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Precanonical Structure of the Schrödinger Wave Functional of a Quantum Scalar Field in Curved Space-Time

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Abstract: A relationship between the functional Schrödinger representation and the precanonical quantization of a nonlinear scalar field theory is extended to arbitrary curved space-times. The canonical functional derivative Schrödinger equation is derived from the manifestly covariant precanonical Schrödinger equation in a singular limiting case when the ultraviolet parameter \varkappa introduced by precanonical quantization is identified with the invariant delta-function at equal spatial points. In the same limiting case, the Schrödinger wave functional is expressed as the trace of the multidimensional product integral of Clifford-algebra-valued precanonical wave functions restricted to a certain field configuration. Thus the standard QFT in curved space-time in functional Schrödinger representation emerges from the precanonical formulation of quantum fields as a singular limiting case.

Keywords: quantum field theory in curved space-time; De Donder-Weyl Hamiltonian formalism; precanonical quantization; canonical quantization; functional Schrödinger representation

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1. Introduction

Since the theoretical discovery of the Hawking radiation of black holes, quantum field theory in curved space-time [1] often has been considered as an opportunity to study an interplay between gravitation, space-time and quantum theory in order to gain insights and intuitions into the quantum geometry of space-time and the quantum theory of gravity. The consideration of quantum fields on curved backgrounds also allows us to understand what concepts and mathematical structures are important in quantum field theory beyond the simplifying framework of the Poincaré-invariant Minkowski spacetime.

Recently we have put forward an approach to quantization of fields called precanonical quantization [2–5] which is based on mathematical structures of the De Donder-Weyl (DW) Hamiltonian theory known in the calculus of variations [6]. This Hamiltonian-like formulation does not require a space-time decomposition and treats all space-time variables on equal footing. In this sense it is an intermediate description of classical fields between the Lagrangian one and the canonical Hamiltonian one (hence the name "precanonical") which allows us to avoid the necessity of treating fields as infinite-dimensional Hamiltonian systems at least on the level of formulating the corresponding quantum theory of fields and hence circumvent technical difficulties the canonical quantization of fields brings in.

31 In DW Hamiltonian theory a proper generalization of Poisson brackets is defined on the dynamical
32 variables represented by differential forms rather than functionals. The brackets are defined by the
33 structure we called polysymplectic and they lead to the structure of Poisson-Gerstenhaber algebra
34 [5,7,8] (see also [9–14] for further generalizations). The latter generalizes the Poisson algebra in the
35 canonical Hamiltonian formalism, which underlines canonical quantization and hence underlines a
36 different procedure of quantization of fields which we call precanonical quantization.

37 The DW Hamiltonian formulation is related to a generalization of the Hamilton-Jacobi (HJ)
38 theory which is formulated in terms of partial derivative rather than functional derivative equations
39 [6]. Precanonical quantization clarifies the question as to which formulation of quantum theory
40 of fields reproduces the DWHJ equation in the classical limit [3,4]. We found that quantization
41 of a small Heisenberg-like subalgebra of the aforementioned Poisson-Gerstenhaber algebra leads
42 to a hypercomplex generalization of the formalism of quantum theory where both operators and
43 wave functions are Clifford-algebra-valued. The precanonical analogue of the Schrödinger equation
44 is formulated using the Dirac operator on the space-time which appears as a multidimensional
45 generalization of the time derivative in the left side of the standard Schrödinger equation [2–5].

46 One of the features of precanonical formulation of quantized fields is that it allows us to reproduce
47 the classical field equations in DW Hamiltonian form as the equations for expectation values of
48 operators defined by precanonical quantization and evolving according to the precanonical Schrödinger
49 equation [15,16]. By treating the space-time variables on equal footing it also leads to a formulation of
50 quantum theory of fields on a finite-dimensional space of field and space-time variables thus providing
51 a new framework for the quantum gauge theory [31,34] and the theory of quantum gravity [33,35]
52 which looks more promising both conceptually and from the point of view of a possibility of rigorous
53 mathematical treatment.

54 An important aspect of realization of the potential of precanonical quantization is understanding
55 of how it could be related to more familiar and already working concepts of standard QFT. In this
56 paper, we extend our previous results on the relationship between precanonical quantization and the
57 functional Schrödinger picture in QFT [27–32] to scalar field theory on arbitrary curved space-times.

58 We proceed as follows. In Section 2 we first remind the results of canonical quantization of
59 scalar field theory on curved space-time in the functional Schrödinger representation and precanonical
60 quantization of the same system, and then we discuss drastic differences between them. Section 2 also
61 serves to introduce the notations used throughout the paper. The connection between the functional
62 Schrödinger representation and the results of precanonical quantization in curved space-time is
63 established in Section 3 which consists of several subsections reflecting the multi-step nature of
64 the argument. Namely, we first outline a general idea which allows us to anticipate a connection
65 between the Schrödinger wave functional and precanonical wave function based on the respective
66 probabilistic interpretation of both objects. Second, in sub-section 3.1, we present a restriction of
67 precanonical Schrödinger equation to the section of the bundle of field coordinates over space-time,
68 which represents a field configuration the Schrödinger wave functional is a functional of. The restriction
69 of precanonical Schrödinger equation is formulated in terms of the total covariant derivative introduced
70 in sub-section 3.1.1 and it allows us to write in sub-section 3.2 the equation for the time evolution of
71 the wave functional composed from precanonical wave functions. To proceed with the derivation of
72 the Schrödinger equation for the wave functional from the restriction of the precanonical Schrödinger
73 equation to a field configuration, in sub-section 3.3 we evaluate the functional derivatives of the
74 functional composed from precanonical wave function with respect to the field variables. Then, in
75 Section 4, we analyze different terms in the equation presented in sub-section 3.2 and show how
76 they can be expressed in terms of the functional derivatives of the composed functional or cancelled
77 in a certain limiting case. The result of this section is the derivation of the functional derivative
78 Schrödinger equation from the restricted precanonical Schrödinger equation up to an additional term
79 which involves the commutator of the zeroth component of the spin connection matrix with the
80 precanonical wave function, see eq. (37). In Section 4 we consider static space-times with the vanishing

81 zeroth component of the spin connection and obtain the expression of the Schrödinger wave functional
 82 as the trace of the continual product or product integral of precanonical wave functions restricted
 83 to a field configuration. A more general case of non-static space-times with non-vanishing zeroth
 84 component of the spin connection is considered in Section 5 where we show that the extraneous term
 85 in (37) with the commutator of the zeroth component of spin connection with precanonical wave
 86 function disappears if the wave functional is expressed in terms of transformed precanonical wave
 87 functions with the transformation given by the time-ordered exponential of the zeroth component
 88 of spin connection. This observation allows us to extend the results from the static space-times to
 89 nonstatic ones. In Section 6 we present our conclusions and highlight the main steps of the derivation
 90 of the functional Schrödinger equation from precanonical Schrödinger equation and the expression
 91 of the Schrödinger wave functional as a product integral of precanonical wave functions or their
 92 transforms. We also discuss the physical meaning of the ultra-violet parameter \varkappa whose infinite value
 93 corresponds to the limiting case in which it is shown to be possible to derive the standard functional
 94 Schrödinger representation of QFT from precanonical quantization.

95 2. Quantum scalar field on a curved space-time: the canonical and precanonical descriptions

Let us recall that the conventional **canonical quantization** of scalar field theory in curved space-time can be formulated in the functional Schrödinger representation of QFT [24,25]. It leads [17–23]) to the description of the corresponding quantum field in terms of the Schrödinger wave functional $\Psi([\phi(\mathbf{x})], t)$ satisfying the Schrödinger equation

$$i\hbar\partial_t\Psi = \int d\mathbf{x} \sqrt{g} \left(\frac{\hbar^2}{2} \frac{g_{00}}{g} \frac{\delta^2}{\delta\phi(\mathbf{x})^2} - \frac{1}{2} g^{ij} \partial_i\phi(\mathbf{x})\partial_j\phi(\mathbf{x}) + V(\phi) \right) \Psi, \quad (1)$$

96 where the right hand side is the canonical Hamiltonian operator formulated in terms of functional
 97 derivative operators, $x^\mu = (t, \mathbf{x}) = (t, x^i)$ are space-time coordinates, $g_{\mu\nu}$ is the space-time metric
 98 tensor whose components depend on x^μ , $g = |\det(g_{\mu\nu})|$. In (1) one uses the space-time coordinates
 99 adapted to the space-like foliation such as the induced metric on the space-like leaves of the foliation is
 100 g_{ij} , the lapse $N = \sqrt{g_{00}}$ and the shift functions $N_i = g_{0i} = 0$.

The **precanonical quantization** of a scalar field $\phi(x)$ on a curved space-time background given by the metric tensor $g_{\mu\nu}(x)$ (cf. [15,16]) gives rise to the description in terms of a wave function $\Psi(\phi, x^\mu)$ on the finite-dimensional bundle with the coordinates (ϕ, x^μ) which takes values in the complexified space-time Clifford algebra, i.e.

$$\Psi = \psi + \psi_\mu \gamma^\mu + \frac{1}{2!} \psi_{\mu_1\mu_2} \gamma^{\mu_1\mu_2} + \dots + \frac{1}{n!} \psi_{\mu_1\dots\mu_n} \gamma^{\mu_1\dots\mu_n},$$

and satisfies the partial derivative precanonical Schrödinger equation (pSE)

$$i\hbar\gamma^\mu(x)\nabla_\mu\Psi = \left(-\frac{1}{2}\hbar^2\varkappa\frac{\partial^2}{\partial\phi^2} + \frac{1}{\varkappa}V(\phi) \right) \Psi =: \frac{1}{\varkappa}\widehat{H}\Psi, \quad (2)$$

where $\gamma^\mu(x)$ are the curved space-time Dirac matrices such that

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x), \quad (3)$$

$\gamma^{\mu_1\dots\mu_p}$ are the antisymmetrized products of p Dirac matrices,

$$\nabla_\mu := \partial_\mu + \omega_\mu(x) \quad (4)$$

is the covariant derivative with the spin-connection matrices $\omega_\mu(x) = \frac{1}{4}\omega_{\mu AB}(x)\underline{\gamma}^{AB}$ (see e.g. [26]) acting on Clifford-algebra-valued wave functions by the commutator product [32], and $\underline{\gamma}^A$ are the constant Dirac matrices which factorize the Minkowski metric η^{AB} of the tangent space¹:

$$\underline{\gamma}^A \underline{\gamma}^B + \underline{\gamma}^B \underline{\gamma}^A = 2\eta^{AB}. \quad (5)$$

101 The operator \hat{H} in (2) is the De Donder-Weyl (DW) Hamiltonian operator constructed according
 102 to the procedure of precanonical quantization [3,4,15,16]. In the expression of \hat{H} there appears an
 103 ultraviolet parameter \varkappa of the dimension of the inverse spatial volume. This parameter typically
 104 appears in the representations of precanonical quantum operators [2–4,15]. For the scalar fields on
 105 curved background the DW Hamiltonian operator \hat{H} coincides with its counterpart in flat space-time (cf.
 106 [2–4,15]). Correspondingly, the curved space-time manifests itself only through the curved space-time
 107 Dirac matrices (3) and the spin-connection in the left-hand side of (2).

108 As we have seen, the description of quantum fields obtained from precanonical quantization is
 109 very different from a familiar description of quantum fields derived from the canonical quantization.
 110 In particular, while in the description using the functional Schrödinger picture the role of space
 111 variables \mathbf{x} is different from the role the time variable t , the precanonical description is entirely
 112 space-time symmetric, manifestly covariant and independent of the assumption of global hyperbolicity
 113 of space-time. One can also wonder how the description in terms of precanonical wave function
 114 on a finite-dimensional space and the corresponding partial derivative precanonical Schrödinger
 115 equation can match the description in terms of functionals on an infinite-dimensional space of field
 116 configurations at a fixed time and the corresponding functional derivative Schrödinger equation, or
 117 how the multiparticle states and multi-point correlation functions of standard QFT could be related to
 118 the natural objects within the precanonical description such as the Green function of the precanonical
 119 Schrödinger equation (2).

120 However, one can reduce the perceived gap between those two descriptions by noticing that
 121 already on the classical level the solutions of field equations can be equally well treated using both
 122 the language of partial derivative equation on a finite dimensional space (in the Lagrangian, DW
 123 Hamiltonian and DWHJ descriptions) and the language of functional derivative equations (in the
 124 canonical Hamiltonian and Hamilton-Jacobi description). Moreover, one can derive the canonical
 125 Hamiltonian and HJ equations from the DW Hamiltonian and DWHJ equations, respectively (see
 126 e.g. [27,30]). In the next section, we will show how those relationships between the canonical and
 127 precanonical are extended to the quantum level in curved space-times.

128 3. Relating the precanonical wave function and the Schrödinger wave functional

129 Our preceding work has established a relationship between the functional Schrödinger
 130 representation and precanonical quantization of scalar and Yang-Mills in flat space-time [29–31]. The
 131 familiar QFT in functional Schrödinger representation was derived from the precanonical quantization
 132 as the limiting case when the combination $\gamma_0\varkappa$ is replaced by $\delta(\mathbf{0})$, a regularized value of Dirac delta
 133 function $\delta(\mathbf{x} - \mathbf{x}')$ at coinciding spatial points, which can be understood as the cutoff of the momentum
 134 space volume introduced by a regularization. Here we intend to extend this relationship to curved
 135 space-time using the example of a quantum scalar field.

The Schrödinger wave functional $\Psi([\phi(\mathbf{x})], t)$ is interpreted as the probability amplitude of finding a field configuration $\phi(\mathbf{x})$ at some moment of time t . The precanonical wave function

¹ We chose the signature $+ - - -$ Note that in this paper we depart from our notation in [16] where the flat Dirac matrices were denoted $\tilde{\gamma}^A$. This notation can be confusing when the Dirac conjugate matrix has to be denoted as $\bar{\gamma}^A$. As for the rest, throughout this paper we mostly follow the notation and conventions used in [15,16,29,30]. In particular, the plane capital Greek letters like Ψ and Φ denote wave functions on a finite dimensional space of ϕ and x^μ and the boldface capital Greek letters like $\mathbf{\Psi}$ and $\mathbf{\Phi}$ denote functionals of field configurations $\phi(\mathbf{x})$.

$\Phi(\phi, x)$ is the probability amplitude of observing the field value ϕ at the space-time point x . Then the time-dependent complex functional probability amplitude $\Psi([\phi(\mathbf{x})], t)$ can be expected to be a composition of space-time dependent Clifford-valued probability amplitudes given by the precanonical wave function $\Psi(\phi, x)$. It means that the Schrödinger wave functional $\Psi([\phi(\mathbf{x})], t)$ is a functional of precanonical wave functions $\Psi(\phi, x)$ restricted to a specific field configuration which is represented by a section Σ in the total space of the bundle with the coordinates (ϕ, x) , which is defined by the equation $\Sigma : \phi = \phi(\mathbf{x})$ at time t . Thus by denoting the restriction of precanonical wave function $\Psi(\phi, x)$ to Σ as

$$\Psi_{\Sigma}(\mathbf{x}, t) := \Psi(\phi = \phi(\mathbf{x}), \mathbf{x}, t)$$

we assume that

$$\Psi([\phi(\mathbf{x})], t) = \Psi([\Psi_{\Sigma}(\mathbf{x}, t), \phi(\mathbf{x})]), \quad (6)$$

so that the time dependence of the wave functional Ψ is totally controlled by the time dependence of precanonical wave function restricted to Σ . Then the chain rule differentiation yields the time derivative of Ψ

$$i\partial_t \Psi = \text{Tr} \int dx \left\{ \frac{\delta \Psi}{\delta \Psi_{\Sigma}^T(\mathbf{x}, t)} i\partial_t \Psi_{\Sigma}(\mathbf{x}, t) \right\}, \quad (7)$$

136 where Ψ^T denotes the transpose of the matrix Ψ . In the following we will be avoiding unnecessarily
137 cumbersome notation by denoting $\Psi_{\Sigma}(\mathbf{x}, t)$ also as $\Psi_{\Sigma}(\mathbf{x})$ or even Ψ_{Σ} .

138 3.1. Restriction of precanonical Schrödinger equation to Σ

The time derivative of Ψ_{Σ} is determined by the restriction of pSE (2) rewritten in space+time split form to Σ :

$$i\partial_t \Psi_{\Sigma} = -i\gamma_0 \gamma^i \left(\frac{d}{dx^i} - \partial_i \phi(\mathbf{x}) \frac{\partial}{\partial \phi} \right) \Psi_{\Sigma} - i\gamma_0 \gamma^i [\omega_i, \Psi_{\Sigma}] - i[\omega_0, \Psi_{\Sigma}] + \frac{\gamma_0}{\varkappa} \hat{H}_{\Sigma} \Psi_{\Sigma}, \quad (8)$$

where $\frac{d}{dx^i}$ is the total derivative along Σ ,

$$\frac{d}{dx^i} := \partial_i + \partial_i \phi(\mathbf{x}) \frac{\partial}{\partial \phi} + \partial_i \phi_{,k}(\mathbf{x}) \frac{\partial}{\partial \phi_{,k}} + \dots \quad (9)$$

In (9) $\phi_{,k}$ denote the fiber coordinates of the first-jet bundle of the bundle of field variables ϕ over space-time (cf. [40,41]) and \hat{H}_{Σ} is the restriction of the DW Hamiltonian operator \hat{H} to Σ . Since \hat{H} contains no space-time derivatives, $\hat{H}_{\Sigma} = \hat{H}$ and

$$\frac{1}{\varkappa} \hat{H} = -\frac{\varkappa}{2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\varkappa} V(\phi). \quad (10)$$

139 3.1.1. Total covariant derivative

Let us introduce the notion of the total covariant derivative acting on Clifford-algebra-valued tensors, particularly on those restricted to Σ . The derivative will be called "total" in the sense that (i) when acting on a Clifford-valued tensor function $T_{\nu_1 \nu_2 \dots}^{\mu_1 \mu_2 \dots}$ it includes both the spin-connection matrix ω_{μ} and the Christoffel symbols $\Gamma_{\beta\gamma}^{\alpha}$ (c.f. [36]) and (ii) when a tensor quantity with the components

depending both on x and ϕ is restricted to Σ , its derivative with respect to x -s is understood in the sense of the total derivative (9):

$$\begin{aligned} \nabla_{\alpha}^{\text{tot}} T_{\nu_1 \nu_2 \dots}^{\mu_1 \mu_2 \dots} &:= \frac{d}{dx^{\alpha}} T_{\nu_1 \nu_2 \dots}^{\mu_1 \mu_2 \dots} + [\omega_{\alpha}, T_{\nu_1 \nu_2 \dots}^{\mu_1 \mu_2 \dots}] \\ &+ \Gamma_{\alpha\beta}^{\mu_1} T_{\nu_1 \nu_2 \dots}^{\beta \mu_2 \dots} + \Gamma_{\alpha\beta}^{\mu_2} T_{\nu_1 \nu_2 \dots}^{\mu_1 \beta \dots} + \dots \\ &- \Gamma_{\alpha\nu_1}^{\beta} T_{\beta \nu_2 \dots}^{\mu_1 \mu_2 \dots} - \Gamma_{\alpha\nu_2}^{\beta} T_{\nu_1 \beta \dots}^{\mu_1 \mu_2 \dots} - \dots \end{aligned} \quad (11)$$

The commutator in the second term guarantees that the total covariant derivative of the Clifford product of two Clifford-valued tensor quantities fulfills the Leibniz rule. The Christoffel symbols appear in the covariant derivative of non-scalar Clifford quantities, e.g. in the condition of metric compatibility

$$\nabla_{\alpha}^{\text{tot}} \gamma^{\mu} = 0, \quad (12)$$

140 where only the first partial derivative term in (9) is non-vanishing when acting on x -dependent
141 γ -matrices.

Now, in terms of the total covariant derivative ∇^{tot} acting on Ψ_{Σ} eq. (8) takes the form

$$i\partial_t \Psi_{\Sigma} = -i\gamma_0 \gamma^i \nabla_i^{\text{tot}} \Psi_{\Sigma} - i[\omega_0, \Psi_{\Sigma}] + i\gamma_0 \gamma^i \partial_i \phi(\mathbf{x}) \partial_{\phi} \Psi_{\Sigma} + \frac{1}{\varkappa} \gamma_0 \widehat{H}_{\Sigma} \Psi_{\Sigma}. \quad (13)$$

142 3.2. Time evolution of the Schrödinger wave functional from pSE

From (7), (8) and (10) the equation of the time evolution of the wave functional (6) constructed from precanonical wave functions takes the form

$$\begin{aligned} i\partial_t \Psi = \text{Tr} \int d\mathbf{x} \left\{ \frac{\delta \Psi}{\delta \Psi_{\Sigma}^T(\mathbf{x}, t)} \left[\underbrace{-i\gamma_0 \gamma^i \frac{d}{dx^i} \Psi_{\Sigma}(\mathbf{x})}_I + \underbrace{i\gamma_0 \gamma^i \partial_i \phi(\mathbf{x}) \partial_{\phi} \Psi_{\Sigma}(\mathbf{x})}_{II} \right. \right. \\ \left. \left. - \underbrace{\frac{i}{4} \gamma_0 \gamma^i [\omega_i, \Psi_{\Sigma}(\mathbf{x})]}_{IIIa} - \underbrace{\frac{i}{4} [\omega_0, \Psi_{\Sigma}]}_{IIIb} - \underbrace{\frac{\varkappa}{2} \gamma_0 \partial_{\phi\phi} \Psi_{\Sigma}(\mathbf{x})}_{IV} + \underbrace{\frac{1}{\varkappa} \gamma_0 V(\phi(\mathbf{x})) \Psi_{\Sigma}(\mathbf{x})}_{V} \right] \right\}. \end{aligned} \quad (14)$$

143 In order to derive from this equation the functional Schrödinger equation (1) we need to try
144 to express the terms in the right hand side of (14) in terms of the functional derivatives of the composite
145 functional Ψ in (6) with respect to $\phi(\mathbf{x})$. Those are calculated in the following section.

146 3.3. Functional derivatives of Ψ

By using the chain rule for the functional differentiation and introducing the notations

$$\Phi(\mathbf{x}) := \frac{\delta \Psi}{\delta \Psi_{\Sigma}^T(\mathbf{x})} \quad (15)$$

and

$$\partial_{\phi} \Psi_{\Sigma}(\mathbf{x}) := (\partial \Psi / \partial \phi)|_{\Sigma}(\mathbf{x}), \quad \partial_{\phi\phi} \Psi_{\Sigma}(\mathbf{x}) := (\partial^2 \Psi / \partial \phi^2)|_{\Sigma}(\mathbf{x}), \quad (16)$$

we obtain

$$\frac{\delta\Psi}{\delta\phi(\mathbf{x})} = \text{Tr} \left\{ \Phi(\mathbf{x}) \partial_\phi \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta}\Psi}{\bar{\delta}\phi(\mathbf{x})}, \quad (17)$$

$$\begin{aligned} \frac{\delta^2\Psi}{\delta\phi(\mathbf{x})^2} &= \text{Tr} \left\{ \delta(\mathbf{0}) \Phi(\mathbf{x}) \partial_{\phi\phi} \Psi_\Sigma(\mathbf{x}) + 2 \frac{\bar{\delta}\Phi(\mathbf{x})}{\bar{\delta}\phi(\mathbf{x})} \partial_\phi \Psi_\Sigma(\mathbf{x}) \right\} \\ &+ \text{Tr} \text{Tr} \left\{ \frac{\delta^2\Psi}{\delta\Psi_\Sigma^T(\mathbf{x}) \otimes \delta\Psi_\Sigma^T(\mathbf{x})} \partial_\phi \Psi_\Sigma(\mathbf{x}) \otimes \partial_\phi \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta}^2\Psi}{\bar{\delta}\phi(\mathbf{x})^2}. \end{aligned} \quad (18)$$

147 where $\bar{\delta}$ denotes the *partial* functional derivative with respect to $\phi(\mathbf{x})$, as opposite to the *total* functional
148 derivative δ , and $\delta(\mathbf{0})$ is a regularized value of $\delta\Psi_\Sigma(\mathbf{x})/\delta\Psi_\Sigma^T(\mathbf{x}')$ at $\mathbf{x} = \mathbf{x}'$ which can be defined using a
149 point splitting or lattice regularization to make sense of $(n-1)$ -dimensional delta function $\delta(\mathbf{x} - \mathbf{x}')$ at
150 equal points. This is the simplest regularization one may use to make sense of the second functional
151 derivative at equal points which appears in the functional derivative Schrödinger equation (1).

152 3.4. The correspondence between terms $I - V$ in Eq. (14) and the canonical Hamiltonian operator in (1)

153 3.4.1. The potential term V

Our starting observation will be that the term V in (14) has to reproduce the last term in the functional derivative Schrödinger equation (1). This means that there exists a mapping \mapsto such that

$$\int d\mathbf{x} \text{Tr} \left\{ \Phi(\mathbf{x}) \frac{1}{\varkappa} \gamma_0 V(\phi(\mathbf{x})) \Psi_\Sigma(\mathbf{x}) \right\} \mapsto \int d\mathbf{x} \sqrt{g} V(\phi(\mathbf{x})) \Psi. \quad (19)$$

The existence of the map in (19) implies that the following relation should be fulfilled at any spatial point \mathbf{x} :

$$\text{Tr} \left\{ \Phi(\mathbf{x}) \frac{1}{\varkappa} \gamma_0 \Psi_\Sigma(\mathbf{x}) \right\} \mapsto \sqrt{g} \Psi. \quad (20)$$

Then the functional differentiation of both sides of (20) with respect to $\Psi_\Sigma^T(\mathbf{x})$ yields

$$\text{Tr} \left\{ \frac{\delta^2\Psi}{\delta\Psi_\Sigma^T(\mathbf{x}) \otimes \delta\Psi_\Sigma^T(\mathbf{x})} \frac{1}{\varkappa} \gamma_0 \Psi_\Sigma(\mathbf{x}) \right\} + \Phi(\mathbf{x}) \frac{1}{\varkappa} \gamma_0 \delta(\mathbf{0}) \mapsto \sqrt{g} \Phi(\mathbf{x}), \quad (21)$$

where again, $\delta(\mathbf{0}) = \delta\Psi_\Sigma(\mathbf{x})/\delta\Psi_\Sigma^T(\mathbf{x})$. This type of relation is possible if

$$\frac{\delta^2\Psi}{\delta\Psi_\Sigma^T(\mathbf{x}) \otimes \delta\Psi_\Sigma^T(\mathbf{x})} = 0 \quad (22)$$

and

$$\frac{1}{\varkappa} \gamma_0(x) \delta(\mathbf{0}) - \sqrt{g}(x) \mapsto 0. \quad (23)$$

The latter **relation** can be understood as the condition

$$\gamma^0 \sqrt{g} \varkappa \mapsto \delta(\mathbf{0}). \quad (24)$$

By taking into account that $\sqrt{g} = \sqrt{g_{00}h}$, where $h := |\det(g_{ij})|$, and $\gamma^0 \sqrt{g_{00}} = \underline{\gamma}_0$ is the time-like tangent Minkowski space Dirac matrix, eq. (24) can be rewritten as

$$\underline{\gamma}_0 \varkappa \mapsto \delta(\mathbf{0})/\sqrt{h} = \delta^{\text{inv}}(\mathbf{0}), \quad (25)$$

154 where $\delta^{\text{inv}}(\mathbf{x})$ is the invariant $(n-1)$ -dimensional delta function defined by the property
 155 $\int d\mathbf{x} \sqrt{h(\mathbf{x})} \delta^{\text{inv}}(\mathbf{x}) = 1$. This formula generalizes to curved space-times the limiting map $\gamma_0 \varkappa \mapsto \delta(\mathbf{0})$
 156 already found in flat space-time [29,30], with the $(n-1)$ -dimensional delta function replaced by the
 157 invariant one.

158 3.4.2. The second variational derivative term

Our next observation is that the term IV in (14) is able to reproduce the first term in the right-hand side of (18) in the limiting case (24)

$$IV : -\frac{\varkappa}{2} \gamma_0 \partial_{\phi\phi} \Psi_{\Sigma} \mapsto -\frac{1}{\sqrt{g}} g_{00} \delta(\mathbf{0}) \partial_{\phi\phi} \Psi_{\Sigma}. \quad (26)$$

A comparison with (18) shows that the term IV in (14) leads to the following expression in functional derivatives of Ψ :

$$IV : \text{Tr} \left\{ \frac{1}{2} \Phi(\mathbf{x}) \varkappa \gamma_0 \partial_{\phi\phi} \Psi_{\Sigma}(\mathbf{x}) \right\} \mapsto \frac{1}{2} \frac{g_{00}}{\sqrt{g}} \left(\frac{\delta^2 \Psi}{\delta \phi(\mathbf{x})^2} - 2 \text{Tr} \left\{ \frac{\bar{\delta} \Phi(\mathbf{x})}{\bar{\delta} \phi(\mathbf{x})} \partial_{\phi} \Psi_{\Sigma}(\mathbf{x}) \right\} - \frac{\bar{\delta}^2 \Psi}{\bar{\delta} \phi(\mathbf{x})^2} \right). \quad (27)$$

159 The first term in the right-hand side of (27) correctly reproduces the first term in the functional
 160 derivative Schrödinger equation (1). However, the second and the third term need further investigation.

161 3.4.3. The non-ultralocality term and the wave functional Ψ in terms of precanonical Ψ_{Σ}

Since the right hand side of (14) is expected to lead to a functional derivative operator acting on the wave functional Ψ , as in the right hand side of the functional Schrödinger equation (1), the second term in (27) with $\partial_{\phi} \Psi_{\Sigma}$ has to be cancelled by the term II in (14) which also contains $\partial_{\phi} \Psi_{\Sigma}$. Therefore, it is required that

$$II + 2\text{nd term of (27)} : \mathbf{i} \Phi(\mathbf{x}) \gamma_0 \gamma^i \partial_i \phi(\mathbf{x}) + \frac{g_{00}}{\sqrt{g}} \frac{\bar{\delta} \Phi(\mathbf{x})}{\bar{\delta} \phi(\mathbf{x})} \mapsto 0, \quad (28)$$

where the sign \mapsto stresses the fact that it is sufficient that the left hand side vanishes under the condition (24) rather than as an equality. In fact, by functionally differentiating both sides of (28) with respect to $\phi(\mathbf{x}')$ we can see that (28) with \mapsto replaced by the equality is not an integrable equation in functional derivatives. Nevertheless, by bearing in mind that (28) has to be valid only under the condition (24), the solution for $\Phi(\mathbf{x})$ can be written in the form

$$\Phi(\mathbf{x}) = \Xi([\Psi_{\Sigma}]; \check{\mathbf{x}}) e^{-i\phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x}) / \varkappa}, \quad (29)$$

where the "integration constant" $\Xi([\Psi_{\Sigma}]; \check{\mathbf{x}})$ is a functional of $\Psi_{\Sigma}(\mathbf{x}')$ on the punctured space with the removed point \mathbf{x} such that that $\mathbf{x}' \neq \mathbf{x}$. By construction, this functional satisfies the identity

$$\frac{\bar{\delta} \Xi([\Psi_{\Sigma}]; \check{\mathbf{x}})}{\bar{\delta} \phi(\mathbf{x})} \equiv 0.$$

Indeed, by differentiating (29) with respect to $\phi(\mathbf{x})$, replacing \varkappa according to the limiting map (24), and taking into account that $\gamma_0(\mathbf{x}) \gamma_0(\mathbf{x}) =: g_{00}(\mathbf{x})$ and $\partial_i \delta(\mathbf{0}) = 0$ (that restricts the admissible class of regularizations of delta-function $\delta(\mathbf{x})$) we conclude that (29) solves (28) under the condition (24). Note also that (29) by construction fulfills

$$\frac{\delta \Phi(\mathbf{x})}{\delta \Psi_{\Sigma}^T(\mathbf{x})} = \frac{\delta^2 \Psi}{\delta \Psi_{\Sigma}^T(\mathbf{x}) \otimes \delta \Psi_{\Sigma}^T(\mathbf{x})} \equiv 0, \quad (30)$$

which is consistent with (22). Thus the required cancellation of the terms with $\partial_\phi \Psi_\Sigma(\mathbf{x})$ (under the condition (24)) fixes the form of the functional $\Phi(\mathbf{x})$ introduced in (15). This allows us to express the wave functional Ψ in the form

$$\Psi \sim \text{Tr} \left\{ \Xi([\Psi_\Sigma]; \ddot{\mathbf{x}}) e^{-i\phi(\mathbf{x})\gamma^i \partial_i \phi(\mathbf{x})/\varkappa} \frac{\gamma_0}{\sqrt{g}\varkappa} \Psi_\Sigma(\mathbf{x}) \right\} \Big|_{\varkappa \rightarrow \gamma_0 \delta(\mathbf{0})/\sqrt{g}}, \quad (31)$$

valid at any point \mathbf{x} . Here the equality up to a normalization factor which will depend on \varkappa and \sqrt{g} is denoted as \sim . The notation $\{\dots\} \Big|_{\varkappa \rightarrow \gamma_0 \delta(\mathbf{0})/\sqrt{g}}$ indicates that every appearance of \varkappa in the expression inside braces is replaced by $\gamma_0 \delta(\mathbf{0})/\sqrt{g}$ as prescribed by the limiting map (24).

Using (31) we can now evaluate the last term in (27) in the limit (24):

$$\text{3-rd term of (27): } \frac{1}{2} \frac{g_{00}}{\sqrt{g}} \frac{\delta^2 \Psi}{\delta \phi(\mathbf{x})^2} \mapsto -\frac{1}{2} \sqrt{g} g^{ij} \partial_i \phi(\mathbf{x}) \partial_j \phi(\mathbf{x}) \Psi. \quad (32)$$

The right hand side of (32) correctly reproduces the second term in the functional derivative Schrödinger equation (1), thus correctly accounting for the inherent non-ultralocality of quantum relativistic scalar field theory (cf. [42]) in curved space-time.

Thus, all terms in the functional derivative Schrödinger equation (1) are now derived from pSE restricted to Σ , eq. (8). However, there are still unaccounted for terms *I*, *IIIa* and *IIIb* in (14)

$$I + IIIa + IIIb : \quad -i \int d\mathbf{x} \text{Tr} \left\{ \Phi(\mathbf{x}) \gamma_0 \gamma^i \nabla_i^{\text{tot}} \Psi_\Sigma + \Phi(\mathbf{x}) [\omega_0, \Psi_\Sigma] \right\}. \quad (33)$$

In flat space-time [29–31], those terms are reduced to the term *I* with with the total derivative $d\Psi_\Sigma(\mathbf{x})/dx^i$ which does not contribute to the equation for the functional Ψ if $\Psi_\Sigma(\mathbf{x})$ vanishes at the spatial infinity. Let us see if or how this property extends to curved space-times.

3.4.4. The vanishing contribution from the terms *I* and *IIIa*

At first we consider the first term in (33). Using the covariant Stokes theorem we obtain

$$\begin{aligned} I + IIIa : \quad & -i \int d\mathbf{x} \text{Tr} \left\{ \Phi(\mathbf{x}) \gamma_0 \gamma^i \nabla_i^{\text{tot}} \Psi_\Sigma \right\} = -i \int d\mathbf{x} \sqrt{h} \left(\text{Tr} \left\{ \frac{1}{\sqrt{h}} \Phi(\mathbf{x}) \gamma_0 \gamma^i \nabla_i^{\text{tot}} \Psi_\Sigma \right\} \right) \\ & = -i \int d\mathbf{x} \sqrt{h} \text{Tr} \left\{ \nabla_i^{\text{tot}} \left(\frac{1}{\sqrt{h}} \Phi(\mathbf{x}) \gamma_0 \gamma^i \Psi_\Sigma \right) \right\} + i \int d\mathbf{x} \left(\sqrt{h} \text{Tr} \left\{ \nabla_i^{\text{tot}} \left(\frac{1}{\sqrt{h}} \Phi(\mathbf{x}) \gamma_0 \gamma^i \right) \Psi_\Sigma \right\} \right) \\ & = -i \oint_{\partial\Sigma} dx_i \text{Tr} \left\{ \Phi \gamma_0 \gamma^i \Psi_\Sigma \right\} + i \int d\mathbf{x} \text{Tr} \left\{ \Phi (\nabla_i^{\text{tot}} (\gamma_0 \gamma^i)) \Psi_\Sigma \right\} \\ & \quad + i \int d\mathbf{x} \left(\frac{-\nabla_i \sqrt{h}}{\sqrt{h}} \text{Tr} \left\{ \Phi \gamma_0 \gamma^i \Psi_\Sigma \right\} + \text{Tr} \left\{ (\nabla_i^{\text{tot}} \Phi(\mathbf{x})) \gamma_0 \gamma^i \Psi_\Sigma \right\} \right) \end{aligned} \quad (34)$$

where $dx_i = d^{n-2}\mathbf{x} |_{\partial\Sigma} n_i(\mathbf{x})$ is the measure of $(n-2)$ -dimensional integration over the boundary $\partial\Sigma$ with the normal vector $n_i(\mathbf{x})$ tangent to Σ . In the right hand side of (34),

- (i) the first boundary term is the result of the covariant Stokes theorem and it vanishes if Ψ_Σ vanishes on the boundary $\partial\Sigma$;
- (ii) the following three terms follow from the Leibniz rule for the total covariant derivative ∇_i^{tot} with respect to the Clifford products of tensor Clifford-algebra-valued functions;
- (iii) in the second term, $\nabla_i^{\text{tot}} (\gamma_0 \gamma^i) = 0$ due to the covariant constancy of Dirac matrices (12);
- (iv) in the third term, the metric compatibility yields $\nabla_i \sqrt{h} = 0$;
- (v) in the fourth term, the explicit formula for $\Phi(\mathbf{x})$ in (29) yields

$$\nabla_i^{\text{tot}} \Phi(\mathbf{x}) = \frac{-i}{\varkappa} \Phi(\mathbf{x}) \left(\partial_i \phi \gamma^l \partial_l \phi + \phi \gamma^l \partial_{il} \phi + \phi (\nabla_i^{\text{tot}} \gamma^l) \partial_l \phi \right). \quad (35)$$

By noticing that the last term in (35) vanishes due to (12) and substituting (35) into the last term in (34), using the covariant Stokes theorem and the assumption that the field configurations $\phi(\mathbf{x})$ vanish at the spatial infinity, we obtain

$$\begin{aligned} \int d\mathbf{x} \operatorname{Tr} \left\{ \Phi(\mathbf{x}) \frac{1}{\varkappa} \gamma_0 \Psi_\Sigma(\mathbf{x}) (g^{il} \partial_i \phi \partial_l \phi + \phi g^{il} \partial_{il} \phi) \right\} &= \Psi \int d\mathbf{x} \sqrt{g} (g^{il} \partial_i \phi \partial_l \phi + \phi g^{il} \partial_{il} \phi) \\ &= -\Psi \int d\mathbf{x} \sqrt{h} \nabla_i (\sqrt{g_{00}} g^{il}) \frac{1}{2} \partial_l \phi^2 = 0, \end{aligned} \quad (36)$$

180 where we use in the first equality the fact that $\operatorname{Tr} \left\{ \Phi(\mathbf{x}) \frac{1}{\varkappa \sqrt{g}} \gamma_0 \Psi_\Sigma(\mathbf{x}) \right\} = \Psi$ (c.f. (20)) and the
 181 covariant Stokes theorem in the second equality. The result is that the right-hand side of (36)
 182 vanishes because of the metricity of space-time: $\nabla_\alpha g^{\mu\nu} = 0$.

183 Therefore, it is demonstrated that in the limiting case (24) all four terms in the right-hand side of
 184 (34) vanish, so that the terms I and IIIa in (14) do not contribute to the equation for the functional Ψ .

By combining the above considerations we obtain from (14) the following equation for the functional Ψ :

$$i\hbar \partial_t \Psi = \int d\mathbf{x} \sqrt{g} \left(\frac{\hbar^2}{2} \frac{g_{00}}{g} \frac{\delta^2}{\delta \phi(\mathbf{x})^2} - \frac{1}{2} g^{ij} \partial_i \phi(\mathbf{x}) \partial_j \phi(\mathbf{x}) + V(\phi) \right) \Psi - \frac{i}{4} \operatorname{Tr} \left\{ \Phi(\mathbf{x}) [\omega_0, \Psi_\Sigma] \right\}. \quad (37)$$

185 We see that the first three terms in the right hand side reproduce the canonical Hamiltonian operator
 186 in the functional derivative Schrödinger equation (1). However, the last term, which does not vanish
 187 in arbitrary non-static space-times where $\omega_0 \neq 0$, still can not be expressed in terms of Ψ alone. For
 188 this reason, we will treat static space-times with $\omega_0 = 0$ and non-static ones with $\omega_0 \neq 0$ separately.

189 4. Static space-times with $\omega_0 = 0$

In static space-times, when $\omega_0 = 0$, equation (37) coincides with the canonical functional derivative Schrödinger equation (1). Thus the latter is derived from the precanonical Schrödinger equation as the limiting case corresponding to (24). In this case, we can also specify the functional $\Xi([\Psi_\Sigma(\mathbf{x})], \check{\mathbf{x}})$ in (31) by combining the observations presented above together and noticing that the relation (31) is valid at *any* given point \mathbf{x} . This is possible only if the functional Ψ is the continual product of identical terms at all points \mathbf{x} , namely,

$$\Psi \sim \operatorname{Tr} \left\{ \prod_{\mathbf{x}} e^{-i\phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x}) / \varkappa} \underline{\gamma}_0 \Psi_\Sigma(\phi(\mathbf{x}), \mathbf{x}, t) \right\} \Big|_{\varkappa \rightarrow \gamma_0 \delta(\mathbf{0}) / \sqrt{g}}, \quad (38)$$

190 where \sim means an equality up to a normalization factor which includes \varkappa and \sqrt{h} .

The formal continual product expression in (38) can be understood as the multidimensional product integral [37,38]

$$\Psi \sim \operatorname{Tr} \left\{ \prod_{\mathbf{x}} e^{-i\phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x}) / \varkappa} \underline{\gamma}_0 \Psi_\Sigma(\phi(\mathbf{x}), \mathbf{x}, t) \right\} \Big|_{\frac{1}{\varkappa} \underline{\gamma}_0 \rightarrow \sqrt{h} dx}, \quad (39)$$

where the notation of the product integral of matrix-valued functions $F(\mathbf{x})$ as proposed by R. Gill [39] (and implemented in the \LaTeX package `prodint`) is used

$$\prod_{\mathbf{x}} e^{F(\mathbf{x}) dx} = \prod_{\mathbf{x}} (1 + F(\mathbf{x}) dx). \quad (40)$$

191 The expression in (39) generalizes a similar result obtained in flat space-time earlier [30]. The only
 192 difference is that in curved space-time the spatial integration measure dx is replaced by the invariant
 193 one $\sqrt{h} dx$ and the Dirac matrices are \mathbf{x} -dependent.

In $(1 + 1)$ -dimensional space-time, the product integral above is given by the well known path-ordered exponential, or the Peano-Baker series (also known as the Dyson series in the context of perturbative QFT and the path-ordered phase related to the Wilson loop in gauge theory), cf. eq. (53) below. A multidimensional generalization is briefly discussed in the books [37,38] and probably needs further refinement. However, in our case, instead of defining the product integral of arbitrary non-commutative matrices, we need only the trace of the product integral of Clifford-algebra valued functions. This significantly simplifies the task of defining the expression (39) mathematically. For example, in the one-dimensional case, the taking of the trace of each of the terms in the series expansion of the ordered exponential in (53) implies that the matrices under the integrals in the series expansion of the trace of product integral are multiplied in the cycling permuted way, which can be generalized to the multidimensional case, rather than a time-ordered one, which implies a one-parameter ordering whose multidimensional generalization is problematic. Then, if the corresponding limit exists,

$$\text{Tr} \prod_{\mathbf{x} \in V} e^{F(\mathbf{x})d\mathbf{x}} := \lim_{N \rightarrow \infty} \frac{1}{N!} \text{Tr} \sum_{P(N)} e^{F(\mathbf{x}_1)\Delta\mathbf{x}_1} e^{F(\mathbf{x}_2)\Delta\mathbf{x}_2} \dots e^{F(\mathbf{x}_N)\Delta\mathbf{x}_N}, \quad (41)$$

194 where $P(N)$ denotes all permutations of $(1, 2, \dots, N)$, the volume of integration $V \ni \mathbf{x}$ is partitioned into
 195 N small sub-volumes $\Delta\mathbf{x}_1, \dots, \Delta\mathbf{x}_N$ whose volumes are taken to zero as $N \rightarrow \infty$, and $F(\mathbf{x}_i)$ denotes the
 196 matrix F at a point $\mathbf{x}_i \in \Delta\mathbf{x}_i$. The existence of the limit in (41) and its independence on the partitioning
 197 of V into $N \rightarrow \infty$ sub-volumes $\Delta\mathbf{x}_i$ and the choice of points \mathbf{x}_i within the subvolumes $\Delta\mathbf{x}_i$ imply a
 198 certain continuity of the dependence of the matrix elements of F of \mathbf{x} , similarly to the definition of the
 199 Riemann integral of functions.

200 Now, by taking into account the fact that some of the terms in (34) are proven to not contribute to
 201 the time evolution of Ψ we can write the effective equation which governs the time evolution of Ψ_Σ
 202 which does contribute to the time evolution of the wave functional Ψ :

$$i\partial_t \Psi_\Sigma = \gamma_0 \left(-\frac{\varkappa}{2} \partial_{\phi\phi} + i\gamma^i \partial_i \phi(\mathbf{x}) \partial_\phi + \frac{1}{\varkappa} V(\phi) \right) \Psi_\Sigma =: \gamma_0 \mathcal{E} \Psi_\Sigma. \quad (42)$$

By substituting Ψ_Σ in the form

$$\Psi_\Sigma = e^{+\frac{1}{\varkappa} \phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x})} \Phi_\Sigma, \quad (43)$$

we obtain

$$i\partial_t \Psi_\Sigma = e^{+\frac{1}{\varkappa} \phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x})} i\partial_t \Phi_\Sigma \quad (44)$$

in the left hand side of (42) and

$$\gamma_0 e^{+\frac{1}{\varkappa} \phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x})} \left(-\frac{\varkappa}{2} \partial_{\phi\phi} - \frac{1}{2\varkappa} g^{ij}(\mathbf{x}) \partial_i \phi(\mathbf{x}) \partial_j \phi(\mathbf{x}) + \frac{1}{\varkappa} V(\phi) \right) \Phi_\Sigma \quad (45)$$

in the right hand side. Hence, Φ_Σ obeys

$$i\partial_t \Phi_\Sigma = \tilde{\gamma}_0(\mathbf{x}) \left(-\frac{\varkappa}{2} \partial_{\phi\phi} - \frac{1}{2\varkappa} g^{ij}(\mathbf{x}) \partial_i \phi(\mathbf{x}) \partial_j \phi(\mathbf{x}) + \frac{1}{\varkappa} V(\phi) \right) \Phi_\Sigma, \quad (46)$$

where

$$\tilde{\gamma}_0(\mathbf{x}) := e^{-\frac{1}{\varkappa} \phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x})} \gamma_0(\mathbf{x}) e^{+\frac{1}{\varkappa} \phi(\mathbf{x}) \gamma^i \partial_i \phi(\mathbf{x})}. \quad (47)$$

203 Obviously, $\tilde{\gamma}_0(\mathbf{x}) \tilde{\gamma}_0(\mathbf{x}) = \gamma_0(\mathbf{x}) \gamma_0(\mathbf{x}) = g_{00}(\mathbf{x})$, hence the transformation in (47) is a Clifford algebra
 204 isomorphism.

From (46) one can conclude that Φ_Σ can be written in the form

$$\Phi_\Sigma = (1 + \underline{\gamma}^0) \Phi_\Sigma^\times, \quad (48)$$

where Φ_{Σ}^{\times} is a scalar function such that

$$i\partial_t\Phi_{\Sigma}^{\times} = \sqrt{g_{00}} \left(-\frac{\varkappa}{2}\partial_{\phi\phi} - \frac{1}{2\varkappa}g^{ij}(\mathbf{x})\partial_i\phi(\mathbf{x})\partial_j\phi(\mathbf{x}) + \frac{1}{\varkappa}V(\phi) \right) \Phi_{\Sigma}^{\times}. \quad (49)$$

In terms of scalar function Φ_{Σ}^{\times} eq. (39) takes the form

$$\Psi \sim \text{Tr} \left\{ \prod_{\mathbf{x}} \left(1 + \underline{\gamma}_0 \right) \Phi_{\Sigma}^{\times}(\phi(\mathbf{x}), \mathbf{x}, t) \right\} \Big|_{\frac{1}{\varkappa}\underline{\gamma}_0 \mapsto \sqrt{\hbar}d\mathbf{x}} \sim \prod_{\mathbf{x}} \Phi_{\Sigma}^{\times}(\phi(\mathbf{x}), \mathbf{x}, t) \Big|_{\frac{1}{\varkappa} \mapsto \sqrt{\hbar}d\mathbf{x}'}, \quad (50)$$

205 where we use the projector property of the matrix $(1 + \underline{\gamma}_0)$. Obviously, the multidimensional product
 206 integral of the scalar function Φ_{Σ}^{\times} is defined without any complications related to the definition of the
 207 product integral of non-commutative matrix functions.

208 5. Non-static space-times with $\omega_0 \neq 0$

In non-static space-times, when $\omega_0 \neq 0$, the last term in (37) does not allow us to obtain a close equation for the functional Ψ . In order to find a way out, let us write the effective equation similar to (42) which governs the time evolution of Ψ_{Σ} , with the term I and the spatial part of the term $IIIa$ in (14), which are proven in (34) to have no contribution to $\partial_t\Psi$, removed:

$$i\partial_t\Psi_{\Sigma} = \gamma_0 \left(-\frac{\varkappa}{2}\partial_{\phi\phi} + i\gamma^i\partial_i\phi(\mathbf{x})\partial_{\phi} + \frac{1}{\varkappa}V(\phi) \right) \Psi_{\Sigma} - i[\omega_0, \Psi_{\Sigma}] =: \hat{H}_0 - i[\omega_0, \Psi_{\Sigma}]. \quad (51)$$

We first note that by transforming Ψ_{Σ} as follows:

$$\Psi_{\Sigma} := U\Psi'_{\Sigma}U^{-1}, \quad (52)$$

where

$$\begin{aligned} U(\mathbf{x}, t) &= \mathcal{T}e^{-\int_0^t ds \omega_0(\mathbf{x}, s)} = \prod_{t=0}^t (1 - \omega_0(\mathbf{x}, s)) ds \\ &:= 1 - \int_0^t dt_1 \omega_0(\mathbf{x}, t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \omega_0(\mathbf{x}, t_1)\omega_0(\mathbf{x}, t_2) \\ &\quad - \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \omega_0(\mathbf{x}, t_1)\omega_0(\mathbf{x}, t_2)\omega_0(\mathbf{x}, t_3) + \dots \end{aligned} \quad (53)$$

is the tranformation determined by the time-ordered exponential, we obtain

$$i\partial_t\Psi = -i[\omega_0, \Psi] + Ui\partial_t\Psi'_{\Sigma}U^{-1}. \quad (54)$$

Then

$$i\partial_t\Psi'_{\Sigma} = U^{-1}\hat{H}_0\Psi_{\Sigma}U = \hat{H}'_0\Psi', \quad (55)$$

where

$$\Psi' := U^{-1}\Psi_{\Sigma}U, \quad \hat{H}'_0 := U^{-1}\hat{H}_0U. \quad (56)$$

As the transformation U affects only the terms with γ^{μ} -s,

$$\hat{H}'_0 = \gamma'_0 \left(-\frac{\varkappa}{2}\partial_{\phi\phi} + i\gamma'^i\partial_i\phi(\mathbf{x})\partial_{\phi} + \frac{1}{\varkappa}V(\phi) \right), \quad (57)$$

where

$$\gamma'^{\mu}(\mathbf{x}, t) := U^{-1}(\mathbf{x}, t)\gamma^{\mu}(x)U(\mathbf{x}, t). \quad (58)$$

It is easy to check that

$$\gamma'^{\mu}\gamma'^{\nu} + \gamma'^{\nu}\gamma'^{\mu} = 2U^{-1}g^{\mu\nu}U = 2g^{\mu\nu}. \quad (59)$$

209 Hence the U -transformation is just a local automorphism of the Clifford algebra of space-time.

Using (55) one can write

$$i\partial_t\Psi = \text{Tr} \int d\mathbf{x} \frac{\delta\Psi}{\delta\Psi'_{\Sigma}(\mathbf{x})} i\partial_t\Psi'_{\Sigma} \quad (60)$$

$$= \text{Tr} \int d\mathbf{x} \frac{\delta\Psi}{\delta\Psi'_{\Sigma}(\mathbf{x})} \hat{H}'_{0\Sigma}\Psi'_{\Sigma}. \quad (61)$$

210 By comparing it with (7) and (8) we conclude that the results in static space-times with $\omega_0 = 0$ are

211 generalized to non-static space-times with $\omega_0 \neq 0$ using the dictionary:

$$\gamma^{\mu} \rightarrow \gamma'^{\mu} = U^{-1}\gamma^{\mu}U, \quad (62)$$

$$\Psi_{\Sigma} \rightarrow \Psi'_{\Sigma} = U^{-1}\Psi_{\Sigma}U, \quad (63)$$

$$\hat{H}_{0\Sigma} \rightarrow \hat{H}'_{0\Sigma} = U^{-1}H_{0\Sigma}U \quad (64)$$

with U given by the path-ordered exponential in (53). Then, the wave functional (39) rewritten in terms of the primed objects:

$$\Psi \sim \text{Tr} \left\{ \prod_{\mathbf{x}} e^{-i\phi(\mathbf{x})\gamma'^i(\mathbf{x},t)\partial_i\phi(\mathbf{x})/\varkappa} \underline{\gamma}_0\Psi'_{\Sigma}(\phi(\mathbf{x}), \mathbf{x}, t) \right\} \Big|_{\underline{\gamma}_0 \frac{1}{\varkappa} \mapsto \sqrt{\hbar} \text{d}\mathbf{x}} \quad (65)$$

represents, up to a normalization factor, the Schrödinger wave functional in terms of precanonical wave functions in an arbitrary curved space-time and it satisfies (37) without the last term, i.e. the functional derivative Schrödinger equation (1). Using the same steps as in the static case, this complicated expression can be transformed to the product integral of the scalar function Φ_{Σ}^{\times} ,

$$\Psi \sim \prod_{\mathbf{x}} \Phi_{\Sigma}^{\times}(\phi(\mathbf{x}), \mathbf{x}, t) \Big|_{\frac{1}{\varkappa} \mapsto \sqrt{\hbar} \text{d}\mathbf{x}'} \quad (66)$$

212 the only difference being the metric tensor components in eq. (49) defining Φ_{Σ}^{\times} can now be also
213 time-dependent.

214 In summary, we have demonstrated that in curved space-times the canonical functional derivative
215 Schrödinger equation (1) and the explicit product integral formula (39) relating the Schrödinger wave
216 functional with the Clifford-valued precanonical wave function can be derived from the precanonical
217 Schrödinger equation (2) in the singular limiting case when $\underline{\gamma}_0\varkappa$ is replaced by $\delta(\mathbf{0})/\sqrt{\hbar}$, a regularized
218 invariant delta-function at coinciding spatial points. A natural interpretation of the latter is that it
219 represents the UV cutoff of the total volume of the momentum space which one has to introduce
220 in order to make sense of the second variational derivative at coinciding points in (1). As in the
221 previously considered case of quantum fields in flat space-time [29–31], the standard unregularized
222 formulation of QFT in curved space-time in functional Schrödinger representation thus emerges from
223 the precanonical description as a singular limiting case.

224 6. Conclusion

225 We explored a connection between the description of an interacting quantum scalar field in curved
226 space-time derived from precanonical quantization and the standard description in the functional
227 Schrödinger picture resulting from the canonical quantization.

228 We have demonstrated that the functional derivative Schrödinger equation (1) can be derived
229 from the partial derivative precanonical Schrödinger equation (2) in the limiting case (24). Namely, the
230 restriction of the precanonical Schrödinger equation to the subspace Σ representing a field configuration

231 at time t , eq. (8), governs the time evolution of the wave functional according to (7) and (14). Then, in
 232 the limiting case (24),

- 233 (i) the potential term V in (14) reproduces the potential term in (1);
- 234 (ii) the term IV in (14) reproduces the second functional derivative term in (1) up to some additional
 235 terms which have no obvious counterpart in (1);
- 236 (iii) by noticing that one of those additional terms can be cancelled by the term II in (14) we obtain
 237 an expression of the Schrödinger wave functional as a trace of the continuous product of the
 238 precanonical wave functions (38) which we suggested to interpret as a multidimensional analogue
 239 of the product integral, eq. (39);
- 240 (iv) using the expression of the wave functional in terms of precanonical wave functions in the other
 241 additional term mentioned in (ii) we reproduce the second term in the right-hand side of (1);
- 242 (v) this explicit expression also allows us to show that for the fields $\phi(\mathbf{x})$ and $\Psi_{\Sigma}(\phi(\mathbf{x}), \mathbf{x}, t)$ vanishing
 243 at the spatial infinity the remaining terms I and $IIIa$ do not contribute to the functional
 244 Schrödinger equation (1);
- 245 (vi) in static space-times when $\omega_0 = 0$ the remaining term $IIIb$ vanishes and the functional
 246 Schrödinger equation (1) emerges from the precanonical Schrödinger equation (2) and the
 247 Schrödinger wave functional is expressed in terms of precanonical wave functions;
- 248 (vii) in non-static space-times with $\omega_0 \neq 0$, we argue that the transformation (53) absorbs the
 249 contribution of the term $IIIb$ in (14) thus allowing us to obtain the functional Schrödinger
 250 equation (1) from the precanonical Schrödinger equation (2) and to express the Schrödinger wave
 251 functional in terms of transformed precanonical wave functions, eq. (66);
- 252 (viii) both in static and non-static space-times the Schrödinger wave functional can be represented as
 253 the product integral of a scalar wave function derived from the precanonical wave function by a
 254 series of transformations and satisfying equation (49).

255 These results generalize to arbitrary curved space-times (whose metric is represented in Gaussian
 256 coordinates with $g_{0i} = 0$) the statement from [29–31] that the standard functional Schrödinger
 257 representation of QFT is a certain (symbolic) limiting case of the theory of quantum fields obtained by
 258 precanonical quantization.

259 The symbolic or singular nature of the limiting transition from precanonical quantization to the
 260 standard formulation of QFT in functional Schrödinger representation is related to the fact that the
 261 latter, due to the presence of the second functional derivative at coinciding points, is not a well-defined
 262 theory unless a regularization is introduced. The regularization typically introduces a UV cutoff scale
 263 as an additional element of the theory removed by a subsequent renormalization. In precanonical
 264 quantization, the ultraviolet scale \varkappa appears as an inherent element quantization, which, unlike
 265 other theories introducing an ultraviolet fundamental length or cutoff, does not alter the relativistic
 266 space-time at smaller scales.

267 Whether \varkappa is a fundamental scale or an auxiliary element of precanonical quantization of fields
 268 remains an open question so far. On the one hand, one can show that in free scalar theory \varkappa disappears
 269 from the observable characteristics of a quantum field because the spectrum of DW Hamiltonian
 270 operator is proportional to \varkappa . In interacting scalar theory, powers of \varkappa enter in the perturbative
 271 corrections to the spectrum of DW Hamiltonian \hat{H} thus suggesting that \varkappa can be renormalized away
 272 by absorbing the expressions with the bare mass and \varkappa in the "observed mass". On the other hand,
 273 an estimation of the mass gap in the quantum pure SU(2) gauge theory [34] derived by precanonical
 274 quantization and a naive estimation of the cosmological constant based on the precanonically quantized
 275 pure Einstein gravity [35] seem to consistently point to the estimation of the scale of \varkappa at roughly
 276 $\sim 10^2 MeV$. We hope to clarify the nature of \varkappa in our forthcoming work.

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