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A 2D Electrostatic Non-Linear Second-Order Differential Model for Electrostatic Circular Membrane MEMS Devices: A Result of Existence and Uniqueness

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Abstract: In the framework of 2D circular membrane Micro-Electric-Mechanical-Systems (MEMS), a new non-linear second-order differential model with singularity in the steady-state case is presented in this paper. In particular, starting from the fact that the electric field magnitude is locally proportional to the curvature of the membrane, the problem is formalized in terms of the mean curvature. Then, a result of existence of at least one solution is achieved. Finally, two different approaches prove that the uniqueness of the solutions is not ensured.

Keywords: Circular Membrane MEMS Devices, Electrostatic Actuator, Boundary Non-Linear Second-Order Differential Problems, Singularities, Mean Curvature.

1. Introduction to the Problem

In recent years, a growing demand for embedded engineering applications has convinced researchers to develop low-cost micro-nano-sized components in which actuators and transducers play important roles. This represents the link between the physical nature of the problem under study and the need to use machine languages to manage interfaces with other devices [1]. The interest of the scientific community in MEMS (Micro-Electro-Mechanical Systems) technology, which was born when Nathanson et al. produced the first batch device in 1964 [1,2], has been growing exponentially. At present, MEMS technology is fully part of multi-disciplinary mathematical physics, allowing for highly varied engineering applications [1,3]. This is mainly due to the fact that it has been supported by sophisticated theoretical models, both static and dynamic [4,5]. However, even if these models appear to be adherent to reality, they are often structured in an implicit form which does not provide solutions in a closed form (except in particular cases), for which numerical solutions are necessarily sought [6] or analytical conditions that ensure the existence, uniqueness, and regularity (up to the desired order) of the solution are derived [7,8]. Both approaches (analytical and numerical) can be used to obtain numerical solutions that do not represent ghost solutions [9–11]. In the field of MEMS technologies, the scientific community has been actively engaged both in the development of theoretical models and in technology transfer. In particular, some important models of coupled problems have been developed: from magnetically actuated systems [1,12,13] to thermo-elastic models [14]; from electro-elastic models [15] to micro-fluidic models [16] with highly complex formulations that manage MEMS dynamics of different natures (with plates, membranes, and so on). These theoretical models have had excellent feedback in technology transfer through the design, realization, and distribution of the MEMS devices used in the various applications and fields [17]. Biomedical diagnostics increasingly require efficient and reliable micro-components that, with contained costs, can be of help to health-care personnel, both

in on-line and off-line modes [18,19]. In the theoretical field, many mathematical models have been conceived in particular functional spaces, in order to provide otherwise difficult to detect conditions of existence and uniqueness of the solutions [20,21]. Cassani et al. [22] built a sophisticated mathematical model of a MEMS device consisting of two metallic plates, where one is fixed (the lower plate) and the other is deformable (the upper plate) but anchored at the boundary of a region $\Omega \in \mathbb{R}^N$ and subjected to a drop voltage deforming the lower plate (at $u = 0$) towards the upper plate (at $u = 1$). This steady-state model assumes the following structure [22]:

$$\begin{cases} \alpha \Delta^2 u = \left(\beta \int_{\Omega} |\nabla u|^2 dx + \gamma \right) \Delta u + \frac{\lambda_1 f_1(x)}{(1-u)^\sigma \left(1 + \zeta \int_{\Omega} \frac{dx}{(1-u)^{\sigma_1-1}} \right)} \\ u = \Delta u - du_\nu = 0, \quad x \in \partial\Omega, \quad d \geq 0 \\ 0 < u < 1, \quad x \in \Omega, \end{cases} \quad (1)$$

where f_1 is a bounded function depending on the dielectric properties of the material constituting the deformable plate and the positive parameters α , β , γ , and χ are related to the mechanical and electrical properties. It is worth noting that σ_1 is the Coulomb's exponent, such that $\sigma_1 \geq 2$ takes into account higher-order Coulombian behaviors. It is obvious that determining a solution for (1) is a difficult task, as the choice of boundary conditions for (1) is also an extremely delicate problem [22,23]. From model (1), considering the inertial and non-local effects to be irrelevant and setting $\alpha = 1$, $\beta = \gamma = 0$, one obtains the following model:

$$\begin{cases} \Delta^2 u(x) = \frac{\lambda_1 f_1(x)}{(1-u(x))^2} \\ 0 < u(x) < 1 \quad \text{in } \Omega, \\ u = \Delta u - \hat{d}u_\nu, \quad \text{on } \partial\Omega, \quad \hat{d} \geq 0. \end{cases} \quad (2)$$

Model (2) has been studied extensively by Cassani et al. in [23], in which the existence of a solution was studied using Steklov boundary conditions to obtain Dirichlet and Navier boundary conditions when $\hat{d} = 0$ or $\hat{d} = +\infty$. In the past, starting from (2), the authors have studied the following new elliptical semi-linear dimensionless model for a 1D-membrane MEMS:

$$\begin{cases} \frac{d^2 u(x)}{dx^2} = -\frac{1}{\theta \lambda^2} \left(1 + \left(\frac{du(x)}{dx} \right)^2 \right)^3 (1 - u(x))^2 \quad \text{in } \Omega = [-1, 1] \\ 0 < u(x) < 1 \\ u = 0 \quad \text{on } \Omega, \end{cases} \quad (3)$$

where the membrane replaces the deformable lower plate (the lower plate is exploited to anchor the membrane to the edge). Model (3) is based on the proportionality between the electric field magnitude, $|\mathbf{E}|$, and the curvature of the membrane, K , obtaining interesting existence and uniqueness results for the solution (for details, see [7]). It is worth noting that, in [7], the uniqueness condition was independent of the electro-mechanical properties of the material constituting the membrane; so, the authors in [10] obtained a new condition of uniqueness for problem (3) which took into account these properties. Moreover, model (3) has been solved numerically, using the shooting approach, in both [24] and [9], highlighting the range of $\theta \lambda^2$ and ensuring convergence of the numerical procedure without ghost solutions. Finally, in [25], the shooting procedure and Keller-box scheme have been compared to achieve an optimal range of $\theta \lambda^2$ while ensuring the absence of ghost solutions of problem (3). It is worth noting that θ takes into account the applied voltage V and λ^2 takes also into account the electro-mechanical properties of the material constituting the membrane [9,10].

In this work, for applicative reasons, the authors focus their attention on 2D circular membrane MEMS, which are useful in a lot of industrial and/or biomedical applications [1,19]. First, we observe that, for physical reasons, there is an axial symmetry in the geometry of the membrane. Then, considering the z

axis as the rotation axis, the profile u of the circular membrane (which is fixed to the edge of a metallic disk with radius R) can be considered as a surface generated by rotating a curve C around z on the vertical plane rz located in the first quadrant, with $0 \leq r \leq R$. Then, the deflection u of the membrane is only dependent on the radial co-ordinate r , such that the 2D problem here can be considered as a 1D problem (model (2)) in which the independent variable x is replaced by r . Then, just considering the radial part of the Laplace operator, model (2) is as follows:

$$\begin{cases} \frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} = -\frac{\lambda^2(r)}{(1-u(r))^2} \\ u(R) = 0 \\ \frac{du(0)}{dr} = 0, \end{cases} \quad (4)$$

where $\lambda^2(r) = \lambda_1(r)f_1(r)$. Thus, a new model with singularity is obtained, due to the presence of the factor $1/r$, in which in (4) $|\mathbf{E}|$ is represented by $\frac{\lambda(r)}{(1-u(r))}$, as in [7]. In addition, we also will consider $|\mathbf{E}|$ to be proportional to the curvature of the membrane itself (with profile $u(r) \in C^2(\Omega)$) and take into account the physical properties of the membrane and its deformation. Moreover, we exploit the expression of mean curvature to achieve, from model (4), the following model:

$$\begin{cases} \frac{d^2 u(r)}{dr^2} = -\frac{1}{r} \frac{du(r)}{dr} - \frac{(1-u(r)-d^*)^2}{\theta\lambda^2} \\ u(R) = 0 \\ \frac{du(0)}{dr} = 0 \\ 0 < u(r) < d, \end{cases} \quad (5)$$

where d^* represents the critical security distance, which is used to ensure that the deflection of the membrane does not produce contact between the membrane itself and the fixed upper plate. Then, for model (5), we prove the following existence theorem:

Theorem 1. *Let us consider the problem (5) and let $u_1(r)$ and $u_2(r)$ two functions, defined in $[0, R]$ and twice continuously differentiable, such that*

$$u_1(r) < u_2(r) \quad (6)$$

and

$$\frac{d^2 u_1(r)}{dr^2} + \frac{1}{r} \frac{du_1(r)}{dr} + \frac{(1-u_1(r)-d^*)^2}{\theta\lambda^2} > 0 \quad (7)$$

$$\frac{d^2 u_2(r)}{dr^2} + \frac{1}{r} \frac{du_2(r)}{dr} + \frac{(1-u_2(r)-d^*)^2}{\theta\lambda^2} < 0 \quad (8)$$

for $r \in (0, R)$. In addition, let $\frac{1}{r} \frac{du(r)}{dr} + \frac{(1-u(r)-d^*)^2}{\theta\lambda^2}$ be a continuous function (except for $r = 0$) satisfying the Lipschitz condition in $U \times (-\infty, +\infty)$, where

$$U = \{(r, u) : 0 < r < R \text{ and } u_1(r) \leq u(r) \leq u_2(r)\}. \quad (9)$$

If

$$\frac{du_1(0)}{dr} \geq \frac{du_2(0)}{dr}, \quad (10)$$

$$u_1(R) = u_2(R) = 0, \quad (11)$$

and, indicating by ϵ_0 the permittivity of the free space, we have:

$$\theta\lambda^2 > \frac{R^2(d^*)^2}{2\epsilon_0 k}, \quad (12)$$

there exists one at least one solution for the problem (5).

However, uniqueness of the solution for the problem (5) is not guaranteed, as proved in the following theorem:

Theorem 2. *Let us consider the problem (5). Let us also suppose that the conditions of the Theorem 1 are satisfied and that $u_1(r)$ and $u_2(r)$ satisfy the given boundary conditions. Then, the uniqueness of the solution $u(r)$, such that $u_1(r) \leq u(r) \leq u_2(r)$, is not guaranteed.*

The paper is structured as follows. After presenting some preliminary Lemmas in Section 2, Section 3 describes the 2D circular membrane MEMS from two points of view: as an actuator and also as a transducer. In the latter case, starting from the theory of elasticity of the plates, the most important elasticity formulas of the membrane are presented, which will be useful in the following. Moreover, an important link between mechanical pressure and electrostatic pressure is detailed for our purposes. The formulation of the problem under study is detailed in Section 4: from the geometry of the problem, the model is formulated in terms of the mean curvature of the membrane, obtaining a non-linear second-order differential equation with singularity (Section 5). Once the problem under study has been formulated in a general manner (Section 6), Theorem 1 is presented in Section 7, proving that problem (5) admits at least one solution. However, although the existence of at least one solution is ensured, uniqueness is not guaranteed, as proved by means of two different approaches in Section 8. Finally, some conclusions and perspectives complete this work.

2. Preliminary Lemmas

We first present the following well-known result, which will be exploited to prove the existence of a solution for the problem (5).

Lemma 1. *We consider the following problem:*

$$\begin{cases} \frac{d^2 u(r)}{dr^2} + F\left(r, u(r), \frac{du(r)}{dr}\right) = 0 \\ u(b) = B \\ \frac{du(a)}{dr} = m. \end{cases} \quad (13)$$

Let $u_1(r)$ and $u_2(r)$ be twice continuously differentiable functions, such that

$$u_1(r) < u_2(r) \quad r \in (a, b) \quad (14)$$

and

$$\frac{d^2 u_1(r)}{dr^2} + F\left(r, u_1(r), \frac{du_1(r)}{dr}\right) > 0 \quad (15)$$

$$\frac{d^2 u_2(r)}{dr^2} + F\left(r, u_2(r), \frac{du_2(r)}{dr}\right) < 0 \quad (16)$$

for $r \in (a, b)$. Let $F\left(r, y, \frac{dy}{dr}\right)$ be a continuous function and satisfying the following Lipschitz condition:

$$\begin{aligned} & K_1(r)(u(r) - v(r)) + L_2(r) \left(\frac{du(r)}{dr} - \frac{dv(r)}{dr} \right) \leq \\ & \leq F\left(r, u(r), \frac{du(r)}{dr}\right) - F\left(r, v(r), \frac{dv(r)}{dr}\right) \leq \\ & \leq K_2(r)(u(r) - v(r)) + L_1(r) \left(\frac{du(r)}{dr} - \frac{dv(r)}{dr} \right), \end{aligned} \quad (17)$$

in $U \times (-\infty, +\infty)$, where

$$U = \{(r, u) : a < r < b \text{ and } u_1(r) \leq u(r) \leq u_2(r)\} \quad (18)$$

and $K_i(r)$ and $L_i(r)$ ($i = 1, 2$) are continuous functions in (a, b) .

If

$$\frac{du_1(a)}{dr} \geq \frac{du_2(a)}{dr}, \quad (19)$$

with

$$u_1(b) = B = u_2(b), \quad (20)$$

then the problem (13) has at least one solution, $u(r)$, such that

$$u_1(r) \leq u(r) \leq u_2(r) \quad \text{in } [a, b] \quad (21)$$

holds

(for details, see [26]). Furthermore, we will use the following Lemmas to prove that the solution of the problem (5) is not unique.

Lemma 2. Let us suppose that the conditions of Lemma 1 are satisfied and that the functions $u_1(r)$, $u_2(r)$ satisfy the given boundary conditions. If the differential equation

$$\frac{d^2u(r)}{dr^2} + K_2(r)u(r) + L_1(r)\frac{du(r)}{dr} = 0 \quad (22)$$

has no trivial solution satisfying zero boundary conditions on any sub-interval of $[a, b]$, then the given boundary value problem has only one solution $u(r)$ such that $u_1(r) \leq u(r) \leq u_2(r)$

(see [26]). Before we present Lemma 3, we need to introduce the following definition.

Definition 1. A point r_0 is a regular singular point if only if

$$\begin{cases} \lim_{r \rightarrow r_0} (r - r_0)p(r) & \text{is finite, and} \\ \lim_{r \rightarrow r_0} (r - r_0)^2q(r) & \text{is finite,} \end{cases} \quad (23)$$

where $p(r)$ and $q(r)$ are functions depending on r .

Lemma 3. Let us consider the second-order ordinary differential equation (for details, see [27]):

$$P(r)\frac{d^2u(r)}{dr^2} + Q(r)\frac{du(r)}{dr} + R(r)u(r) = 0, \quad (24)$$

in which $P(r)$, $Q(r)$, and $R(r)$ are functions depending on r . As $P(r) \neq 0$, and setting $p(r) = \frac{Q(r)}{P(r)}$ and $q(r) = \frac{R(r)}{P(r)}$, we have:

$$r^2\frac{d^2u(r)}{dr^2} + r\{rp(r)\}\frac{du(r)}{dr} + \{r^2q(r)\}u(r) = 0, \quad (25)$$

in which $r = 0$ is a regular singular point. As it is well-known that $rp(r)$ and $r^2q(r)$ are both analytic functions at $r = 0$ with convergent power series expansions

$$rp(r) = \sum_{n=0}^{\infty} p_n r^n \quad (26)$$

and

$$r^2q(r) = \sum_{n=0}^{\infty} q_n r^n, \quad (27)$$

for $|r| < \rho$, and as $\rho > 0$ is the minimum of the radii of convergence of the power series for $rp(r)$ and $r^2q(r)$, considering t_1 and t_2 as the roots of the following indicial equation:

$$F(t) = t(t-1) + p_0t + q_0 = 0, \quad (28)$$

we have that:

- if t_1 and t_2 are real, then, in either the interval $-\rho < r < 0$ or the interval $0 < r < \rho$, there exists a solution of the form

$$u_1(r) = |r|^{t_1} \left[1 + \sum_{n=1}^{\infty} a_n(t_1) r^n \right], \quad (29)$$

in which $a_n(t_1)$ can be obtained by the following recurrence relation:

$$F(t+n)a_n + \sum_{k=0}^{n-1} a_k [(t+k)p_{n-k} + q_{n-k}] = 0, \quad n \geq 1, \quad (30)$$

in which $a_0 = 1$ and $t = t_1$.

- If, in addition, $t_1 = t_2$, then the second solution of (25) the equation is

$$u_2(r) = u_1(r) \ln|r| + |r|^{t_1} \sum_{n=1}^{\infty} b_n(t_1) r^n, \quad (31)$$

in which $b_n(t_1)$ can be determined by substituting the form of the series solution u in equation (25). Finally, the general solution for equation (25) can be achieved by a linear combination of $u_1(r)$ and $u_2(r)$.

3. An Overview of Circular Membrane MEMS Devices

3.1. The Circular Actuator Membrane MEMS

Let us consider a circular membrane MEMS, constituted of two parallel disks with radius R with mutual distance d (see, Figure 1). On the edge of the lower disk, a deformable circular membrane of the same radius is clumped, which is free to deform towards the upper disk (which is fixed). Upon applying an external electric voltage V , the membrane deforms closer to the upper disk without touching it. In particular, when V is externally applied (the lower disk can be considered at $V = 0$), the field \mathbf{E} between the disks generates an electrostatic pressure $p_{el} = 0.5\epsilon_0|\mathbf{E}|^2$ [3], where ϵ_0 is the permittivity of free space, which moves the membrane, producing a deflection u . While deformation of the membrane occurs, \mathbf{E} changes such that its direction is always locally orthogonal to the tangent line of the membrane and $|\mathbf{E}|$ depends on the local distance between the membrane and the upper disk [3]. It is worth noting that, when the membrane deforms, the capacitance C_{el} of the device also varies, as the distance between the membrane and the upper disk locally varies. It is evident that, the bigger $|\mathbf{E}|$ is, the bigger the curvature of the membrane will be. Hence, it is clear that $|\mathbf{E}|$ can be locally considered to be proportional to the curvature of the membrane.

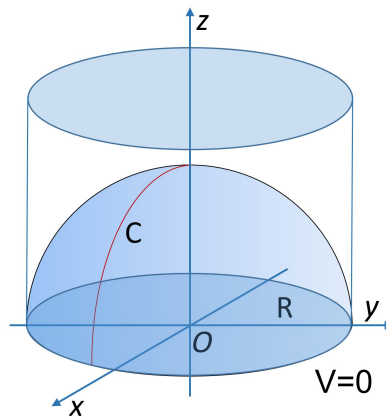


Figure 1. Representation of the deformation of the membrane in a circular MEMS.

3.2. The Circular Transducer Membrane MEMS

To study our model, we will utilize some analogies to the model of a circular transducer plate MEMS, in which the device works when subjected to a mechanical pressure p .

3.2.1. Circular Plate MEMS Device: behavior under the effect of p

As has been shown, when a material plate is subjected to a mechanical pressure p , it deforms with deflection u satisfying, in the dynamical case, the following partial differential equation [1]:

$$\rho h \frac{\partial^2 u}{\partial t^2} - T \Delta u + D \Delta^2 w = 0, \quad (32)$$

where ρ is the density of the material constituting the plate, h is its thickness, and T and D are the mechanical tension in the plate and the flexural rigidity of the plate, respectively. Obviously, in the steady-state case, the term $\rho h \frac{\partial^2 u}{\partial t^2}$ vanishes. Usually, D is defined as:

$$D = \frac{Y h^3}{12(1 - \nu^2)}, \quad (33)$$

in which Y is Young's modulus and ν is the Poisson ratio. In addition, for a circular plate, the deflection u , for reasons of symmetry, only depends on the radial co-ordinate r . Then, in the steady-state case, $u(r)$ can be calculated as [28]:

$$u(r) = \frac{R^4}{64D} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}^2 p \quad 0 \leq r \leq R \quad (34)$$

for the z-directed displacement u , according to [28]. Obviously, if $r = 0$, one obtains the displacement at the center of the plate:

$$u_0 = \frac{R^4}{64D} p. \quad (35)$$

Thus, inserting (35) into (34) yields:

$$u(r) = u_0 \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}^2. \quad (36)$$

In other words, in contrast with the behaviour of an actuator, here, the device behaves as a transducer because a mechanical pressure generates a deflection $u(r)$, with consequent variation of the electrostatic

capacitance C_{el} . Moreover, calculating the integral of the capacitance of an elementary ring over the extent of the circular plate yields:

$$C_{el}(u_0) = \int_0^R \epsilon_0 \frac{2\pi r}{d} \left\{ 1 - \frac{u(r)}{d} \right\}^{-1} dr, \quad (37)$$

which is acceptable if $|u_0| \ll d$. It is worth noting that, being a material plate with a thickness h and a flexural rigidity D , its deformation $u(r)$ is very small, such that the distance between the upper and deformable (lower) plates can be retained constant (equal to d). By expanding the integrand of (37) in a Taylor series (with three terms retained), we obtain:

$$C_{el}(u_0) \approx 2C_0 \int_0^R \left[1 + \frac{u_0}{d} \left(1 - \frac{r^2}{R^2} \right)^2 + \frac{u_0^2}{d^2} \left(1 - \frac{r^2}{R^2} \right)^4 \right] \frac{r}{R^2} dr, \quad (38)$$

where

$$C_0 = \epsilon_0 \frac{\pi R^2}{d} \quad (39)$$

represents the equilibrium capacitance ($p = 0$). After integration of (38), one obtains:

$$C_{el}(u_0) \approx C_0 \left(1 + \frac{u_0}{3d} + \frac{u_0^2}{5d^2} \right). \quad (40)$$

The relation (40) is very interesting, because it gives us the possibility to achieve the co-energy function, the charge on the circular plate, and the force of electrical origin as well. In fact, the co-energy function of the capacitance system is:

$$W' \simeq \frac{1}{2} C_{el}(u_0) v^2 = \frac{1}{2} C_0 \left(1 + \frac{u_0}{3d} + \frac{u_0^2}{5d^2} \right) v^2, \quad (41)$$

in which v represents the applied voltage (not necessary constant), while the charge on the membrane is:

$$q = \frac{\partial W'}{\partial v} \simeq C_0 V \left(1 + \frac{u_0}{3d} + \frac{u_0^2}{5d^2} \right). \quad (42)$$

Moreover, the force of electrical origin is:

$$f_{el} = \frac{\partial W'}{\partial u_0} \simeq \frac{1}{2} C_0 V^2 \left(\frac{1}{3d} + \frac{2u_0}{5d^2} \right). \quad (43)$$

It can be noted that the capacitance $C_{el}(u_0)$ is a non-linear function of u_0 , according to (40). The capacitance change $\frac{dC_{el}}{du_0}$ is proportional to the pressure through the electrical force:

$$\frac{dC}{du_0} \simeq C_0 \left(\frac{1}{3d} + \frac{2u_0}{5d^2} \right) \simeq \frac{2}{V^2} f_{el}. \quad (44)$$

The capacitance change can be easily detected. Under the assumption $|u_0| \ll d$, $|\mathbf{E}(r)|$ turns out to be [1,28]:

$$|\mathbf{E}(r)| \simeq \frac{VR^4}{R^4d - u_0(R^2 - r^2)^2} = \frac{V}{d - u_0(1 - (r/R)^2)^2}. \quad (45)$$

Remark 1. It is worth noting that C_{el} , W' , q , f_{el} , and $\frac{dC}{du_0}$ depend on d . This is due to the fact that the circular plate has a relevant value of D , such that the deflection $u(r)$ is not accentuated in fact, $|u_0| \ll d$. Then, it follows that the electrical quantities only depend on d and do not depend on $d - u(r)$ [3].

3.2.2. Circular Membrane MEMS Device: Behavior under the effect of p

Being interested in a circular MEMS membrane device with radius R , we need to modify the relations presented in Subsection 3.2.1. In particular, for circular membranes, the thickness h is very small; so, that the flexural rigidity D is greater than when a circular plate MEMS is considered (see (33)). It is worth nothing that the greater D is, more flexible the membrane will be. Then, u_0 (as given by (35)) grows, such that the membrane is closer to the upper disk. In this case, the condition $|u_0| \ll d$ does not hold and, so, it is necessary to consider the quantity $d - u(r)$ in the denominator of (43). If the membrane is subjected to a mechanical pressure p , one can prove that the deflection of the membrane assumes the following form [1]:

$$u(r) = u_0 \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}, \quad (46)$$

where

$$u_0 = \frac{pR^2}{4T} \quad (47)$$

and T is the mechanical tension of the membrane. In addition, (43) becomes [1,28]:

$$f_{el} = \frac{1}{2} \frac{\epsilon_0 \pi R^2 V^2}{(d - u(r))^2}, \quad (48)$$

as the membrane deforms such that the electrostatic quantities depend on $d - u(r)$. Then, the electrostatic pressure, p_{el} , is justified as:

$$p_{el} \cong \frac{f_{el}}{\pi R^2} = \frac{1}{2} \frac{\epsilon_0 V^2}{(d - u(r))^2}. \quad (49)$$

We observe that, to calculate f_{el} and p_{el} (by (48) and (49), respectively), we approximate the surface of the membrane as πR^2 , even when the membrane deforms. This approximation is justified by the fact that $d \ll R$, so that the surface of the deformed membrane is almost equal to the surface of the membrane in the resting position.

3.3. Link between Mechanical Pressure p and Electrostatic Pressure p_{el}

As specified above, the device under study is an actuator that, under the effect of V , produces a field \mathbf{E} that generates p_{el} . This pressure deforms the membrane, achieving the deflection $u(r)$. It is clear that there exists a link between p and p_{el} . However, by physical considerations, u_0 depends on the mechanical pressure, p . Particularly, the link between u_0 and p is linear. In fact, from (47), we can write

$$u_0 = \frac{pR^2}{4T} = k_1 p, \quad (50)$$

with $k_1 = \frac{R^2}{4T}$ constant. Moreover, in the absence of further causes, the mechanical pressure p arises exclusively from the electrostatic pressure p_{el} due to $|\mathbf{E}|$. Then, p can be considered dependent on p_{el} , such that the following chain of equalities holds:

$$u_0 = k_1 p = k_1 k_2 p_{el} = k p_{el}, \quad (51)$$

where k_2 and k are constant. By (49), we can write

$$u_0 = \frac{k \epsilon_0 V^2}{2(d - u(r))^2}. \quad (52)$$

Remark 2. In (52), the quantity $d - u(r)$ represents the distance between the profile of the membrane and the upper disk. Considering that the profile of the membrane does not touch the upper disk and that, when the deformation occurs, it is far from the upper disk with a distance d^* , it makes sense to write

$$u(r) \leq d - d^* \quad (53)$$

from which, we obtain

$$\frac{1}{(d - u(r))^2} \leq \frac{1}{d^{*2}}. \quad (54)$$

For details, see Fig. 2.

Again, (46), by (52) and taking into account Remark 2, becomes:

$$u(r) \leq \bar{u}(r) = \frac{k\epsilon_0 V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}. \quad (55)$$

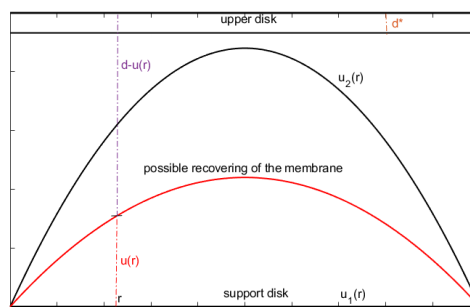


Figure 2. The functions $u_1(r)$ and $u_2(r)$ for the problem under study.

Remark 3. It is worth nothing that the link between p and p_{el} makes the transducer-actuator model dual. In other words, the behavior of the transducer, as we have seen in this Subsection, helps to understand how the actuator operates and vice versa.

4. Formulation of the Problem

4.1. The Proposed Model

As specified in the Introduction, we have already described the geometry of the device. As shown above, $\lambda^2 / (1 - u(r))^2$ in model (4) is considered to be proportional to $|\mathbf{E}|^2$. Thus, it makes sense to rewrite model (4) as follows:

$$\begin{cases} \frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} = -\theta |\mathbf{E}|^2; & \theta \in \mathbb{R}^+ \\ u(1) = 0 \\ \frac{du(0)}{dr} = 0. \end{cases} \quad (56)$$

Appealing to evident physical motivation, we can suppose θ to be a continuous function depending on r on $\Omega = [-R, R]$. Moreover, indicating by $K(r, u(r))$ the curvature of the deformed membrane, the proportionality between $|\mathbf{E}(r)|$ and $K(r, u(r))$ can be expressed by

$$|\mathbf{E}(r)| = \mu(r, u(r))K(r, u(r)), \quad (57)$$

in which $\mu(r, u(r))$ represents the proportionality function. Then, if we pose

$$\mu(r, u(r)) = \lambda / (1 - u(r) - d^*), \quad (58)$$

where λ is a continuous function proportional to the tension V , model (56) can be rewritten as follows:

$$\begin{cases} \frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} = -\theta \mu^2(r, u(r)) K^2(r, u(r)) = -\theta \lambda^2 \frac{K^2(r, u(r))}{(1-u(r)-d^*)^2} & \theta \in \mathbb{R}^+ \\ u(1) = 0 \\ \frac{du(0)}{dr} = 0, \end{cases} \quad (59)$$

where d^* represents the critical security distance, which is used to ensure that the deflection of the membrane does not produce contact between the membrane itself and the fixed upper plate (obviously, $\mu(r, u(r)) \in C^0(A)$ with $A = [-R, R] \times [0, 1)$). To completely define the proposed model, we need to explicitly find the curvature K in (59), which, in this paper, is expressed as the mean curvature [29].

5. Formulation of the Proposed Model in Terms of Mean Curvature

Let us consider a surface S generated by rotating around the vertical axis z a curve C located in a plane orthogonal to the xy plane forming an angle t with the zx plane (see Fig. 1) [29]. To simplify our calculations, we suppose that C is parametrized with a generic parameter r , differing from the curvilinear co-ordinate s , such that

$$P(r) = (\bar{f}(r), 0, \bar{g}(r)), \quad r \in I \subset \mathbb{R}, \quad (60)$$

with $\bar{f}(r)$ and $\bar{g}(r)$ regular functions such that

$$\left(\frac{d\bar{f}(r)}{dr} \right)^2 + \left(\frac{d\bar{g}(r)}{dr} \right)^2 \geq 0 \quad (61)$$

for each $r \in I = [0, R]$. The parametrization of the surface S is

$$\begin{cases} P(t, r) = (\bar{f}(r) \cos t, \bar{f}(r) \sin t, \bar{g}(r)) \\ (t, r) \in [0, 2\pi) \times I. \end{cases}$$

We observe that $P(r)$, as a natural parametrization, ensures that C is regular everywhere and so, by rotation, S is regular. Then, we easily obtain

$$\begin{cases} \frac{\partial P}{\partial t} = (-\bar{f}(r) \sin t, \bar{f}(r) \cos t, 0) \\ \frac{\partial P}{\partial r} = \left(\frac{d\bar{f}(r)}{dr} \cos t, \frac{d\bar{f}(r)}{dr} \sin t, \frac{d\bar{g}(r)}{dr} \right). \end{cases}$$

Thus, the coefficients of the first fundamental form are

$$E = \left\| \frac{\partial P(t, r)}{\partial r} \right\|^2 = \bar{f}^2(r) \quad F = \frac{\partial P(t, r)}{\partial r} \cdot \frac{\partial P(t, r)}{\partial t} = 0 \quad G = \left\| \frac{\partial P(t, r)}{\partial t} \right\|^2 = 1. \quad (62)$$

We observe that, as $F = 0$ everywhere, the co-ordinate lines are everywhere orthogonal. Furthermore deriving, we obtain

$$\begin{cases} \frac{\partial^2 P(t, r)}{\partial t^2} = (-\bar{f}(r) \cos t, -\bar{f}(r) \sin t, 0) \\ \frac{\partial^2 P(t, r)}{\partial t \partial r} = \left(-\frac{d\bar{f}(r)}{dr} \sin t, \frac{d\bar{f}(r)}{dr} \cos t, 0 \right) \\ \frac{\partial^2 P(t, r)}{\partial r^2} = \left(\frac{\partial^2 \bar{f}(r)}{\partial r^2} \cos t, \frac{\partial^2 \bar{f}(r)}{\partial r^2} \sin t, \frac{\partial^2 \bar{g}(r)}{\partial r^2} \right). \end{cases}$$

Finally, we have:

$$\frac{\partial P(t,r)}{\partial t} \wedge \frac{\partial P(t,r)}{\partial r} = \bar{f}(r) \left(\frac{dg(r)}{dr} \cos t, \frac{dg(r)}{dr} \sin t, -\frac{d\bar{f}(r)}{dr} \right), \quad (63)$$

and so the unit normal vector to S in $P(t,r)$ is

$$\hat{\mathbf{n}} = \frac{\frac{\partial P(t,r)}{\partial t} \wedge \frac{\partial P(t,r)}{\partial r}}{\left\| \frac{\partial P(t,r)}{\partial t} \wedge \frac{\partial P(t,r)}{\partial r} \right\|} = \left(\frac{dg(r)}{dr} \cos t, \frac{dg(r)}{dr} \sin t, -\frac{d\bar{f}(r)}{dr} \right). \quad (64)$$

Then, the coefficients of the second fundamental form are

$$\mathbf{e} = \frac{\partial^2 P(t,r)}{\partial r^2} \cdot \hat{\mathbf{n}} = -\bar{f}(r) \frac{dg(r)}{dr} \quad (65)$$

$$\mathbf{f} = \frac{\partial^2 P(t,r)}{\partial r \partial t} \cdot \hat{\mathbf{n}} = 0 \quad (66)$$

$$\mathbf{g} = \frac{\partial P(t,r)}{\partial t^2} = \frac{d^2 \bar{f}(r)}{dr^2} \frac{dg(r)}{dr} - \frac{d\bar{f}(r)}{dr} \frac{d^2 \bar{g}(r)}{dr^2}. \quad (67)$$

To achieve the principal curvatures $k_1(t,r)$ and $k_2(t,r)$, it is sufficient to solve the following algebraic equation

$$\begin{vmatrix} \mathbf{e} - k\mathbf{E} & \mathbf{f} - k\mathbf{F} \\ \mathbf{f} - k\mathbf{F} & \mathbf{g} - k\mathbf{G} \end{vmatrix} = 0, \quad (68)$$

from which we can easily obtain

$$k_1(t,r) = -\frac{\frac{dg(r)}{dr}}{\bar{f}(r)}; \quad k_2(t,r) = \frac{d^2 \bar{f}(r)}{dr^2} \frac{dg(r)}{dr} - \frac{d\bar{f}(r)}{dr} \frac{d^2 \bar{g}(r)}{dr^2}. \quad (69)$$

Thus, the mean curvature $H(t,r)$ assumes the following form

$$H(t,r) = \frac{1}{2}(k_1(t,r) + k_2(t,r)) = \frac{1}{2} \left(-\frac{\frac{dg(r)}{dr}}{\bar{f}(r)} + \frac{d^2 \bar{f}(r)}{dr^2} \frac{dg(r)}{dr} - \frac{d\bar{f}(r)}{dr} \frac{d^2 \bar{g}(r)}{dr^2} \right). \quad (70)$$

In our case, assuming that the curve C lies in the plane $y = 0$, we set:

$$\begin{cases} \bar{f}(r) = r \\ \frac{d\bar{f}(r)}{dr} = 1 \\ \frac{d^2 \bar{f}(r)}{dr^2} = 0 \end{cases} \quad \begin{cases} \bar{g}(r) = u(r) \\ \frac{d\bar{g}(r)}{dr} = \frac{du(r)}{dr} \\ \frac{d^2 \bar{g}(r)}{dr^2} = \frac{d^2 u(r)}{dr^2} \end{cases} \quad (71)$$

where $\frac{d\bar{f}(r)}{dr}$ and $\frac{d^2 \bar{g}(r)}{dr^2}$ satisfy (61). Thus,

$$H(r) = -\frac{1}{2} \left(\frac{1}{r} \frac{du(r)}{dr} + \frac{d^2 u(r)}{dr^2} \right). \quad (72)$$

Taking into account (72), the model (59) can be written as follows:

$$\begin{cases} \frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} = -\frac{\theta\lambda^2}{4(1-u(r)-d^*)^2} \left(\frac{1}{r} \frac{du(r)}{dr} + \frac{d^2u(r)}{dr^2} \right)^2 \\ u(R) = 0 \\ \frac{du(0)}{dr} = 0 \\ 0 < u(r) < d. \end{cases} \quad (73)$$

We observe that, in (73),

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} \quad (74)$$

is always non-zero if $u(r) \neq 0$. In fact, let us suppose that

$$\begin{cases} \frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} = 0 \\ u(R) = 0 \\ \frac{du(0)}{dr} = 0 \\ 0 < u(r) < d; \end{cases} \quad (75)$$

the equation of model (75) admits, as solution, the trivial one $u(r) = 0$ that cannot be considered a solution of the problem (75). In fact, let us suppose absurdly that a solution of the model (73) also satisfies

$$\frac{dy(r)}{dr} + \frac{1}{r}y(r) = 0; \quad (76)$$

from which

$$\frac{dy(r)}{dr} = -\frac{1}{r}y(r) \quad (77)$$

and, again,

$$\int \frac{dy}{y} = -\int \frac{dr}{r} + c_1, \quad (78)$$

obtaining

$$y = y(r) = -e^{\ln r} = c_2 \left(\frac{1}{e} \right)^{\ln r} = \frac{c_2}{r}. \quad (79)$$

Taking into account the substitution $y(r) = \frac{du(r)}{dr}$, we easily achieve

$$u(r) = c_2 \ln(r) + c_3, \quad (80)$$

which is a solution for the differential equation of the model (73), which also satisfies the equation of the model (75). However, although both the differential equations (73) and (75) are satisfied by the solution (80), this does not satisfy the second boundary condition $\frac{du(0)}{dr} = 0$. Then, we can deduce that, in the problem (75), $\frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} \neq 0$. Then, we can divide both sides of the differential equation of model (73) by $\frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr}$, obtaining

$$1 = -\frac{\theta\lambda^2}{4(1-u(r)-d^*)^2} \left(\frac{1}{r} \frac{du(r)}{dr} + \frac{d^2u(r)}{dr^2} \right). \quad (81)$$

Then, we obtain:

$$\frac{1}{r} \frac{du(r)}{dr} + \frac{d^2u(r)}{dr^2} = -\frac{(1-u(r)-d^*)^2}{\theta\lambda^2} \quad (82)$$

and, again,

$$\frac{d^2u(r)}{dr^2} = -\frac{1}{r} \frac{du(r)}{dr} - \frac{(1-u(r)-d^*)^2}{\theta\lambda^2}. \quad (83)$$

Finally, problem (73) can be written as

$$\begin{cases} \frac{d^2u(r)}{dr^2} = -\frac{1}{r} \frac{du(r)}{dr} - \frac{(1-u(r)-d^*)^2}{\theta\lambda^2} \\ u(R) = 0 \\ \frac{du(0)}{dr} = 0 \\ 0 < u(r) < 1 \end{cases} \quad (84)$$

that is the model (5).

6. General Formulation of the Problem

Model (5) can be considered as a special case of the following general problem. Specifically, let us consider a closed interval $\Omega = [a, b]$ in which a singularity takes place at a . Let us consider $u(r) : (a, b) \rightarrow \mathbb{R}$, such that $u(r) \in C^2(\bar{\Omega})$. Then, (84) is a particular case of the general following model:

$$\begin{cases} \frac{d^2u(r)}{dr^2} + \bar{F}\left(r, u(r), \frac{du(r)}{dr}\right) = 0 \\ u(b) = B \\ \frac{du(a)}{dr} = m, \end{cases} \quad (85)$$

where $\bar{F} \in C^0((a, b] \times \mathbb{R} \times \mathbb{R})$ and $B, m \in \mathbb{R}$. If we set

$$\bar{F}\left(r, u(r), \frac{du(r)}{dr}\right) = \frac{1}{r} \frac{du(r)}{dr} + \frac{(1-u(r)-d^*)^2}{\theta\lambda^2}, \quad (86)$$

$B = 0$, and $m = 0$, we observe that the obtained model is (5). We focus our attention on achieving conditions ensuring both existence and uniqueness of the solution for model (5). Section 7 describes a very interesting existence result for problem (5), which is useful for our purposes.

7. A Result of Existence for the Problem Under Study

We now expose and prove our principal goal, regarding the existence of at least one solution for the problem (5).

Theorem. 1 *Let us consider the problem (5) and let $u_1(r)$ and $u_2(r)$ two functions, defined in $[0, R]$ and twice continuously differentiable, such that*

$$u_1(r) < u_2(r) \quad (87)$$

and

$$\frac{d^2u_1(r)}{dr^2} + \frac{1}{r} \frac{du_1(r)}{dr} + \frac{(1-u_1(r)-d^*)^2}{\theta\lambda^2} > 0 \quad (88)$$

$$\frac{d^2u_2(r)}{dr^2} + \frac{1}{r} \frac{du_2(r)}{dr} + \frac{(1-u_2(r)-d^*)^2}{\theta\lambda^2} < 0 \quad (89)$$

for $r \in (0, R)$. In addition, let $\frac{1}{r} \frac{du(r)}{dr} + \frac{(1-u(r)-d^*)^2}{\theta\lambda^2}$ be a continuous function (except for $r = 0$) satisfying the Lipschitz condition in $U \times (-\infty, +\infty)$, where

$$U = \{(r, u) : 0 < r < R \text{ and } u_1(r) \leq u(r) \leq u_2(r)\}. \quad (90)$$

If

$$\frac{du_1(0)}{dr} \geq \frac{du_2(0)}{dr}, \quad (91)$$

$$u_1(R) = u_2(R) = 0, \quad (92)$$

and

$$\theta\lambda^2 > \frac{R^2(d^*)^2}{2\epsilon_0 k}, \quad (93)$$

there exists one at least one solution for the problem (5).

Proof. To prove the theorem, we verify that the hypotheses of Lemma 1 are verified. We can assume the following expressions as $u_1(r)$ and $u_2(r)$:

$$u_1(r) = 0 \quad \forall r \in [0, R] \quad (94)$$

and

$$u_2(r) = \bar{u}(r) = \frac{k\epsilon_0 V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}, \quad (95)$$

derived from (55). Fig. 2 depicts both $u_1(r)$ and $u_2(r)$, as well as a possible recovery of the membrane. Clearly, $u_1(r) < u_2(r)$ and both are twice continuously differentiable functions. Now, we must verify the inequalities (88) and (89); that is

$$\frac{d^2 u_1(r)}{dr^2} + F\left(r, u_1(r), \frac{du_1(r)}{dr}\right) = \frac{d^2 u_1(r)}{dr^2} + \frac{1}{r} \frac{du_1(r)}{dr} + \frac{(1 - u_1(r))^2}{\theta\lambda^2} > 0, \quad (96)$$

$$\frac{d^2 u_2(r)}{dr^2} + F\left(r, u_2(r), \frac{du_2(r)}{dr}\right) = \frac{d^2 u_2(r)}{dr^2} + \frac{1}{r} \frac{du_2(r)}{dr} + \frac{(1 - u_2(r) - d^*)^2}{\theta\lambda^2} < 0. \quad (97)$$

To verify (96), we observe that, for $u_1(r) = 0 \quad \forall r \in [0, R]$, we trivially have

$$\frac{du_1(r)}{dr} = \frac{d^2 u_1(r)}{dr^2} = 0. \quad (98)$$

Then, if we assume that

$$\frac{1}{\theta\lambda^2} > 0, \quad (99)$$

or, equivalently,

$$\theta\lambda^2 > 0, \quad (100)$$

(96) is verified. To verify (97) (see (95)), with

$$u_2(r) = \frac{\epsilon_0 k}{2} \frac{V^2}{(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\}, \quad (101)$$

we have

$$u_2'(r) = -\frac{\epsilon_0 k V^2}{(d^*)^2} \left\{ \frac{r}{R^2} \right\} \quad (102)$$

and

$$u_2''(r) = -\frac{\epsilon_0 k V^2}{R^2 (d^*)^2}. \quad (103)$$

Then, taking into account both (102) and (103), the inequality (96) becomes

$$-\frac{\epsilon k V^2}{R^2(d^*)^2} - \frac{\epsilon k V^2}{R^2(d^*)^2} + \frac{1}{\theta \lambda^2} \left\{ 1 - \frac{\epsilon_0 k V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\} \right\}^2 < 0, \quad (104)$$

from which we get

$$-\frac{2\epsilon k V^2}{R^2(d^*)^2} + \frac{1}{\theta \lambda^2} \left\{ 1 - \frac{\epsilon_0 k V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\} \right\}^2 < 0, \quad (105)$$

so that:

$$\frac{1}{\theta \lambda^2} \left\{ 1 - \frac{\epsilon_0 k V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\} \right\}^2 < \frac{2\epsilon k V^2}{R^2(d^*)^2}. \quad (106)$$

We note that, in inequality (106), the quantity

$$\left\{ 1 - \frac{\epsilon_0 k V^2}{2(d^*)^2} \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\} \right\}^2 < 1, \quad (107)$$

such that, if we impose

$$\frac{1}{\theta \lambda^2} < \frac{2\epsilon k V^2}{R^2(d^*)^2}, \quad (108)$$

and so

$$\theta \lambda^2 > \frac{R^2(d^*)^2}{2\epsilon_0 k}, \quad (109)$$

it follows that the inequality (97) is surely satisfied. Then, for both the inequalities (96) and (97) to be verified, we must take into account both inequalities (100) and (109), which implies that

$$\theta \lambda^2 > \frac{R^2(d^*)^2}{2\epsilon_0 k} \quad (110)$$

satisfies both the inequalities (96) and (97). As above mentioned, T represents the mechanical tension of the membrane; θ is a parameter proportional to the applied voltage V and λ^2 takes also into account the electro-mechanical properties of the material constituting the membrane. Figure 3 depicts, in the plane $d^* - \theta \lambda^2$, the zone of existence of at least one solution for the problem (5). Particularly, the line of equation

$$\theta \lambda^2 = \frac{R^2(d^*)^2}{2\epsilon_0 k}, \quad (111)$$

shown in Figure 3 as a black line, separates the area of existence at least one solution for (5) (light green area) from that which ensure the non-existence of solution for (5) (light red area). As Lemma 1 requires, we also need to prove that

$$F\left(r, u(r), \frac{du(r)}{dr}\right) = \frac{1}{r} \frac{du(r)}{dr} + \frac{(1-u(r))^2}{\theta \lambda^2} \quad (112)$$

satisfies the Lipschitz's condition (17). Then we easily prove that:

$$\begin{aligned}
 F\left(r, u(r) - v(r), \frac{du(r)}{dr} - \frac{dv(r)}{dr}\right) &= \frac{1}{r} \frac{du(r)}{dr} + \frac{(1-u(r))^2}{\theta\lambda^2} - \frac{1}{r} \frac{dv(r)}{dr} - \frac{(1-v(r))^2}{\theta\lambda^2} = \quad (113) \\
 &= \frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) + \frac{1}{\theta\lambda^2} \left\{ (1-u(r) - 1 + v(r))(1-u(r) + 1 - v(r)) \right\} = \\
 &= \frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) + \frac{(-u(r) + v(r))(2 - (u(r) + v(r)))}{\theta\lambda^2} = \\
 &= \frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) - \frac{(u(r) - v(r))(2 - (u(r) + v(r)))}{\theta\lambda^2} \geq \\
 &\geq \frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) - \frac{2}{\theta\lambda^2} (u(r) - v(r)) = L_2(r) \left(\frac{du(r)}{dr} - \frac{dv(r)}{dr} \right) + K_1(r) (u(r) - v(r)).
 \end{aligned}$$

In addition,

$$\begin{aligned}
 &\frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) - \frac{1}{\theta\lambda^2} ((u(r) - v(r))(2 - (u(r) + v(r)))) \leq \quad (114) \\
 &\leq \frac{1}{r} \left(\frac{u(r)}{dr} - \frac{dv(r)}{dr} \right) - \frac{1}{\theta\lambda^2} \{Z(u - v)\} = L_1(r) \left(\frac{du(r)}{dr} - \frac{dv(r)}{dr} \right) + K_2(r) (Z(u(r) - v(r))).
 \end{aligned}$$

As $2 - (u(r) + v(r)) \geq 0$, then there exists a constant Z such that $0 < Z < 2 - (u(r) + v(r))$. Finally, Lemma 1 requires that $\frac{du_1(a)}{dr} \geq \frac{du_2(a)}{dr}$. For this purpose, as $a = r = 0$, we obtain:

$$\frac{du_1(a)}{dr} = \frac{du_1(0)}{dr} = 0. \quad (115)$$

In addition,

$$\frac{du_2(a)}{dr} = \frac{du_2(0)}{dr} = 0. \quad (116)$$

Moreover, $u_1(R) = u_2(R) = 0$. Thus, the proof of the theorem is completed.

□

Remark 4. It is worth noting that condition (110) has an important physical meaning. In fact, taking into account (51), (110) can be written as follows:

$$\theta\lambda^2 > \frac{R^2(d^*)^2}{2\epsilon_0 k} = \frac{R^2(d^*)^2}{2\epsilon_0 k_1 k_2}. \quad (117)$$

As $k_1 = \frac{R^2}{4T}$, (117) becomes:

$$\theta\lambda^2 > \frac{2T(d^*)^2}{\epsilon_0 k_2}. \quad (118)$$

Thus, the greater k_2 is, the lower the value of $\theta\lambda^2$ will be and, so, in the problem under study (5), $\frac{d^2 u(r)}{dr^2}$ will be smaller; that is, the concavity of the membrane will rise. In other words, the greater the value of k_2 is, the greater the influence of the electrostatic pressure will be. Then, the mechanical pressure will rise (as $p = k_1 k_2 p_{el}$), with a consequent increase of deformation in the membrane.

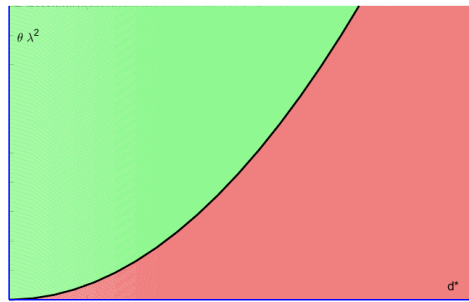


Figure 3. The plane $d^*\theta\lambda^2$ is divided in two zones by the line of equation $\theta\lambda^2 = \frac{R^2(d^*)^2}{2\epsilon_0 k}$ (black line): the light green area represents the zone of existence at least one solution for (5); the light red area ensures the non-existence of solutions for (5).

8. On the Uniqueness of the Solution

Although the problem (5) admits at least one solution $u(r)$ such that $u_1(r) < u(r) < u_2(r)$ with $u_1(r)$ and $u_2(r)$, verifying the hypothesis of the Theorem 1, its uniqueness has not been admitted. This Section proves this fact through two alternative approaches.

Theorem. 2 Let us consider the problem (5). Let us also suppose that the conditions of the Theorem 1 are satisfied and that $u_1(r)$ and $u_2(r)$ satisfy the given boundary conditions. Then, the uniqueness of the solution $u(r)$, such that $u_1(r) \leq u(r) \leq u_2(r)$, is not guaranteed.

Proof. As specified in (114), we can write:

$$L_1(r) \frac{du(r)}{dr} + K_2(r)Zu(r) = \frac{1}{r} \left(\frac{du(r)}{dr} \right) - \frac{Z}{\theta\lambda^2} u(r). \quad (119)$$

Thus, exploiting Lemma 2, we can consider the ordinary differential equation:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r} \left(\frac{du(r)}{dr} \right) - \frac{Z}{\theta\lambda^2} u(r) = 0, \quad (120)$$

which can be considered as a particular case of the following Bessel equation

$$r^2 \frac{d^2u(r)}{dr^2} + (2\bar{k} + 1)r \frac{du(r)}{dr} + (\alpha^2 r^{2s} + \beta^2)u(r) = 0 \quad (121)$$

with $2s \neq 0$ and $\bar{k}, \alpha, s, \beta \in \mathbb{C}$. In fact, from the above equation, we can write:

$$\frac{d^2u(r)}{dr^2} + \frac{2\bar{k} + 1}{r} \frac{du(r)}{dr} + \left(\frac{\alpha^2 r^{2s}}{r^2} + \frac{\beta^2}{r^2} \right) u(r) = 0, \quad (122)$$

from which (in our case, assuming $\beta = 0$ and $\bar{k} = 0$), equation (122) becomes:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \alpha^2 r^{2s-2} u(r) = 0. \quad (123)$$

If, then, $2s - 2 = 0$, it follows that $r = 1$, and so we obtain:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \alpha^2 u(r) = 0. \quad (124)$$

Finally, setting $\alpha^2 = -\frac{Z}{\theta\lambda^2} \in \mathbb{C}$, we obtain the equation (120). As known from the Bessel theory [27,30], the general solution for (120) can be written as a linear combination of two linearly independent Bessel functions of the first and second kind of zeroth order, respectively; $J_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)$ and $Y_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)$:

$$u(r) = c_1 J_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right) + c_2 Y_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right), \quad (125)$$

where c_1 and c_2 are constant [27,30]. It is known that

$$J_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)^{2m}}{2^{2m}(m!)^2} \quad (126)$$

and

$$Y_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right) = \frac{2}{\pi} \left[\left(\gamma + \ln\left(0.5\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)\right) \right) J_0\left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} \left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)^{2m} \right], \quad (127)$$

in which γ is the Euler–Mascheroni constant (its value is 0.5772). Moreover,

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}. \quad (128)$$

From the above relations, we obtain the general solution for equation (120):

$$\begin{aligned} u(r) = & c_1 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)^{2m}}{2^{2m}(m!)^2} \right] + \\ & + c_2 \frac{2}{\pi} \left[\left(\gamma + \ln\left(0.5\sqrt{\frac{Z}{\theta\lambda^2}}r\right) \right) \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)^{2m}}{2^{2m}(m!)^2} \right] + \right. \\ & \left. + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} \left(r \sqrt{\frac{Z}{\theta\lambda^2}}r \right)^{2m} \right]. \end{aligned} \quad (129)$$

It is worth noting that, as $r \rightarrow 0$, we have $J_0 \rightarrow 1$. Meanwhile, due to the presence of $\ln\left(0.5\sqrt{\frac{Z}{\theta\lambda^2}}r\right)$, Y_0 presents a logarithmic singularity as $r = 0$. However, taking a linear combination with $c_1 \neq 0$ and $c_2 = 0$, we find that the general integral of the differential equation assumes the form:

$$u(r) = c_1 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\sqrt{\frac{Z}{\theta\lambda^2}}r\right)^{2m}}{2^{2m}(m!)^2} \right]. \quad (130)$$

We have also found a non-trivial solution for the ordinary differential equation (120) which differs from the trivial one $u(r) = 0$. By Lemma 2, we deduce that uniqueness of the solution of the problem (5) is not guaranteed.

Alternatively, we studied an alternative approach to search a non-trivial solution for the differential equation (120). In fact, multiplying equation (120) by r , we obtain

$$r \frac{d^2 u(r)}{dr^2} + \frac{du(r)}{dr} - \bar{K}ru(r) = 0, \quad (131)$$

with $\bar{K} = \frac{Z}{\theta\lambda^2}$. Equation (131) can be considered as a particular case of the following equation:

$$P(r)\frac{d^2u(r)}{dr^2} + Q(r)\frac{du(r)}{dr} + R(r)u(r) = 0, \quad (132)$$

in which, we can write

$$\begin{cases} P(r) = r \\ Q(r) = 1 \\ R(r) = -\bar{K}r. \end{cases} \quad (133)$$

Moreover, we define

$$\begin{cases} p(r) = \frac{Q(r)}{P(r)} = \frac{1}{r} \\ q(r) = \frac{R(r)}{P(r)} = -\bar{K}. \end{cases} \quad (134)$$

In our case $r_0 = 0$, and so (see Definition 1):

$$\begin{cases} \lim_{r \rightarrow 0} r \frac{1}{r} = 1 \quad \text{is finite, and} \\ \lim_{r \rightarrow 0} r^2(-\bar{K}) = 0 \quad \text{is finite,} \end{cases} \quad (135)$$

from which, we deduce that $r_0 = 0$ is a regular singular point for (131). In addition, equation (132), taking into account (134), can be easily written as follows:

$$r^2u'' + r\{rp(r)\}u' + \{r^2q(r)\}u = 0, \quad (136)$$

such that Lemma 3 can be applied [27]. In our case, by (26), we can write

$$rp(r) = 1 = p_0 + p_1r + p_2r^2 + \dots, \quad (137)$$

from which

$$p_0 = 1; \quad p_1 = p_2 = p_3 = \dots = 0. \quad (138)$$

Again,

$$r^2q(r) = -\bar{K}r^2 = q_0 + q_1r + q_2r^2 + q_3r^3 + \dots, \quad (139)$$

so that

$$q_0 = q_1 = q_3 = q_4 = q_5 = \dots = 0; \quad q_2 = -\bar{K}. \quad (140)$$

Then, the indicial equation (28) becomes

$$F(t) = t(t-1) + t = 0, \quad (141)$$

whose roots are $t_1 = t_2 = 0$. Then, applying Lemma 3, we obtain

$$u_2(r) = u_1(r) \ln |r| + \sum_{n=1}^{\infty} b_n(t_1)r^n. \quad (142)$$

To obtain $b_n(t_1)$, it is sufficient to substitute the series solution for u in (136). Then, we can finally conclude that:

$$u(r) = \bar{c}_1 u_1(r) + \bar{c}_2 u_2(r) = \quad (143)$$

$$= \bar{c}_1 |r|^{t_1} \left[1 + \sum_{n=1}^{\infty} a_n(t_1) r^n \right] + \bar{c}_2 \left[u_1(r) \ln |r| + \sum_{n=1}^{\infty} b_n(t_1) r^n \right], \quad (144)$$

such that \bar{c}_2 may turn out to be zero, in which case there is no logarithmic term in the solution. As equation (120) admits solutions different from the trivial one, we deduce that uniqueness of the solution of the problem (5) is not guaranteed. \square

9. Conclusion and Perspectives

In this work, a 2D electrostatic non-linear II-order differential model for circular membrane MEMS devices is presented and studied as well. In particular, a rapid introduction opens the paper introducing the reader to the proposed 2D model based on the proportionality between $|\mathbf{E}|$ and the mean curvature of the membrane. After to have presented some preliminary Lemmas, the device is detailed both from the point of view of the actuator and from the point of view of the transducer. Moreover, the link between mechanical pressure p and electrostatic pressure p_{el} here exploited gives the possibility to consider as dual the transducer-actuator model. Then, the proposed model is detailed in terms of mean curvature exploiting both the first e second differential forms as differential geometry suggests. Exploiting a result well-known in scientific literature, we have proved the existence at least a solution for the proposed model giving, on a suitable 2D plane and exploiting an achieved algebraic condition, the area where the existence at least one solution is ensured and the area in which the non-existence of the solution is guaranteed. However, as detailed at the end of the paper, the uniqueness of the solution has not been admitted. Although the achieved results are encouraging, it makes sense to classify the problem as ill-posed in the sense of Hadamard's second statement and ask, as future perspective, whether any additional conditions does not make it regular.

Author Contributions: The contributions of the authors are to be considered equal.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MEMS Micro-Electro-Mechanical Systems

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