Article

A note on discrete degenerate random variables†

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Abstract: In this paper, we introduce two discrete degenerate random variables, namely the degenerate binomial and degenerate Poisson random variables. We deduce the expectations of the degenerate binomial random variables. We compute the generating function of the moments of the degenerate Poisson random variables, which leads us to define the new type degenerate Bell polynomials, and hence obtain explicit expressions for the moments of those random variables in terms of such polynomials. We also get the variances of the degenerate Poisson random variables. Finally, we illustrate two examples of the degenerate Poisson random variables.

Keywords: discrete degenerate random variables, degenerate binomial random variable, degenerate Poisson random variable, new type degenerate Bell polynomials.

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1. Introduction

As is well known, the sample space S is the set of all possible outcomes of an experiment and an event is any subset of the sample space. For each evant E of the sample space, we assume that a number $P(E)$ is defined and satisfies the following three conditions:

$$(i)\ \ 0 \le P(E) \le 1,\ \ (ii)\ P(S) = 1,$$

$$(iii)\ \text{For any sequence of events } E_1, E_2, \cdots \text{ with } E_i \cap E_j = \emptyset\ (i \neq j),\ P(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n).$$

We refer to $P(E)$ as the probability of the evant E, (see [4,7,13,16]). Two events E and F are said to be independent if $P(EF) = P(E)P(F)$, (see [11,16]). A random variable X is a real valued function defined on a sample space. If X takes any values in a countable set, then X is called a discrete random variable. If X takes any values in an interval on the real line, then X is called a continuous random variable. For a discrete random variable X, we define the probability mass function $p(a)$ of X by

$$p(a) = P\{X = a\}, \qquad \text{(see [8,9,11,16]).} \tag{1}$$

Suppose that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability $1 - p$, are to be performed. If X represents the number of successes that occur in n trials, then X is called the binomial random variable with parameters n, p, which is denoted by $X \sim B(n, p)$. For $X \sim B(n, p)$, the probability mass function is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i},\ \ i = 0, 1, 2, \cdots, \qquad \text{(see [7,13,16]).} \tag{2}$$

A Poisson random variable indicates how many events occured within a given period of time. A random variable X, taking on one of the values $0, 1, 2, \cdots$, is said to be a Poisson random variable with parameter $\alpha(>0)$ if the probability mass function of X is given by

$$p(i) = P\{X = i\} = e^{-\alpha}\frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \cdots. \tag{3}$$

Note that $\sum_{i=0}^{\infty} p(i) = e^{-\alpha}\sum_{i=0}^{\infty}\frac{\alpha^i}{i!} = e^{-\alpha}e^{\alpha} = 1$. For $n \geq 1$, the quantity $E[X^n]$ of the Poisson random variable X, which is called the n-th moment of X, is defined by

$$E[X^n] = \sum_{i=0}^{\infty} i^n p(i), \quad \text{(see [8,9,11,16]).} \tag{4}$$

When $n = 1$, $E[X]$ is refered to as the mean or the expectation or the first moment of X.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t), \quad \text{(see [3,5,10,12,14]).} \tag{5}$$

When $t = 1$, we write $e_\lambda(1) = e_\lambda = (1 + \lambda)^{\frac{1}{\lambda}}$. It is known that the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n, l)(x)_l, \quad (n \geq 0), \quad \text{(see [1,2,6,15]),} \tag{6}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x - \lambda)\cdots(x - (n - 1)\lambda) \ (n \geq 1)$, $(x)_0 = 1$, $(x)_n = x(x - 1)\cdots(x - n + 1), \ (n \geq 1)$. From (6), we note that

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [9,10]).} \tag{7}$$

From (7), we get

$$\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda} = S_{2,\lambda}(n, k), \quad (n \geq k). \tag{8}$$

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n, k) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^n = S_2(n, k)$, where $S_2(n, k)$ are the ordinary Stirling numbers of the second kind. In [10], the degenerate Bell polynomials are defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{9}$$

From (7) and (9), we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n, k)x^k, \quad (n \geq 0). \tag{10}$$

Let us assume that the probability of success in an experiment is p. Then we might wonder if the probability of success in the ninth trial is still p after failing eight times in a ten trial experiment. Because there is a psychological burden for one to be succesful. It seems plausible that the probability is less than p. This speculation motivated our study of discrete degenerate random variables. In view of this, we will define and study the discrete degenerate random variables in relation to the assumption that the more successful the experiment is, the more likely it is to succeed. In this paper, we consider the degenerate binomial and the degenerate Poisson random variables and obtain expressions for their moments.

2. Degenerate random variable

In this section, we assume that p is the probability of success in an exprement and that the higher the number of successes in the experiment, the higher the probability of success. For $\lambda \in \mathbb{R}$, X_λ is the *degenerate binomial random variable with parameters* n, p if the probability mass function is given by

$$P_\lambda(i) = P\{X_\lambda = i\} = \binom{n}{i}(p)_{i,\lambda}(1-p)_{n-i,\lambda}\frac{1}{(1)_{n,\lambda}}, \tag{11}$$

where $i = 0, 1, 2, \cdots$. If X_λ is the degenerate binomial random variable with parameters n, p, then we denote it by

$$X_\lambda \sim B_\lambda(n, p). \tag{12}$$

From (11), we note that

$$\sum_{i=0}^{\infty} p_\lambda(i) = \frac{1}{(1)_{n,\lambda}}\sum_{i=0}^{\infty}\binom{n}{i}(p)_{i,\lambda}(1-p)_{n-i,\lambda}. \tag{13}$$

Now, we observe that

$$(x+y)_{n,\lambda} = \sum_{i=0}^{n}\binom{n}{i}(x)_{i,\lambda}(y)_{n-i,\lambda}, \quad (n \geq 0). \tag{14}$$

By (13) and (14), we get

$$\begin{aligned}\sum_{i=0}^{\infty} p_\lambda(i) &= \frac{1}{(1)_{n,\lambda}}\sum_{i=0}^{n}\binom{n}{i}(p)_{i,\lambda}(1-p)_{n-i,\lambda}\\ &= \frac{1}{(1)_{n,\lambda}}(p+1-p)_{n,\lambda} = 1.\end{aligned} \tag{15}$$

Let $X_\lambda \sim B_\lambda(n, p)$. Then we have

$$\begin{aligned}E[X_\lambda] &= \sum_{i=0}^{\infty} i p_\lambda(i)\\ &= \frac{1}{(1)_{n,\lambda}}\sum_{i=0}^{\infty} i\binom{n}{i}(p)_{i,\lambda}(1-p)_{n-i,\lambda}\\ &= \frac{n}{(1)_{n,\lambda}}\sum_{i=1}^{n}\binom{n-1}{i-1}(p)_{i,\lambda}(1-p)_{n-i,\lambda}\\ &= \frac{n}{(1)_{n,\lambda}}\sum_{i=0}^{n-1}\binom{n-1}{i}(p)_{i,\lambda}(p-i\lambda)(1-p)_{n-1-i,\lambda}\\ &= \frac{np}{(1)_{n,\lambda}}(1)_{n-1,\lambda} - \frac{n\lambda}{(1)_{n,\lambda}}\sum_{i=0}^{n-1} i\binom{n-1}{i}(p)_{i,\lambda}(1-p)_{n-1-i,\lambda}\\ &= \frac{np}{(1)_{n,\lambda}}(1)_{n-1,\lambda} - \frac{\lambda n(n-1)}{(1)_{n,\lambda}}p(1)_{n-2,\lambda} + \frac{\lambda^2 n(n-1)}{(1)_{n,\lambda}}\sum_{i=0}^{n-2} i\binom{n-2}{i}(p)_{i,\lambda}(1-p)_{n-2-i,\lambda}\\ &= \cdots\\ &= \frac{p}{(1)_{n,\lambda}}\sum_{k=1}^{n}(n)_k(-\lambda)^{k-1}(1)_{n-k,\lambda}.\end{aligned} \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, let $X_\lambda \sim B_\lambda(n,p)$. Then we have*

$$E[X_\lambda] = \frac{p}{(1)_{n,\lambda}} \sum_{k=1}^{n} (n)_k (-\lambda)^{k-1} (1)_{n-k,\lambda}.$$

Note that $\lim_{\lambda \to 0} E[X_\lambda] = np = E[X]$, where X is the binomial random variable with parameters n, p.

For $\lambda \in \mathbb{R}$, X_λ is the *degenerate Poisson random variable with papramenter* $\alpha > 0$ if the probability mass function of X is given by

$$P_\lambda(i) = P\{X_\lambda = i\} = e_\lambda^{-1}(\alpha) \frac{\alpha^i (1)_{i,\lambda}}{i!}, \tag{17}$$

where $i = 0, 1, 2, 3, \cdots$. From (17), we note that

$$\sum_{i=0}^{\infty} P_\lambda(i) = e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{i!} = e_\lambda^{-1}(\alpha) e_\lambda(\alpha) = 1.$$

We observe that

$$\begin{aligned} e^{x(e_\lambda(t)-1)} &= e^{-x} e^{x e_\lambda(t)} = e^{-x} \sum_{k=0}^{\infty} e_\lambda^k(t) \frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty} \left(e^{-x} \sum_{k=0}^{\infty} (k)_{n,\lambda} \frac{x^k}{k!} \right) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

From (9) and (18), we have

$$Bel_{n,\lambda}(x) = e^{-x} \sum_{n=0}^{\infty} (k)_{n,\lambda} \frac{x^k}{k!}, \quad (n \geq 0), \tag{19}$$

where $Bel_{n,\lambda}(x)$ are the degenerate Bell polynomials given by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x) \frac{x^n}{n!}.$$

Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. Then the expectation of X_λ is given by

$$\begin{aligned} E[X_\lambda] &= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{i!} i \\ &= e_\lambda^{-1}(\alpha) \sum_{i=1}^{\infty} \frac{\alpha^i}{(i-1)!} (1)_{i,\lambda} \\ &= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{i!} (1)_{i+1,\lambda} \\ &= \alpha e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} (1 - i\lambda) \\ &= \alpha e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} - \alpha\lambda e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} i \\ &= \alpha - \alpha\lambda E[X_\lambda]. \end{aligned} \tag{20}$$

Thus, by (20), we get

$$E[X_\lambda] = \frac{\alpha}{1+\alpha\lambda}. \tag{21}$$

Therefore, by (21), we obtain the following theorem.

Theorem 2.2. *Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. Then we have*

$$E[X_\lambda] = \frac{\alpha}{1+\alpha\lambda}.$$

Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. For $n \in \mathbb{N}$, the moments of X_λ are given by

$$E[X_\lambda^n] = \sum_{i=0}^{n} i^n p_\lambda(i) = e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{i!} i^n. \tag{22}$$

Let X be the Poisson random variable with parameter $\alpha > 0$. Then the probability mass function of X is given by

$$P(i) = P\{X = i\} = e^{-\alpha}\frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \cdots.$$

Note that, by (19), we have

$$\begin{aligned} E[(X)_{n,\lambda}] &= E[X(X-\lambda)\cdots(X-(n-1)\lambda)] \\ &= \sum_{k=0}^{\infty} (k)_{n,\lambda} \frac{e^{-\alpha}\alpha^k}{k!} = Bel_{n,\lambda}(\alpha). \end{aligned} \tag{23}$$

From (22), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E[X_\lambda^n]\frac{t^n}{n!} &= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{i!} \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} \\ &= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} e^{it} \\ &= e_\lambda^{-1}(\alpha) e_\lambda(\alpha e^t). \end{aligned} \tag{24}$$

In view of (24), it is natural to define the *new type degenerate Bell polynomials* by

$$e_\lambda^{-1}(x) e_\lambda(x e^t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{25}$$

Here we note that the generating function in (25) is obtained from that in (9) by replacing e^x by $e_\lambda(x)$ and vice versa. When $x = 1$, $\beta_{n,\lambda}(1)$ are called the *new type degenerate Bell numbers*. From (25), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} &= e_\lambda^{-1}(x) e_\lambda(x e^t) \\ &= e_\lambda^{-1}(x) \sum_{k=0}^{\infty} (1)_{k,\lambda} x^k \frac{e^{kt}}{k!} \\ &= e_\lambda^{-1}(x) \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (1)_{k,\lambda} x^k \frac{k^n}{k!} \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

Thus, by comparing the coefficients on both sides of (26), we get

$$\beta_{n,\lambda}(x) = e_\lambda^{-1}(x) \sum_{k=0}^{\infty} (1)_{k,\lambda} x^k \frac{k^n}{k!}. \tag{27}$$

Therefore, by (22) and (27), we obtain the following theorem.

Theorem 2.3. *Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. Then we have*

$$E[X_\lambda^n] = \beta_{n,\lambda}(\alpha), \quad (n \in \mathbb{N}).$$

In addition, if X is the Poisson random variable with parameter $\alpha > 0$, then

$$E[(X)_{n,\lambda}] = Bel_{n,\lambda}(\alpha), \quad (n \in \mathbb{N}).$$

Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. Then we have

$$\begin{aligned}
E[X_\lambda^2] &= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} i^2 \\
&= e_\lambda^{-1}(\alpha) \sum_{i=1}^{\infty} \frac{\alpha^i (1)_{i,\lambda}}{(i-1)!} i \\
&= e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^{i+1} (1)_{i+1,\lambda}}{i!} (i+1) \\
&= e_\lambda^{-1}(\alpha)\alpha \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} i(1-i\lambda) + e_\lambda^{-1}(\alpha)\alpha \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda}(1-i\lambda) \\
&= e_\lambda^{-1}(\alpha)\alpha \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} + \alpha(1-\lambda) e_\lambda^{-1}(\alpha) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} i - e_\lambda^{-1}(\alpha)\alpha\lambda \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} (1)_{i,\lambda} i^2 \\
&= \alpha + \alpha(1-\lambda) E[X_\lambda] - \alpha\lambda E[X_\lambda^2].
\end{aligned}$$

By Theorem 2.2, we get

$$\begin{aligned}
(1+\alpha\lambda) E[X_\lambda^2] &= \alpha + \alpha(1-\lambda) E[X_\lambda] \\
&= \alpha + \alpha(1-\lambda) \left(\frac{\alpha}{1+\alpha\lambda} \right) \\
&= \left(\frac{\alpha + \alpha^2}{1+\alpha\lambda} \right).
\end{aligned}$$

Thus, we have

$$E[X_\lambda^2] = \frac{\alpha + \alpha^2}{(1+\alpha\lambda)^2}. \tag{28}$$

The variance of random variable X_λ, denoted by $Var(X_\lambda)$, is defined as

$$\begin{aligned}
Var(X_\lambda) &= E[(X_\lambda - E[X_\lambda])^2] \\
&= E[X_\lambda^2] - (E[X_\lambda])^2
\end{aligned} \tag{29}$$

From (28) and (29), we can derive the following equation (30),

$$\begin{aligned} Var(X_\lambda) &= E[X_\lambda^2] - (E[X_\lambda])^2 \\ &= \frac{\alpha+\alpha^2}{(1+\alpha\lambda)^2} - \frac{\alpha^2}{(1+\alpha\lambda)^2} = \frac{\alpha}{(1+\alpha\lambda)^2}. \end{aligned} \tag{30}$$

Therefore, by (30), we obtain the following theorem.

Theorem 2.4. *Let X_λ be the degenerate Poisson random variable with parameter $\alpha > 0$. Then we have*

$$Var(X_\lambda) = \frac{\alpha}{(1+\alpha\lambda)^2}.$$

Example 2.1. *The number of traffic accidents occuring during the day at some sections of the highway in downtown Seoul is said to follow the degenerate Poisson random variable with parameter 4. The probability of no traffic accident today is*

$$P_\lambda(0) = P\{X_\lambda = 0\} = e_\lambda^{-1}(4).$$

Example 2.2. *Suppose the probability that each guest entering Kim's clothing store in Hapcheon will buy clothes is p. Assuming that the number of guests entering Kim's store follows the degenerate Poisson random variable with parameter $\alpha > 0$, let's compute the probability that the owner won't sell any clothes. Let's say X is the number of clothes sold and N is the number of customers who entered the store. Then*

$$\begin{aligned} P_\lambda(0) = P\{X_\lambda = 0\} &= \sum_{n=0}^{\infty} P\{X=0|\ N=n\} P_\lambda\{N=n\} \\ &= \sum_{n=0}^{\infty} P\{X=0|\ N=n\} e_\lambda^{-1}(\alpha)\frac{\alpha^n}{n!}(1)_{n,\lambda}. \end{aligned}$$

If n people enter the store, the probability of not selling any clothes is $(1-p)^n$. Hence,

$$\begin{aligned} P_\lambda\{X=0\} &= \sum_{n=0}^{\infty} \frac{(1-p)^n(1)_{n,\lambda}}{n!}\alpha^n e_\lambda^{-1}(\alpha) \\ &= e_\lambda^{-1}(\alpha) e_\lambda(\alpha(1-p)). \end{aligned}$$

3. Conclusions

In this paper, we introduced two discrete degenerate random variables, namely the degenerate binomial and degenerate Poisson random variables. Their details and results are as in the following. We defined the degenerate binomial random variables with parameter n, p in (11) and deduced their expectations in Theorem 2.1. We also defined the degenerate Poisson random variables with parameter $\alpha > 0$ in (17). We obtained their expectations in Theorem 2.2. Then, by computing the generating function of the moments of those random variables in (24), we were naturally led to define the new type degenerate Bell polynomials in (25). We observed explicit expressions for those polynomials in (27) and found explicit expressions for the moments of the degenerate Poisson random variables in terms of the new type degenerate Bell polynomials in Theorem 2.3. Finally, by calculating the second moments of the degenerate Poisson random variables, we were able to get the variances of those random variables in Theorem 2.4. Finally, we illustrated two interesting examples of the degenerate Poisson random variables in Exampes 2.1 and 2.2.

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References

1. Bell, E.T. Exponential polynomials, *Ann. of Math.* **1934**, *35(2)*, 258-277.
2. Brillhart, J. Mathematical notes: note on the single variable Bell polynomials, *Amer. Math. Monthly* **1967**, *74(6)*, 695-696.
3. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Math.* **1979**, *15*, 51-88.
4. Comtet, L. Advanced combinatorics: the art of finite and infinite expansions(translated from the French by J.W. Nienhuys), Dordrecht and Boston: Reidel, 1974.
5. Dolgy, D.V.; Jang, G.-W.; Kim, T. A note on degenerate central factorial polynomials of the second kind, *Adv. Stud. Contemp. Math. (Kyungshang)* **2019**, *29(1)*, 7-13.
6. He, Y.; Pan, J. Some recursion formulae for the number of derangements and Bell numbers, *J. Math. Res. Appl.* **2016**, *36(1)*, 15-22.
7. Haigh, J. Probability models, 2nd ed.; Springer Undergraduate Mathematics Series. Springer, London, 2013; xii+287 , ISBN: 978-1-4471-5342-9; 978-1-4471-5343-6.
8. Kim, T.; Yao, Y.; Kim, D.S.; Kwon, H.-I. Some identities involving special numbers and moments of random variables, *Rocky Mountain J. Math.* **2019**, *49(2)*, 521-538.
9. Kim, T.; Kim, D.S.; Jang, G.-W. Extended Stirling polynomials of the second kind and extended Bell polynomials, *Proc. Jangjeon Math. Soc.* **2017**, *20(3)*, 365-376.
10. Kim, T.; Kim, D. S.; Dolgy, D.V. On partially degenerate Bell numbers and polynomials, *Proc. Jangjeon Math. Soc.* **2017**, *20(3)*, 337-345.
11. Kim, T.; Kim, D. S.; Kim, H.Y.; Kwon, J. Degenerate Stirling polynomials of the second kind and some applications, *Symmetry* **2019**, *11(8)*, 1046; https://doi.org/10.3390/sym11081046.
12. Kim, T.; Kim, D. S. Degenerate Laplace transform and degenerate gamma function, *Russ. J. Math. Phys.* **2017**, *24(2)*, 241-248.
13. Leon-Garcia, A. Probability and random processes for electronical engineering, Addition Wesley Longman; Massachusetts, USA, 2018; ISBN:0-321-18963-9.
14. Pyo, S.S. Degenerate Cauchy numbers and polynomials of the fourth kind, *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, *28(1)*, 127-138.
15. Roman, S. The umbral calculus, Pure and applied mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
16. Ross, S.M. Introduction to probability models, Twelfth edition of [MR0328973]; Academic Press; London, 2019; xv+826, ISBN:978-0-12-814346-960-01.