_2}{2}$; $u_1, u_2 \in \mathbb{R}$, with $u_1, u_2 > 0$.

2. The generalized log-mean:
   $L_m(u_1, u_2) = \left[\dfrac{u_2^{m+1} - u_1^{m+1}}{(m+1)(u_2 - u_1)}\right]^{\frac{1}{m}}$, $m \neq -1, 0$.

The following propositions are obtained from the results in Section 4 and the above applications special means.

Proposition 5. *Let $0 < u_1 < u_2$, where $m \geq 2$, then we have*

$$\left|A(u_1^m, u_2^m) - L_m^m(u_1, u_2)\right| \leq \frac{m(u_2 - u_1)}{2}A(|u_1^{m-1}|, |u_2^{m-1}|)\int_0^1 \frac{|1-2\delta|}{h(\delta)}d\delta.$$

Proof. This inequality is obtained from Corollary(1) applied on the h-Godunova-Levin preinvex function $\psi : \mathbb{R} \to \mathbb{R}, \psi(x) = x^m, m \geq 2$. $\qquad\square$

Proposition 6. *Let $0 < u_1 < u_2$, where $p, q > 1$ and $m \geq 2$, then we get*

$$\left|A(u_1^m, u_2^m) - L_m^m(u_1, u_2)\right| \leq \frac{m(u_2 - u_1)}{(p+1)^{\frac{1}{p}}}A(|u_1|^{\frac{(m-1)p}{p-1}}, |u_2|^{\frac{(m-1)p}{p-1}})^{\frac{1}{q}}\int_0^1 \frac{1}{h(\delta)}d\delta. \qquad (8)$$

Proof. We derived this inequality from Corollary 2 applied to the h-Godunova-Levin preinvex function $\psi : \mathbb{R} \to \mathbb{R}$. $\qquad\square$

6   Conclusion

Since the Hermite-Hadamard type inequalities, due to their importance, can be found in many field of studies, the present study establishes new generalizations of such inequalities. Thus, two classes of function, h-Godunova-Levin and h-Godunova-Levin preinvex functions, along with some their properties were established here. The applications to special means are also discussed in this study.

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