Some Inquiries Into Special Relativity

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Abstract This paper presents research within the topic of special relativity. First, we show that time dilation and length contraction do not necessarily require that the speed of light be the same in all reference frames. It is only necessary to require that the speed of light be finite in any particular frame. Next, we demonstrate that the relativistic Doppler shift for electromagnetic waves has an analogue for extended material objects, and furnish an expression for the relative velocity of such objects in terms of what we call their "rest length shift." An alternative velocity, which we call "rest velocity," is then introduced to simplify the Lorentz transformation formulas. We also show that regular velocity can be regarded as the geometric average of the "rest velocity" and another type of velocity we term "contracted velocity." We conclude by formulating the relationship between regular and "rest" velocity for non-uniform motion in one, two, and three dimensions.

Keywords: physics, relativity

I. INTRODUCTION

This article presents the findings of original research within the area of special relativity. The utmost care has been taken to derive results in a clear and orderly manner. Hence the "textbook" look and feel of the article.

In Sections II and III we use the light clock concept to define time and length units for stationary and moving reference frames. In Section III we then precisely define the relative velocity v of a reference frame (or material object) by means of the previously defined time and length units. Only after v is thus defined, do we employ it to derive the phenomenon of time dilation.

In Section IV we note that the phenomenon of time dilation of moving clocks as observed in any particular reference frame is not strictly contingent on the speed of light being the same in all reference frames. It is only contingent on the the speed of light being finite in that particular reference frame.

In Section V we observe that the conclusion in Section IV also holds for the phenomenon of length contraction. We also show that an equivalent definition of material velocity v can be given that is entirely consistent with the previous definition in Section III.

Section VI provides a derivation of the Lorentz transformation based on length contraction alone.

In Section VII we derive the Lorenz factor from the consideration of two rigid rods in relative motion. If instead we consider the Lorentz factor as already derived from the usual Pythagorean relation, we arrive at a "rest length shift" formula for extended material objects in relative motion, analogous to the relativistic Doppler shift for electromagnetic waves. By inverting this formula, we obtain an expression for the velocity of an extended object in terms of its "rest length shift."

By introducing an alternative velocity definition, which we call "rest velocity," the Lorentz transformation equations may be simplified so that they no longer contain the Lorentz factor radical. This is shown in Section VIII. The expression for the Lorenz factor in terms of "rest velocity" is also derived.

In Section IX, a second alternative velocity definition, which we call "contracted velocity," is introduced and expressed in terms of regular velocity. Regular velocity is then identified as the geometric average of the two alternative velocities introduced Sections VIII and IX. The formula for the Lorenz factor in terms of "contracted velocity" is derived as well.

In Section X we extend Section VIII's description of "rest" velocity to non-uniform motion. For this purpose, the section represents variable-speed motion of a material particle by fixed-speed motion of the endpoint of a "light ray curve" in a reference frame containing an added spatial axis. The endpoint of this "light ray curve" moves at the fixed speed of light c, so that variations of the particle's speed v correspond to bends in its "light ray curve."

In Sections XI and XII the discussion in Section X is broadened to motion in two and three spatial dimensions, respectively. For three-dimensional variable-speed particle motion, the particle's corresponding fixed-speed "light ray curve" unfolds (and bends) in a four-dimensional spatial frame of reference.

II. PERPENDICULAR TIME

The units for time and length have long been connected through the reference-frame-invariant speed of light c: The unit of length (the meter) is defined as the distance traveled by light during a fixed fraction $\frac{1}{n}$ of the unit of time (the second), where $n = 299,792,458$ [1]. So length is currently defined as a **light-time**, such as a "light-second" or "light-year."

Alternatively, one could define the unit of time as the time it takes light to travel n units of length. Here, we adopt the this alternative definition. That is, we define

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time as a **light-length**. For example, a "light-meter" would be the time it takes light to travel one meter.

In Figure 1, there is an inertial reference frame S with orthogonal axes (x, y, z), and an inertial reference frame S' with orthogonal axes (x', y', z'). These axes are not shown in the figure, to avoid clutter. The x'-axis of S' coincides with the x-axis of S, and the y'-axis of S' is parallel with the y-axis of S. The x-axis and x'-axis are directed horizontally to the right. The y-axis and y'-axis are directed vertically upward.

Relative to the origin of S, the origin of S' is moving to the right along the x-axis at a constant velocity $\vec{v}$, as measured in S. S' moves uniformly in the left-to-right direction, entering Figure 1 at its left edge and exiting at its right edge. The x-axis and x'-axis are defined to be parallel with the velocity vector $\vec{v}$ of S' relative to S. The y-axis and y'-axis are similarly defined to be perpendicular to $\vec{v}$. The z and z' axes are perpendicular to the plane of the figure (the xy-plane).

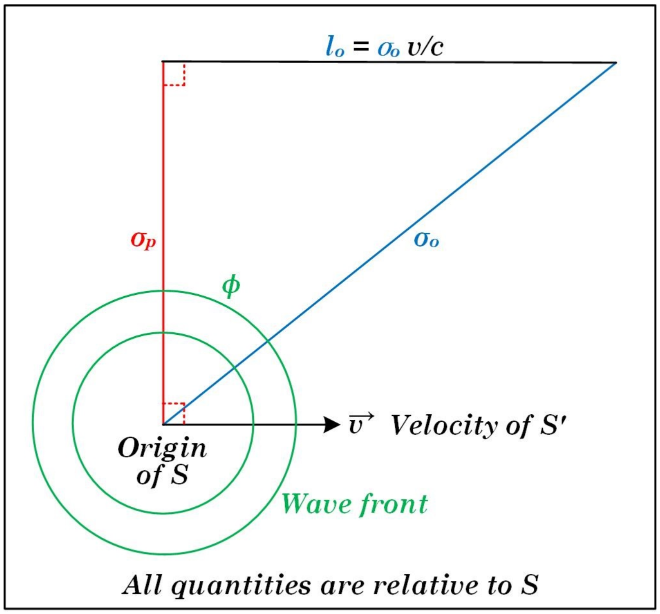

FIG. 1. Light ray segments in S.

In Figure 1, we imagine there are two two-dimensional infinite flat mirrors parallel with $\vec{v}$ and perpendicular to the y-axis and y'-axis. The reflecting surface of the bottom mirror contains the x-axis and x'-axis. The bottom mirror lies in the xz-plane of S and the $x'z'$-plane of S'. In Figure 1, the velocity vector $\vec{v}$ is drawn in the plane of the bottom mirror surface.

The **mirror separation** is the perpendicular distance σ_p between the reflecting surface of the top mirror and the reflecting surface of the bottom mirror. Note that there is no relative motion between S and S' perpendicular to the mirrors.

When the origins of S and S' coincide, let a flash of light be emitted from their common point of origin. The light radiates outward with a spherical wave front, as seen from S, and subsequently bounces off the two mirrors indefinitely.

We now represent time intervals in S and S' in terms of the length of certain light ray segments from the flash in S and S' by means of the concept of a **light clock** [2, 3]. The ticking of the light clock in S is the up-and-down propagation of light along the y-axis between the two mirrors, as seen in S. Similarly, the ticking of the light clock in S' is the up-and-down propagation of light along the y'-axis between the same two mirrors, as seen in S'. See Figure 1.

We take the **unit of perpendicular length** in S to be the length of that light ray segment, as seen from S, that originates from the origin of S at the bottom mirror, is perpendicular to the velocity $\vec{v}$ of S' relative to S, and terminates at the top mirror. This particular light ray segment in S is labeled "σ_p" in Figure 1 (the subscript "p" stands for "perpendicular"). The segment is along the positive y-axis. The length of the segment is obviously equal to the mirror separation σ_p. We denote the speed of light as measured in S by c. In Figure 1, we have labeled the speed of light accordingly as c.

We take the **unit of perpendicular time** in S to be the time

$$\tau_p = \frac{\sigma_p}{c} \tag{1}$$

it takes for light to propagate a unit σ_p of perpendicular length in S. In other words, **we define the unit of perpendicular time τ_p as the light-length σ_p**. Note that the perpendicular light ray segment in S is traced out by the wave front along the positive y-axis in S. Since the y-axis is at rest in S, the unit of perpendicular time is the **unit of rest time** in S.

When the wave front along the perpendicular segment in Figure 1 impinges on the top mirror, the wave is reflected perpendicularly back towards the bottom mirror. The reflected wave is a spherical wave emanating from the point of impingement. The reflected wave travels down the positive y-axis, and the duration of travel from the top mirror to the bottom mirror is again the unit τ_p of perpendicular time in S.

A **perpendicular time tick** (or **rest time tick**) in S is the event when the light propagating along the perpendicular segment in S reflects off the top mirror or off the bottom mirror. The tick itself has no duration. The time elapsed between two successive perpendicular time ticks in S is the unit τ_p of perpendicular time in S. The perpendicular light ray continues to bounce off the two mirrors indefinitely along the positive y-axis of S, tracing out perpendicular time in S. The **rate of perpendicular time** (or **rate of rest time**) in S is the frequency

$$\nu_p = \frac{1}{\tau_p} \tag{2}$$

of perpendicular time ticks in S.

III. OBLIQUE TIME. TIME DILATION.

Consider the wave front points from the flash at the common point of origin of S and S' that lie on the perpendicular light ray segment in S'. In S', this perpendicular light ray segment is along the positive y'-axis and its length is the mirror separation σ_p, since there is no relative motion between S and S' perpendicular to the mirrors.

In S, the same wave front points form the oblique light ray segment labeled "σ_o" in Figure 1 (the subscript "o" stands for "oblique"). The obliquity is due to the motion of S' from left to right with uniform velocity $\vec{v}$ relative to S. As measured in S, the length of this oblique segment is σ_o. We call this length the **unit of oblique length** in S.

When the wave front along the oblique segment in Figure 1 impinges on the top mirror, as seen from S, the origin of S' has moved to the right in S a distance l_o to the point on the x-axis where the y'-axis of S' contains the point of impingement, as seen from S. Note that this definition of l_o assumes that l_o is less than σ_o.

For reasons that will become clear in Equation 5 below, we shall refer to l_o as the **fractional unit of oblique length** in S. We may picture the segment of the positive x-axis whose left endpoint is the origin of S and whose length is l_o as a rigid rod in S. Since this rod is at rest in S, we may also refer to l_o as the **fractional unit of rest length** in S.

We take the **unit of oblique time** in S to be the time

$$\tau_o = \frac{\sigma_o}{c} \tag{3}$$

it takes for light to propagate a unit σ_o of oblique length in S. In other words, **we define the unit of oblique time τ_o as the light-length σ_o**. Note that the oblique ray in S is traced out by the wave front along the positive y'-axis in S'. Since the y'-axis is moving to the right in S, we also refer to τ_o as the **unit of moving time** in S.

We define the **velocity** of S' in S to be the ratio of the fractional unit of oblique length (rest length) in S to the unit of oblique time (moving time) in S:

$$v = |\vec{v}| = \frac{l_o}{\tau_o} = c\frac{l_o}{\sigma_o}. \tag{4}$$

As seen from S, the time it takes S' to move the distance l_o is $\tau_o = \frac{\sigma_o}{c}$ (Equation 3). v is the distance l_o that S' travels, as measured in S, divided by the time τ_o of travel, also as measured in S.

From Figure 1, we see that by definition, σ_p, l_o, and σ_o form the first leg, second leg, and hypotenuse, respectively of a right triangle. Therefore,the velocity $v = c\frac{l_o}{\sigma_o}$ of S' cannot exceed c due to the way v is defined. Compare this with the alternatively defined velocity $u = c\frac{l_o}{\sigma_p}$ in Equation 32, which can exceed c.

From Equation 4, l_o is given by

$$l_o = v\tau_o = \sigma_o\frac{v}{c}. \tag{5}$$

Equation 5 shows that l_o is equal to the unit σ_o of oblique length times the velocity fraction $\frac{v}{c}$. This is the reason for calling l_o the fractional unit of oblique length. Equation 5 also shows that l_o is equal to the velocity v of S' in S times the unit τ_o of moving time in S.

According to the right-triangle relation (Pythagorean relation) for the right triangle of lengths in Figure 1 for S, we have

$$\sigma_p^2 + l_o^2 = \sigma_o^2. \tag{6}$$

Substitution of Equation 5 into Equation 6 and solving for $\frac{v}{c}$ gives

$$\frac{v}{c} = \sqrt{1 - \frac{\sigma_p^2}{\sigma_o^2}}. \tag{7}$$

We define the **obliquity factor** in S to be the ratio

$$\gamma = \frac{\sigma_o}{\sigma_p} \tag{8}$$

of the unit of oblique length to the unit of perpendicular length in S.

Solving 7 for the obliquity factor, we get

$$\gamma = \frac{\sigma_o}{\sigma_p} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{9}$$

which is of course the familiar Lorentz factor [4]. From Equation 9, we have

$$\sigma_o = \gamma\sigma_p = \frac{\sigma_p}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{10}$$

From Equation 10, we see that the unit of oblique length in S is a factor of γ longer than the unit of perpendicular length in S. When dividing Equation 10 by c and using Equations 1 and 3, we obtain

$$\tau_o = \frac{\sigma_o}{c} = \gamma\frac{\sigma_p}{c} = \gamma\tau_p. \tag{11}$$

So the unit τ_o of moving time in S is a factor of γ longer than the unit τ_p of rest time in S. This is the familiar **time dilation** result: The elapsed time, as seen from S, between two successive time ticks of a light clock at the origin of S' is a factor of γ longer than the elapsed time, as seen from S, between two successive time ticks of a light clock at the origin of S.

We define the **obliquity angle** in S as the angle ϕ that the oblique light ray segment "σ_o" makes with the perpendicular light ray segment "σ_p," as seen from S in Figure 1.

From the right triangle in Figure 1, we see that ϕ is given by

$$\sin\phi = \frac{l_o}{\sigma_o} = \frac{v}{c}, \tag{12}$$

or equivalently by

$$\cos\phi = \frac{\sigma_p}{\sigma_o} = \frac{1}{\gamma}. \tag{13}$$

When the wave front along the light ray segment labeled "σ_o" in Figure 1 impinges on the top mirror, the ray is reflected obliquely back towards the bottom mirror. The reflected wave is a spherical wave emanating from the point of impingement. The oblique reflected ray makes the angle ϕ with the perpendicular. This oblique reflected ray is not shown in Figure 1, to avoid clutter. When the wave front along the oblique reflected ray impinges on the bottom mirror, as seen from S, the origin of S' has moved to the right a distance l_o again to the point on the x-axis where the origin of S' contains the point of impingement, as seen from S. Relative to S, the duration of travel of the oblique reflected ray from the top mirror to the bottom mirror is again the unit τ_o of oblique time in S.

An **oblique time tick** (or **moving time tick**) in S is the event when the light propagating along an oblique segment in S reflects off the top mirror or off the bottom mirror. The tick itself has no duration. The time elapsed between two successive oblique time ticks in S is the unit τ_o of oblique time in S. The oblique light ray continues to zigzag between the two mirrors indefinitely at a net horizontal velocity $\vec{v}$ relative to S, tracing out oblique time in S. The **rate of oblique time** (or **rate of moving time**) in S is the frequency

$$\nu_o = \frac{1}{\tau_o} \tag{14}$$

of oblique time ticks in S.

IV. LIGHT SPEED FINITUDE

If c were infinite instead of finite in some reference frame, light would traverse any distance in no time, i.e., time would not exist in this reference frame. We owe the existence of time to the fact that c is finite in any particular reference frame. We may refer to this property of electromagnetic radiation as **light speed finitude**.

If S' travels to the right at velocity v relative to S, then S travels to the left at velocity v relative to S'. We may call this fact **frame speed reciprocity**. We also know that the speed c of light in S is the same as the speed of light in S', which we may call **light speed invariance**.

Note that it is not necessary to assume frame speed reciprocity or light speed invariance to deduce the existence of time dilation in any particular reference frame S. One only needs to assume light speed finitude in S. Frame speed reciprocity and light speed invariance are only required to deduce that the amount γ of time dilation is the same in both S and S', a property we may call **time dilation reciprocity**.

In Figure 1, light speed finitude means the wave along the perpendicular light ray segment σ_p in S and the wave along the oblique light ray segment σ_o in S both travel at the same finite speed c. The oblique segment is longer than the perpendicular segment, so it takes light longer to travel along the oblique segment than along the perpendicular segment. This is the reason for time dilation.

V. LENGTH CONTRACTION

Assume there is a rigid rod stationary in S' along the negative x'-axis with its right endpoint at the origin of S'. The rod is thus moving with uniform velocity v to the right across the stationary light clock at the origin of S, as seen from S.

We determine the elapsed time, according to the stationary light clock in S, between the instance in S when the rod's right endpoint is at the origin of S and the instance in S when the rod's left endpoint is at the origin of S. This elapsed time will be the unit τ_p of perpendicular time (i.e., rest time) in S multiplied by a certain number of perpendicular time ticks in S.

For simplicity, let's assume that the length of the rod is such that the elapsed time measured by the stationary light clock in S is exactly the unit τ_p of perpendicular time in S. The length of the moving rod, as measured in S, is then

$$l_p = v\tau_p = \sigma_p \frac{v}{c}. \tag{15}$$

Equation 15 shows that l_p is equal to the unit σ_p of perpendicular length times the velocity fraction $\frac{v}{c}$. We may therefore call l_p the **fractional unit of perpendicular length**. Since the rod is moving in S, we may also call l_p the **fractional unit of moving length** in S.

The rod moving past a light clock at a velocity v relative to the light clock is equivalent to the light clock moving past the rod at a velocity v relative to the rod. So we now assume that the same rod is stationary in S and lies along the positive x-axis with its left endpoint at the origin of S. The light clock in S' is now moving with uniform velocity v to the right across the stationary rod, as seen from S.

We determine the elapsed time, according to the moving light clock, between the instance in S when the origin of S' is at the rod's left endpoint and the instance in S when the origin of S' is at the rod's right endpoint. This elapsed time will be the unit τ_o of moving time in S. The length of the stationary rod, as measured in S, is then given by Equation 5, and as stated before, the fractional unit l_o of rest length in S is the velocity v of S' in S times the unit τ_o of moving time in S.

We may now suitably define the **unit rod** in S to be a rod whose length in S when at rest in S is the fractional unit of rest length in S and whose length in S when moving in S is the fractional unit of moving length in S.

The relationship between the fractional unit l_o of rest length in S and the fractional unit l_p of moving length in S is shown in Figure 2.

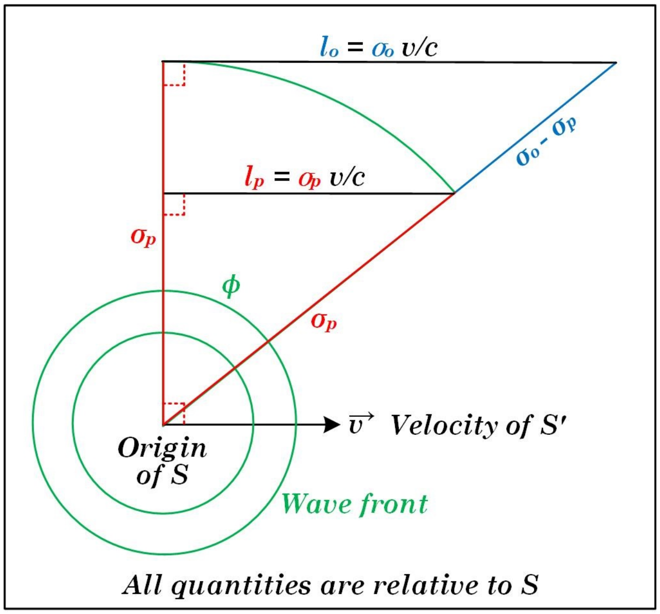

FIG. 2. Length contraction in S.

Since the fractional unit l_p of moving length in S is the fraction $\frac{v}{c}$ times the unit σ_p of perpendicular length in S (Equation 15), and the fractional unit l_o of rest length in S is the same fraction $\frac{v}{c}$ times the unit σ_o of oblique length in S (Equation 5), we may also call σ_p the **unit of moving length** in S, and σ_o the **unit of rest length** in S.

Note that σ_o is the unit of rest length and $\tau_o = \frac{\sigma_o}{c}$ is the unit of moving time, whereas σ_p is the unit of moving length and $\tau_p = \frac{\sigma_p}{c}$ is the unit of rest time.

Equation 15 can be viewed as an alternative definition of the velocity v of S' in S:

$$v = \frac{l_p}{\tau_p} = c\frac{l_p}{\sigma_p}. \tag{16}$$

This definition is obviously equivalent to the definition given by Equation 4, because the two right triangles in Figure 2 are similar (proportional). **So v is oblique length (rest length) over oblique time (moving time), or perpendicular length (moving length) over perpendicular time (rest time)**.

Due to time dilation (Equation 11), the time τ_o it takes for the origin of S' to move across the rod, as seen in S, is longer by a factor of γ than the time τ_p it takes for the rod to move across the origin of S, as seen in S. If we multiply Equation 11 by v, Equations 5 and 15 show that

$$l_o = v\tau_o = \gamma v\tau_p = \gamma l_p. \tag{17}$$

So the fractional unit l_o of rest length in S is a factor of γ longer than the fractional unit l_p of moving length in S. This is the familiar **length contraction** result: The distance the origin of S' travels along the x-axis between two successive time ticks, as seen from S, of a light clock at the origin of S' is a factor of γ longer than the distance the origin of S' travels along the x-axis between two successive time ticks, as seen from S, of a light clock at the origin of S.

Just as for time dilation, it is not necessary to assume frame speed reciprocity or light speed invariance to deduce the existence of length contraction in any particular reference frame S. One only needs to assume light speed finitude in S. Frame speed reciprocity and light speed invariance are only required to deduce that the amount γ of length contraction in S is equal to that in S', a property we may call **length contraction reciprocity**.

VI. SPACETIME TRANSFORMATION

We assume a rigid rod is at rest along the positive x'-axis in S', has its left endpoint at the origin of S', and has a rest length of x'. x' is thus the location of the rod's right endpoint, as seen in S'. As in Figure 1, we assume the origin of S' moves to the right along the x-axis in S with constant velocity v, as seen from S. We define t to be the time elapsed in S after the origins of S and S' coincided. t is the elapsed rest time in S.

Due to light speed finitude, the moving rod undergoes length contraction in S. As seen from S therefore, the location x of the right endpoint of the rod after the time interval t has elapsed is the moving length $\frac{x'}{\gamma}$ of the rod (Equation 17) plus the distance vt the rod has traveled to the right during the time interval t:

$$x = \frac{x'}{\gamma} + vt. \tag{18}$$

Solving this equation for x', we get

$$x' = \gamma(x - vt). \tag{19}$$

We now instead assume the same rigid rod is at rest along the positive x-axis in S, has its left endpoint at the origin of S, and has a rest length of x. x is thus the location of the rod's right endpoint, as seen in S. Relative to S', the origin of S moves to the left along the x'-axis in S' with constant velocity v, as seen from S'. We define t' to be the time elapsed in S' after the origins of S and S' coincided. t' is the elapsed rest time in S'.

Due to light speed finitude, the moving rod undergoes length contraction in S'. We now impose frame speed reciprocity and light speed invariance. As seen from S' therefore, the location x' of the right endpoint of the rod after the time interval t' has elapsed is the moving length $\frac{x}{\gamma}$ of the rod (Equation 17) minus the distance vt' the rod has traveled to the left during the time interval t':

$$x' = \frac{x}{\gamma} - vt'. \tag{20}$$

Solving this equation for x, we get

$$x = \gamma(x' + vt'). \tag{21}$$

By substituting the expression for x in Equation 18 into Equation 20 and then solving for t (while making use of Equation 9), we get

$$t = \gamma(t' + \frac{vx'}{c^2}). \tag{22}$$

Equations 21 and 22 define the familiar **spacetime transformation** (Lorentz transformation) from S' to S.

We see that it is necessary to assume frame speed reciprocity and light speed invariance to deduce the spacetime transformation (Equations 21 and 22). It is not sufficient to only assume light speed finitude.

VII. REST LENGTH SHIFT

Consider two parallel rigid rods R and r of rest lengths L and l respectively. Without loss of generality, we take $l < L$. Let r move to the right with speed v relative to R. See Figure 3. When the rods' left ends are aligned, let a flash of light be emitted from the left alignment point. This is indicated by the photon in Figure 3.

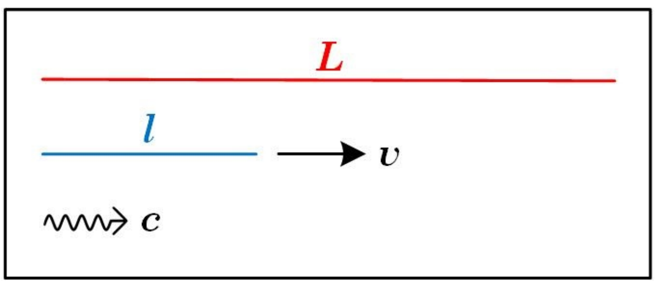

FIG. 3. Rest length shift.

We take v to be the speed required for the right ends of the rods to be aligned when the photon reaches the right alignment point. Note that the required speed $v < c$ can always be found for any L and any $l < L$.

In the rest frame of R, the time elapsed between the event of the rods' left ends being aligned to the event of the rods' right ends being aligned is $\frac{L}{c}$. In this time, r has traveled a distance $v\frac{L}{c}$. We set the "moving length" of r (length of r as measured in the rest frame of R) to be the rest length l of r times some (yet unknown) factor $\frac{1}{\gamma}$. Then

$$L = \frac{1}{\gamma}l + v\frac{L}{c}, \tag{23}$$

or

$$\frac{l}{\gamma} = L(1 - \frac{v}{c}). \tag{24}$$

In the rest frame of r, R is moving to the left with the same speed v (due to frame speed reciprocity). The speed of light in the rest frame of r is the same as that in the rest frame of R (due to light speed invariance). The time elapsed between the event of the rods' left ends being aligned to the event of the rods' right ends being aligned is therefore $\frac{l}{c}$. In that time, R has traveled a distance $v\frac{l}{c}$ to the left. Because of frame speed reciprocity, we may set the "moving length" of R (length of R as measured in the rest frame of r) to be the rest length L of R times the same factor $\frac{1}{\gamma}$. Then

$$l = \frac{1}{\gamma}L - v\frac{l}{c}, \tag{25}$$

or

$$\frac{L}{\gamma} = l(1 + \frac{v}{c}). \tag{26}$$

When multiplying Equations 24 and 26, we get

$$\frac{Ll}{\gamma^2} = lL(1 - \frac{v^2}{c^2}). \tag{27}$$

γ is thus the familiar length contraction factor (Lorentz factor) of Equation 9.

The argument above holds for any rest lengths L and $l < L$, and therefore for any relative speeds v. Thus, length contraction is again proved (in a manner different from that of Section V): The "moving length" of rod R is shorter than its rest length L by the factor $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}$. Similarly, the "moving length" of rod r is shorter than its rest length l by the same factor $\frac{1}{\gamma}$.

By substituting Equation 9 into Equation 24 and then squaring the result, we get

$$l^2(1 - \frac{v^2}{c^2}) = L^2(1 - \frac{v}{c})^2, \tag{28}$$

or

$$l^2(1 + \frac{v}{c}) = L^2(1 - \frac{v}{c}). \tag{29}$$

Solving Equation 29 for $\frac{L}{l}$ yields

$$\frac{L}{l} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}. \tag{30}$$

$\frac{L}{l}$ (> 1) is the rest length of R in units of the rest length of r. Equation 30 is the familiar wavelength shift (Doppler shift) expression: Assume the light source is attached to R's left end and a light detector is attached to r's right end. If l is the wavelength of the flash of light as seen from R's rest frame, then L is the wavelength of the same flash as seen from r's rest frame.

Solving Equation 29 for $\frac{v}{c}$ yields

$$\frac{v}{c} = \frac{L^2 - l^2}{L^2 + l^2} = \frac{(\frac{L}{l})^2 - 1}{(\frac{L}{l})^2 + 1}. \tag{31}$$

Equation 31 expresses the relative velocity v in terms of the "rest length shift" $\frac{L}{l}$ it generates. The functional dependence of v on $\frac{L}{l}$ is graphed in Figure 4.

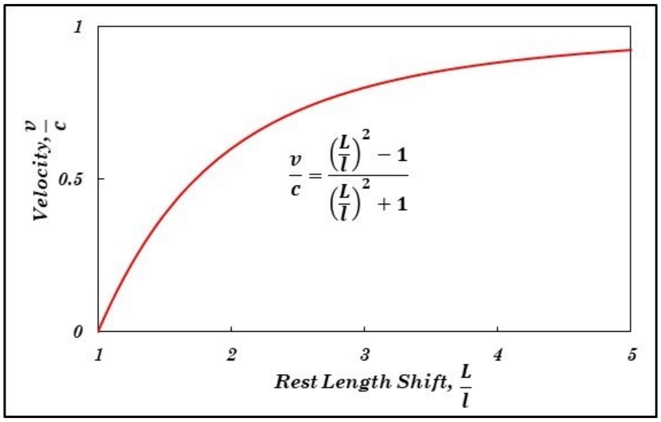

FIG. 4. Dependence of v on L/l.

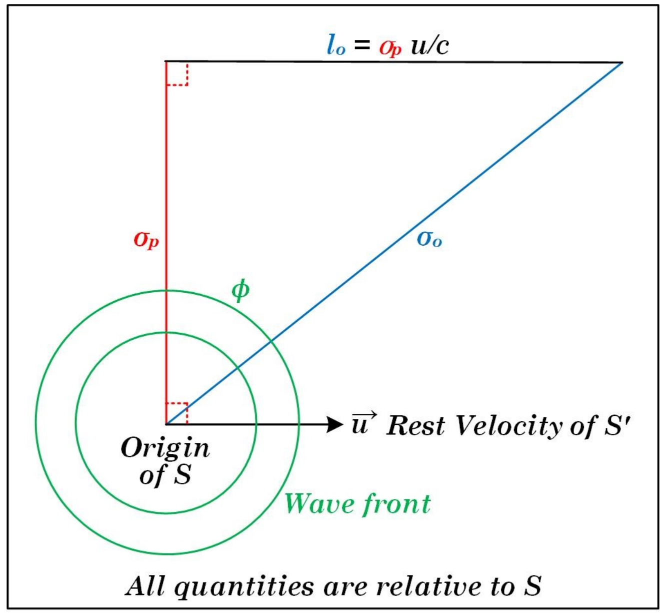

FIG. 5. Rest velocity $\vec{u}$.

VIII. REST VELOCITY

Equation 4 defines the velocity v of S' in terms of oblique length and time, while Equation 16 equivalently defines v in terms of perpendicular length and time. Let's now define a new velocity in terms of oblique length and perpendicular time. We label it u and call it **rest velocity**, to distinguish it from the previously defined velocity v:

$$u = |\vec{u}| = \frac{l_o}{\tau_p} = c\frac{l_o}{\sigma_p}. \tag{32}$$

Figure 5 depicts how u is defined. When S' has moved the distance l_o in S, the unit $\tau_o = \frac{\sigma_o}{c}$ (Equation 3) of moving time in S has elapsed, and the unit $\tau_p = \frac{\sigma_p}{c}$ (Equation 1) of rest time in S has elapsed. To compute u, an observer in S must divide l_o by the rest time unit τ_p rather than the moving time unit τ_o.

From Figure 5, we see that l_o and σ_p are both legs in a right triangle. Therefore, the rest velocity u of S' can exceed c due to the way u is defined. Compare this with the regular velocity v defined in Equations 4 and 16. Note that by definition, σ_p, l_o, and σ_o still form the first leg, second leg, and hypotenuse, respectively of a right triangle. That is, l_o is still defined to be less than σ_o.

From Equation 32, the fractional unit of rest length l_o in S is now given by

$$l_o = u\tau_p = \sigma_p\frac{u}{c}. \tag{33}$$

The right-triangle relation (Equation 6) still holds. Substitution of Equation 33 into Equation 6 and solving for $\frac{u}{c}$ gives

$$\frac{u}{c} = \sqrt{\frac{\sigma_o^2}{\sigma_p^2} - 1}. \tag{34}$$

Solving 34 for the obliquity factor $\gamma = \frac{\sigma_o}{\sigma_p}$ (Equation 8), we get

$$\gamma = \frac{\sigma_o}{\sigma_p} = \sqrt{1 + \frac{u^2}{c^2}}. \tag{35}$$

This expression is of course just the familiar Lorentz factor in Equation 9, expressed in terms of u instead of v.

From Equation 35, we have

$$\sigma_o = \gamma\sigma_p = \sigma_p\sqrt{1 + \frac{u^2}{c^2}}. \tag{36}$$

Just like Equation 10, Equation 36 expresses the phenomenon of time dilation: The elapsed time, as seen from S, between two successive time ticks of a light clock at the origin of S' is a factor of γ longer than the elapsed time, as seen from S, between two successive time ticks of a light clock at the origin of S.

If we divide Equation 4 by Equation 32 and compare with Equation 8, we see that

$$v = \frac{u}{\gamma}. \tag{37}$$

If we substitute Equation 35 into Equation 37, we get the following expression for v in terms of u:

$$v = \frac{u}{\sqrt{1 + \frac{u^2}{c^2}}}. \tag{38}$$

Equation 38 shows that v cannot be superluminal even if u is superluminal.

If we substitute Equation 9 into Equation 37, we get the following expression for u in terms of v:

$$u = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{39}$$

Equation 39 shows that u is luminal when v is $\frac{c}{\sqrt{2}}$, and becomes superluminal when v exceeds $\frac{c}{\sqrt{2}}$. The functional dependence of the rest velocity u on the regular velocity v is graphed in Figure 6.

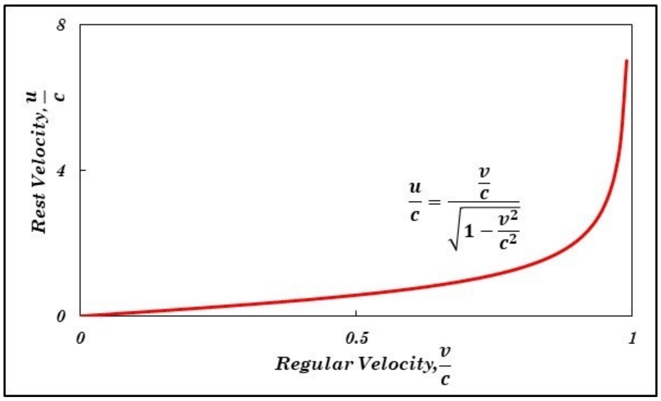

FIG. 6. Dependence of u on v

By multiplying Equations 38 and 39, we see that the fundamental relationship between v and u is

$$(1 - \frac{v^2}{c^2}) \cdot (1 + \frac{u^2}{c^2}) = 1. \tag{40}$$

Equation 37 lets us write the spacetime transformation equations 21 and 22 as

$$\frac{x}{u} = \frac{x'}{v} + t' \tag{41}$$

and

$$\frac{t}{u} = \frac{t'}{v} + \frac{x'}{c^2}. \tag{42}$$

We see of course that the price for the removal of the Lorentz factor γ (Equation 8) is the introduction of the rest velocity u.

IX. CONTRACTED VELOCITY

Equation 32 defines the rest velocity u of S' in terms of oblique length and perpendicular time. Let's now define yet another velocity in terms of perpendicular length and oblique time. We label it q and call it **contracted velocity**, to distinguish it from the previously defined velocities v and u:

$$q = |\vec{q}| = \frac{l_p}{\tau_o} = c\frac{l_p}{\sigma_o}. \tag{43}$$

Figure 7 depicts the definition of q. q is the length-contracted distance l_p that S' travels, as measured in S, divided by the time τ_o of travel, as measured in S.

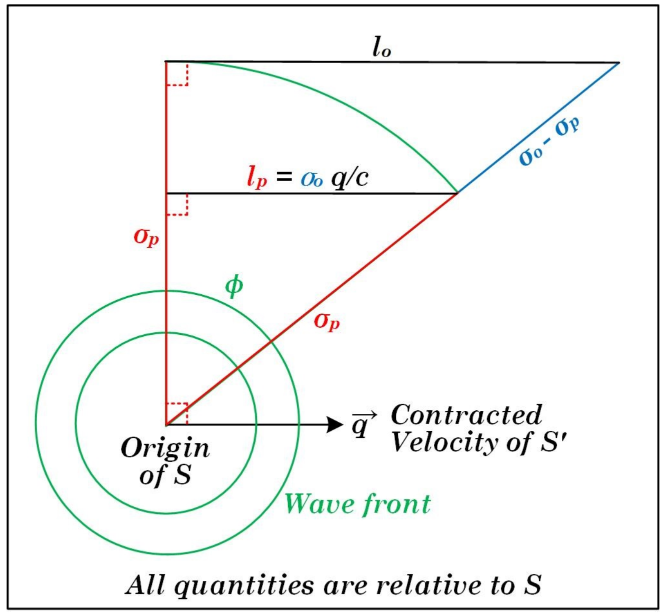

FIG. 7. Contracted velocity $\vec{q}$.

From Figure 7, we see that l_p is restricted geometrically to be less than σ_o. Therefore, **the velocity q of S' is less than c due to the way q is defined**. Compare this with the restricted velocity v defined in Equations 4 and 16, and the unrestricted velocity u defined in Equation 32.

If the length of the vertical leg of the small triangle in Figure 7 is denoted by L, then the right-triangle relation for the small triangle is

$$L^2 + l_p^2 = \sigma_p^2. \tag{44}$$

From Figure 7, we see that the vertical legs of the small and large triangle are in the same ratio as their hypotenuses:

$$\frac{L}{\sigma_p} = \frac{\sigma_p}{\sigma_o}. \tag{45}$$

Therefore

$$L = \frac{\sigma_p^2}{\sigma_o}. \tag{46}$$

From Equation 43, the fractional unit of moving length l_p in S is now given by

$$l_p = q\tau_o = \sigma_o\frac{q}{c}. \tag{47}$$

Substitution of Equations 46 and 47 into Equation 44 and solving for $\frac{q}{c}$ gives

$$\frac{q}{c} = \frac{\sigma_p}{\sigma_o}\sqrt{1 - \frac{\sigma_p^2}{\sigma_o^2}}. \tag{48}$$

The square of Equation 48 is

$$\frac{q^2}{c^2} = \frac{\sigma_p^2}{\sigma_o^2} - (\frac{\sigma_p^2}{\sigma_o^2})^2. \tag{49}$$

By viewing Equation 49 as a quadratic equation in $\frac{\sigma_p^2}{\sigma_o^2}$ and solving for $\frac{\sigma_p^2}{\sigma_o^2}$, we obtain the following expression for the obliquity factor $\gamma = \frac{\sigma_o}{\sigma_p}$ (Equation 8) in terms of the contracted velocity q:

$$\gamma = \frac{\sigma_o}{\sigma_p} = \frac{1}{\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{q^2}{c^2}}}}. \tag{50}$$

We see from the innermost radical in Equation 50 that q cannot exceed $\frac{c}{2}$.

From Equations 7 and 8, we see that Equation 48 relates q to v as follows:

$$v = \gamma q. \tag{51}$$

This is also seen by dividing Equation 16 by Equation 43, and comparing the result with Equation 8. When q attains its maximum value of $\frac{c}{2}$, $\gamma = \sqrt{2}$ (as seen from Equation 50), and $v = \frac{c}{\sqrt{2}}$ (as seen from Equation 51). The dependence of the contracted velocity q on the regular velocity v is illustrated in Figure 8.

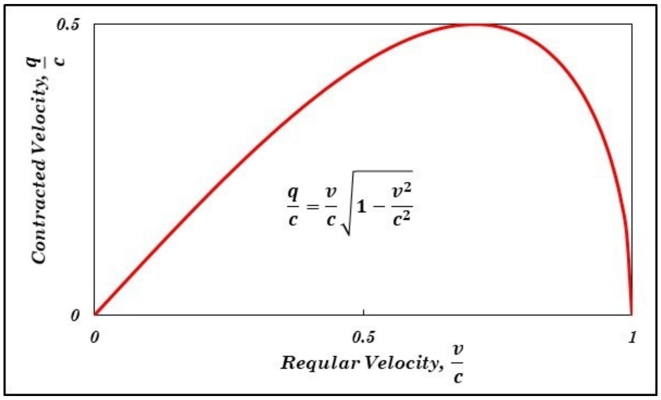

FIG. 8. Dependence of q on v

By substituting Equation 51 into Equation 37, we get the following expression for the rest velocity u in terms of the contracted velocity q:

$$u = \gamma^2 q. \tag{52}$$

By multiplying Equations 37 and 51, we see that v **is the geometric average of** u **and** q:

$$v = \sqrt{uq}. \tag{53}$$

This is also seen by multiplying Equations 4 and 16, and comparing the result with Equations 32 and 43.

X. NON-UNIFORM MOTION

The mirrors in Figures 1, 2, 5, and 7 are of course only a construct that allows us to count time ticks. We may dispose of the mirrors and let the the light wave from the flash of light propagate indefinitely outward. The perpendicular light ray segment in S (along the positive y-axis) now grows longer and longer, and the oblique light ray segment in S likewise grows longer and longer as time passes. The length of either segment is the radius of the expanding spherical wave front surface from the flash in S.

Expressed as a light-length, elapsed perpendicular time (rest time) in S is the length of the growing perpendicular segment as a multiple of the light-length unit σ_p of perpendicular time in S, and elapsed oblique time (moving time) in S is the length of the growing oblique segment as a multiple of the light-length unit σ_o of oblique time in S. Since $\sigma_o > \sigma_p$, and both growing segments have the same length in S (i.e., they are the radius of the same wave front sphere in S), oblique time (moving time) in S proceeds slower than perpendicular time (rest time) in S. This is of course the phenomenon of time dilation.

Let us now assume that the origin of S' is not in uniform motion relative to the origin of S, but rather moves in an arbitrary fashion (including acceleration and deceleration) along the x-axis. The growing oblique light ray segment in the xy-plane in S must then be modified to be a **light ray curve** in the xy-plane, and the curve's endpoint must move as follows: (1) the perpendicular projection of the endpoint onto the x-axis is the origin of S', and (2) the speed of the endpoint in the xy-plane is c. Figure 9 depicts this situation.

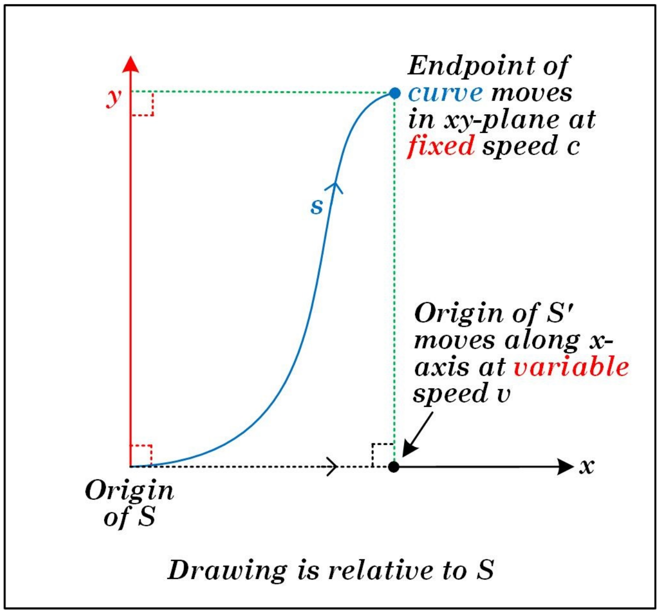

FIG. 9. Light ray curve in S.

Note that the light ray curve is a single-valued function of y, in the sense that for any particular value of y, there is one and only one value of x. This means **the light ray curve fully determines the motion of the origin of S' along the x-axis**. Note that since the endpoint of the light ray curve moves at a fixed speed c, a change in v corresponds to a bend of the light curve.

The y-axis is always perpendicular to the motion of the origin of S'. The length unit of the y axis is the rest time light-length unit σ_p. We may call this perpendicular axis the **light axis**. The light axis is a spatial axis that represents time as a light-length.

In Figure 9, the length of the light ray curve traced out by the endpoint is denoted by s, and the perpendicular projection of the curve onto the y-axis is denoted by y. Both s and y are lengths measured relative to S. In terms of rest time ticks in S, rest time elapsed in S is $\frac{s}{c}$, while moving time elapsed in S is $\frac{y}{c}$. Since y is always less than s, moving time in S proceeds slower than rest time in S, which is of course a manifestation of time dilation in S.

Equation 4 defines the uniform velocity v of the origin of S' along the x-axis. The corresponding **instantaneous velocity** is

$$v = c\frac{dx}{ds}. \tag{54}$$

For uniform motion, $\frac{dx}{ds} = \sin\phi$, and Equation 54 reduces to Equation 12.

The differential version of the right-triangle relation (Equation 6) is

$$(ds)^2 = (dx)^2 + (dy)^2. \tag{55}$$

Solving Equation 55 for $\frac{ds}{dy}$, we get

$$\frac{ds}{dy} = \sqrt{1 + (\frac{dx}{dy})^2}. \tag{56}$$

By the chain rule,

$$\frac{dx}{ds} = \frac{dx}{dy} \cdot \frac{dy}{ds}. \tag{57}$$

For uniform motion, Equation 57 is simply

$$\frac{l_o}{\sigma_o} = \frac{l_o}{\sigma_p} \cdot \frac{\sigma_p}{\sigma_o}, \tag{58}$$

which Equations 4, 8, and 32 show is the same as Equation 37. From Equations 54, 56, and 57, v can be expressed as

$$v = \frac{c\frac{dx}{dy}}{\sqrt{1 + (\frac{dx}{dy})^2}}. \tag{59}$$

For uniform motion, $\frac{dx}{dy} = \tan\phi$, and Equation 59 again reduces to Equation 12.

Equation 32 defines the uniform rest velocity u of the origin of S' along the x-axis. The corresponding **instantaneous rest velocity** is

$$u = c\frac{dx}{dy}. \tag{60}$$

Equation 60 shows that, for uniform motion, Equation 59 reduces to Equation 38.

Equation 8 defines the (uniform) obliquity factor γ. The corresponding **instantaneous obliquity factor** is

$$\gamma = \frac{ds}{dy}. \tag{61}$$

γ is the derivative of moving time light-length s with respect to rest time light-length y. For uniform motion, $\frac{ds}{dy} = \frac{1}{\cos\phi}$, and Equation 61 reduces to Equation 13. Equations 54, 60, and 61 confirm that, for uniform motion, Equation 57 reduces to Equation 37.

Let's divide Equation 61 by c and write it as a relation between the differentials $\frac{ds}{c}$ and $\frac{dy}{c}$:

$$\frac{ds}{c} = \gamma\frac{dy}{c}. \tag{62}$$

Equation 62 is just just the differential counterpart to the time dilation relation of Equation 11. We may call $\frac{ds}{c}$ the **oblique time (moving time) differential**, and $\frac{dy}{c}$ the **perpendicular time (rest time) differential**.

If we divide Equation 62 by the differential dx, and then use Equations 54 and 60, we get

$$\frac{1}{v} = \gamma\frac{1}{u}, \tag{63}$$

which of course is just Equation 37 for the case when v and u are instantaneous velocities. We may call dx the **fractional oblique length (fractional rest length) differential**.

XI. MOTION IN TWO DIMENSIONS

Up until this point, the motion of the origin of S' has only been along the x-axis of S. Let's add motion along the z-axis of S. In Figures 1, 2, 5, 7, and 9, the z axis is perpendicular to the plane of the figure, and the positive z-direction is towards the reader.

We assume that the origin of S' moves in an arbitrary fashion (including acceleration and deceleration) in the xz-plane. The light ray curve in the xy-plane of Figure 9 must then be modified to be a light ray curve in the xyz-space, and the curve's endpoint must move as follows: (1) the perpendicular projection of the endpoint onto the xz-plane is the origin of S', and (2) the speed of the endpoint in the xyz-space is c. Figure 10 depicts this situation.

In Figure 10, the length of the light ray curve traced out by the endpoint is denoted by s, and the light axis is again the y-axis. Note again that the light axis is a spatial axis that represents time as a light-length.

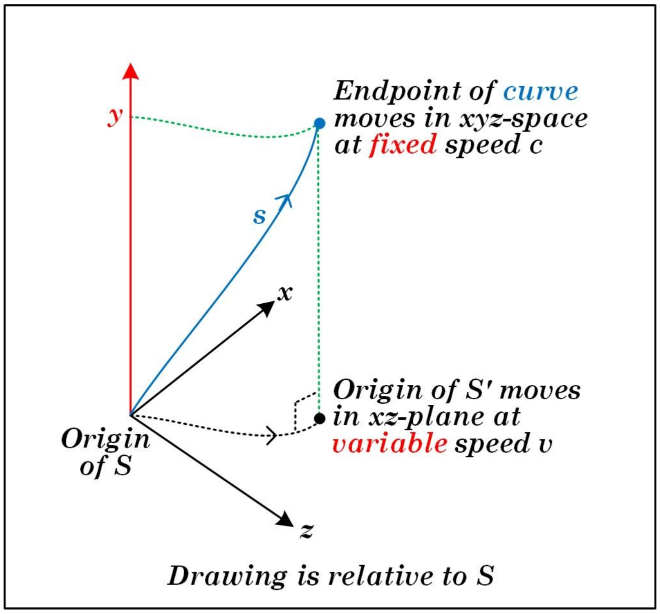

FIG. 10. Light ray curve for 2D motion in S.

The light ray curve is now a single-valued vector function of y, in the sense that for any particular value of y, there is one and only one value of the pair (x, z). This means **the light ray curve fully determines the motion of the origin of S' in the xz-plane**.

The **instantaneous velocity** v (Equation 54) now becomes a planar vector $\vec{v} = (v_x, v_z)$ in the xz-plane, with components

$$v_x = c\frac{dx}{ds},\ v_z = c\frac{dz}{ds}. \tag{64}$$

Note that ds is now a light curve segment in three dimensions, whereas in Equation 54 it is a light curve segment in two dimensions only.

The differential right-triangle relation (Equation 55) becomes

$$(ds)^2 = (dx)^2 + (dz)^2 + (dy)^2. \tag{65}$$

Solving Equation 65 for $\frac{ds}{dy}$, we get

$$\frac{ds}{dy} = \sqrt{1 + (\frac{dx}{dy})^2 + (\frac{dz}{dy})^2}. \tag{66}$$

By the chain rule,

$$\frac{dx}{ds} = \frac{dx}{dy} \cdot \frac{dy}{ds}, \tag{67}$$

$$\frac{dz}{ds} = \frac{dz}{dy} \cdot \frac{dy}{ds}. \tag{68}$$

So from Equations 64 and 66, the components of $\vec{v} = (v_x, v_z)$ can be expressed as

$$v_x = \frac{c\frac{dx}{dy}}{\sqrt{1 + (\frac{dx}{dy})^2 + (\frac{dz}{dy})^2}}, \tag{69}$$

$$v_z = \frac{c\frac{dz}{dy}}{\sqrt{1 + (\frac{dx}{dy})^2 + (\frac{dz}{dy})^2}}. \tag{70}$$

The **instantaneous rest velocity** u (Equation 32) similarly becomes a planar vector $\vec{u} = (u_x, u_z)$ in the xz-plane with components

$$u_x = c\frac{dx}{dy},\ u_z = c\frac{dz}{dy}. \tag{71}$$

The definition for the **instantaneous obliquity factor** γ for motion in the xz-plane is still

$$\gamma = \frac{ds}{dy}, \tag{72}$$

as in Equation 61, with the caveat that ds is now a light curve segment in three dimensions instead of two dimensions.

Equations 64, 71, and 72 show that Equations 67 and 68 can be written as the planar vector equation

$$\vec{v} = \frac{\vec{u}}{\gamma} = \frac{\vec{u}}{\sqrt{1 + \frac{|\vec{u}|^2}{c^2}}}, \tag{73}$$

which is the extension of Equations 37 and 38 to motion in two dimensions.

XII. MOTION IN THREE DIMENSIONS

We now assume that the origin of S' moves in an arbitrary fashion (including acceleration and deceleration) in the three-dimensional xyz-space. This requires us to employ a 4-dimensional space with four independent axes. We label the fourth axis as w. We take the y-axis to be a third spatial axis equivalent to the x and z axes, and take the w-axis be the light axis. Note again that the light axis is also a spatial axis, but "special" in that it represents time as a light-length.

The light ray curve in the xyz-space of Figure 10 now becomes a light ray curve in the $xyzw$-space, and the curve's endpoint moves as follows: (1) the projection parallel to the w-axis of the endpoint onto the xyz-space is the origin of S', and (2) the speed of the endpoint in the $xyzw$-space is c. Figure 11 depicts this situation.

In Figure 11, the length of the light ray curve traced out by the endpoint in the 4-dimensional $xyzw$-space is denoted by s. There are 6 planes total in the $xyzw$-space: xy, xz, yz, xw, yw, and zw. In order to visualize them in three dimensions, the planes cannot be perpendicular. Instead, we visualize them as being at a 60° angle to each other. Figure 11 does not depict the 6 planes, to avoid clutter.

Note that the light ray curve is again a single-valued vector-function of w, in the sense that for any particular value of w, there is one and only one value of the triple (x, y, z). This means **the light ray curve fully**

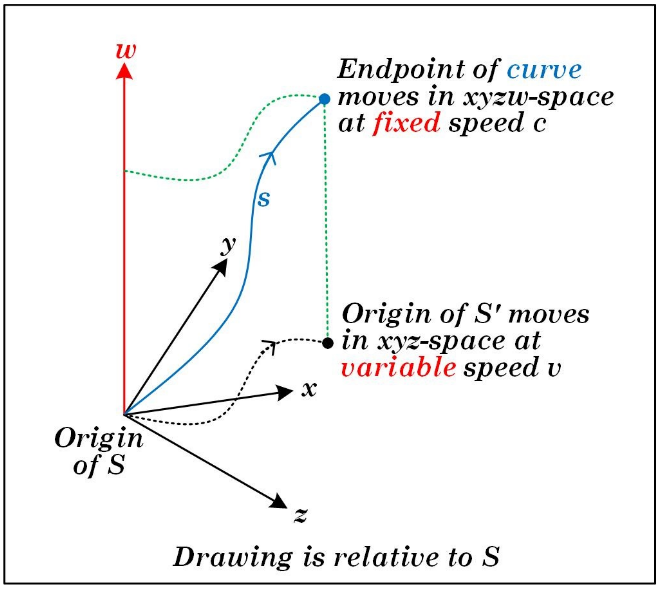

FIG. 11. Light ray curve for 3D motion in S.

determines the motion of the origin of S' in the xyz-space.

The **instantaneous velocity** v (Equation 64) now becomes a spatial vector $\vec{v} = (v_x, v_y, v_z)$ in the xyz-space, with components

$$v_x = c\frac{dx}{ds},\ v_y = c\frac{dy}{ds},\ v_z = c\frac{dz}{ds}. \tag{74}$$

Note that ds is now a light curve segment in four dimensions, whereas in Equation 64 it is a light curve segment in three dimensions only.

The differential right-triangle relation (Equation 65) becomes

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 + (dw)^2. \tag{75}$$

Solving Equation 75 for $\frac{ds}{dw}$, we get

$$\frac{ds}{dw} = \sqrt{1 + (\frac{dx}{dw})^2 + (\frac{dy}{dw})^2 + (\frac{dz}{dw})^2}. \tag{76}$$

By the chain rule,

$$\frac{dx}{ds} = \frac{dx}{dw} \cdot \frac{dw}{ds}, \tag{77}$$

$$\frac{dy}{ds} = \frac{dy}{dw} \cdot \frac{dw}{ds}, \tag{78}$$

$$\frac{dz}{ds} = \frac{dz}{dw} \cdot \frac{dw}{ds}. \tag{79}$$

So from Equations 74 and 76, the components of $\vec{v} = (v_x, v_y, v_z)$ can be expressed as

$$v_x = \frac{c\frac{dx}{dw}}{\sqrt{1 + (\frac{dx}{dw})^2 + (\frac{dy}{dw})^2 + (\frac{dz}{dw})^2}}, \tag{80}$$

$$v_y = \frac{c\frac{dy}{dw}}{\sqrt{1 + (\frac{dx}{dw})^2 + (\frac{dy}{dw})^2 + (\frac{dz}{dw})^2}}, \tag{81}$$

$$v_z = \frac{c\frac{dz}{dw}}{\sqrt{1 + (\frac{dx}{dw})^2 + (\frac{dy}{dw})^2 + (\frac{dz}{dw})^2}}. \tag{82}$$

The **instantaneous rest velocity** $\vec{u} = (u_x, u_z)$ (Equation 71) similarly becomes a spatial vector $\vec{u} = (u_x, u_y, u_z)$ in the xyz-space with components

$$u_x = c\frac{dx}{dw},\ u_y = c\frac{dy}{dw},\ u_z = c\frac{dz}{dw}. \tag{83}$$

The definition for the **instantaneous obliquity factor** γ for motion in the xyz-space becomes

$$\gamma = \frac{ds}{dw}, \tag{84}$$

where ds is a light curve segment in four (spatial) dimensions.

Equations 74, 83, and 84 show that Equations 77, 78, and 79 can be written as the spatial vector equation

$$\vec{v} = \frac{\vec{u}}{\gamma} = \frac{\vec{u}}{\sqrt{1 + \frac{|\vec{u}|^2}{c^2}}}, \tag{85}$$

which is the extension of Equation 73 to motion in three dimensions.

XIII. CONCLUSIONS

We have determined that the least restrictive condition for time dilation and length contraction in a particular reference frame is that the speed of light be finite in that particular frame. This condition is a subset of the postulate of light speed invariance.

We also considered the relative motion of two rigid rods, and concluded that a concept analogous to the Doppler shift for electromagnetic waves may be applied to the relative motion of extended objects. Such a "rest length shift" concept may be used to determine the relative velocity of extended objects in terms of such a "shift." To make this concrete, we furnished the explicit formula for relative velocity in terms of this "shift."

We also found that one may employ two alternative velocity definitions to simplify the Lorentz transformation formulas. Regular velocity is the geometric average of these two alternative velocities.

Finally, we noted that a particle's motion may be uniquely determined by a "light ray curve." We observed that differentiation of the particle's location with respect to the path length of this curve allowed us to determine the relationship between the particle's regular and "rest" velocity for non-uniform motion in one, two, and three dimensions, respectively.

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