Free Surface Flows in Electrohydrodynamics with a Constant Vorticity Distribution

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Abstract

In 1895, Korteweg and de Vries (KdV), [8], derived their celebrated equation describing the motion of waves of long wavelength in shallow water. In doing so they made a number of quite reasonable assumptions, incompressibility of the water and irrotational fluid. The resulting equation, the celebrated KdV equation, has been shown to be a very reasonable description of real water waves. However there are other phenomena which have an impact on the shape of the wave, that of vorticity and viscosity. This paper examines how a constant vorticity affects the shape of waves in electrohydrodynamics. For constant vorticity, the vertical component of the velocity obeys a Laplace equation and also has the usual lower boundary condition. In making the vertical component of the velocity take central stage, the Burns condition can be thus bypassed.

1 Introduction

Water waves constitute a very classical problem in hydrodynamics [3]. However, this problem is traditionally formulated in terms of the velocity potential to achieve some simplifications. In other words, there has always been an implicit assumption of zero vorticity in the flow region. In numerous recent studies this assumption started to be

questioned. One of pioneering studies was made by Burns (1953) [2]. Later, Da Silva and Peregrine (1988) [4] studied steep and steady waves on finite depth with constant vorticity. A constant distribution is the next logical step after identically zero distribution. More recently, this problem was analyzed mathematically in some two-component systems [7]. The effect of the vorticity on travelling wave solutions (solitary and cnoidal) was investigated in the purely hydrodynamic context in [6] using the qualitative phase space analysis methods. A Hamiltonian formulation has been reported in [9]. However, this problem in electrohydrodynamics seems to be still open to the best of our knowledge. The present study should be considered as a further attempt to fill in this gap in the literature.

The current approach to examining flows with constant vorticity in two dimensions is via the use of a stream function, ψ, and it's harmonic conjugate, the velocity potential, φ, so $\mathbf{u} = \nabla\varphi + \nabla^\perp\psi$. This approach is introduces two essentially unnecessary functions which complicates the problem and has the limiting effect in being restricted to fully nonlinear and linear computations. There has been no attempt to undertake a weakly nonlinear analysis which is the purpose of this manuscript.

The present manuscript is organized as follows. The problem is formulated in Section 2. The linear analysis of this problem is performed in Section 3, while the weakly nonlinear analysis is presented in Section 4. Some numerical predictions of the weakly nonlinear theory are presented in Section 5. Finally, the main conclusions and perspectives of this study are outlined in section 6.

2   Formulation

A two dimensional fluid in region 1 is considered which is incompressible and inviscid. The vorticity, ω is constant as is the surface tension σ. Cartesian co-ordinates are introduced as shown in figure 1. Region 1 is defined as $-h < y < \eta(t,x) \; \forall x \in \mathbb{R}$. The moving pressure distribution $\mathcal{P}(t,x)$ is chosen to act along the interface $y = \eta(t,x)$ and $\mathcal{P} \to 0$ as $|x| \to 0$.

In region 2 defined by $\{(x,y)|x \in \mathbb{R}, y > \eta(t,x)\}$ there is an electric field, $\mathbf{E}$ which has no charges and is therefore obtained by a potential $\mathbf{E} = -\nabla V$. The potential is chosen as $V(x,-h) = 0$ and as the fluid is perfectly conducting this also means that $V(x,\eta(t,x)) = 0$. A vertical electric field is set up by imposing:

$$V \sim -E_0 y, \quad y \to \infty \tag{1}$$

The equation for the electric potential is therefore given by:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{2}$$

with the condition:

$$\left[\mathbf{E} \cdot \hat{\mathbf{t}}\right]_1^2 = 0 \tag{3}$$

which is one of the general boundary conditions derivable from Maxwell's equations. In region 1, the Navier-Stokes equations are used with the stress tensor:

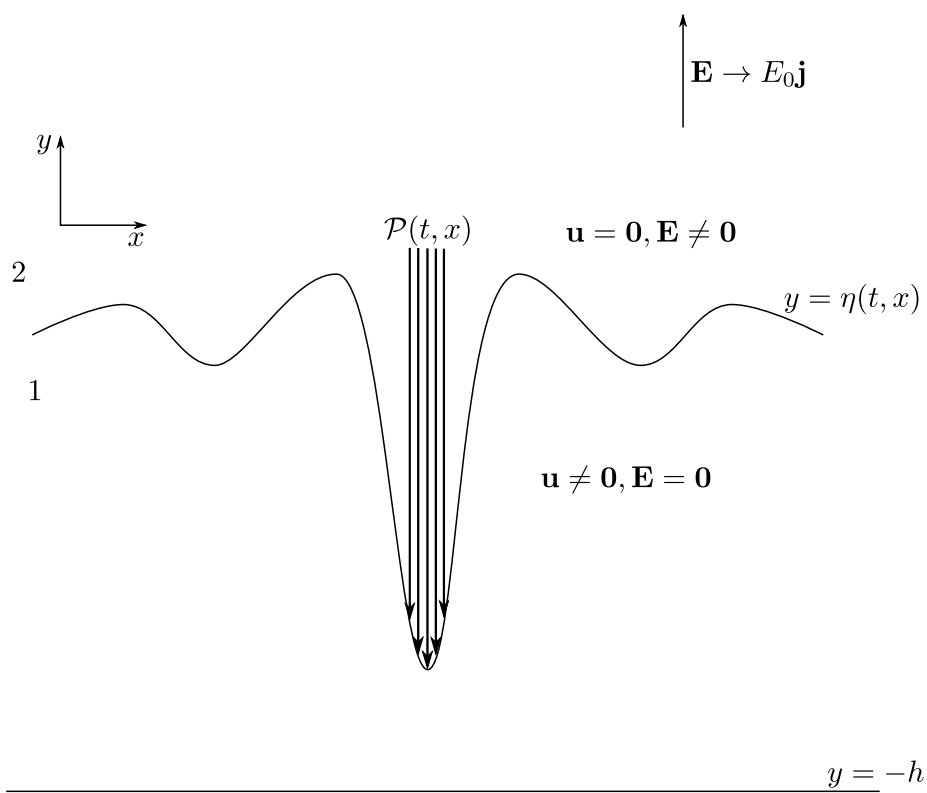$$T_{ij} = -P\delta_{ij} + \Sigma_{ij} \tag{4}$$

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Figure 1: *Physical problem schematic representation.*

Where:

$$\Sigma_{ij} = \epsilon_p \left(\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} - \frac{1}{2} \delta_{ij} \sum_{k=1}^{2} \frac{\partial V}{\partial x_k} \frac{\partial V}{\partial x_k} \right). \tag{5}$$

Where ϵ_p is called the electric permittivity. The tensor Σ_{ij} has various names, in the fluids literature it is known as the Maxwell-stress tensor. It can be shown that:

$$\sum_{i=1}^{2} \frac{\partial \Sigma_{ij}}{\partial x_i} = 0. \tag{6}$$

So The Navier-Stokes equations reduce to the Euler equations.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla P - g\mathbf{j}, \tag{7}$$

where $\mathbf{u} = (u, v)$ is the velocity and P is the pressure in the fluid. The boundary $y = -h$ is taken as impenetrable, so $v(x, -h) = 0$. The fluid is incompressible and has constant vorticity, ω and so:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega. \tag{8}$$

These equations can be cross differentiated to obtain a single equation for v,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{9}$$

The free surface equation is given by:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v, \quad y = \eta(t, x). \tag{10}$$

This gives a boundary condition for v on $y = \eta(t, x)$. The lower boundary for v is given by:

$$v(t, x, -h) = 0 \tag{11}$$

Equations (9)-(11) are the core of the technique in deriving the free surface profiles. The other boundary condition used is the Young-Laplace equation given by:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_1^2 = \sigma \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{\frac{3}{2}}} \tag{12}$$

The Euler equations may be simplified using electrostatic equilibrium. The equilibrium condition is:

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} - g = 0 \tag{13}$$

Integrating this equation shows that $p = -\rho g y + C$ To compute the value of C, use the Young-Laplace equation to see $C = P_a - \epsilon_d E_0^2/2$, here P_a is the atmospheric pressure. So now write:

$$P = P_a - \frac{\epsilon_p E_0^2}{2} - \rho g y + p \tag{14}$$

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The Euler equations now become:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{15}$$

The Young-Laplace equation becomes:

$$p - \rho g \eta - \frac{\epsilon_p E_0^2}{2} = \mathcal{P} - \frac{1}{1 + (\partial_x \eta)^2}\left[(\partial_x \eta)^2 \Sigma_{11} - 2\partial_x \eta \Sigma_{12} + \Sigma_{22}\right] - \sigma \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{\frac{3}{2}}} \tag{16}$$

3 Linear Theory

The scaling for the linear theory is:

$$(x, y, \eta) = h(\hat{x}, \hat{y}, \hat{\eta}), \quad t = \sqrt{\frac{\rho h^3}{\sigma}}\hat{t}, \quad (p, \mathcal{P}) = \frac{\sigma}{h}(\hat{p}, \hat{\mathcal{P}}), \quad \mathbf{u} = \sqrt{\frac{\sigma}{\rho h}}\hat{\mathbf{u}} \tag{17}$$

$$\mathcal{P} = \rho g h \hat{\mathcal{P}} \quad V = h E_0 \hat{V} \tag{18}$$

The equations become:

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} = 0, \quad \hat{y} > \hat{\eta} \tag{19}$$

$$\frac{\partial \hat{V}}{\partial \hat{x}} + \frac{\partial \hat{\eta}}{\partial \hat{x}}\frac{\partial \hat{V}}{\partial \hat{y}} = 0, \quad \hat{y} = \hat{\eta} \tag{20}$$

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\hat{\nabla}\hat{p}, \quad -1 < \hat{y} < \hat{\eta} \tag{21}$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \quad \frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\partial \hat{u}}{\partial \hat{y}} = \Omega, \quad -1 < \hat{y} < \hat{\eta} \tag{22}$$

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{\eta}}{\partial \hat{x}} = \hat{v}, \quad \hat{y} = \hat{\eta} \tag{23}$$

$$\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} = 0 \tag{24}$$

$$\hat{p} - B\hat{\eta} - \frac{E_b}{2} = \hat{\mathcal{P}} - \frac{E_b}{1 + (\partial_{\hat{x}}\hat{\eta})^2}\left[(\partial_{\hat{x}}\hat{\eta})^2 \hat{\Sigma}_{11} - 2\partial_{\hat{x}}\hat{\eta}\hat{\Sigma}_{12} + \hat{\Sigma}_{22}\right] -$$
$$- \frac{\partial_{\hat{x}}^2 \hat{\eta}}{(1 + (\partial_{\hat{x}}\hat{\eta})^2)^{\frac{3}{2}}}, \quad \hat{y} = \hat{\eta} \tag{25}$$

Where:

$$\Omega = \omega\sqrt{\frac{\rho h^3}{\sigma}}, \quad B = \frac{\rho g h^2}{\sigma}, \quad E_b = \frac{\epsilon_p E_0^2}{\sigma} \tag{26}$$

Expand according to:

$$\hat{u} = -\Omega \hat{y} + \varepsilon u_1 + o(\varepsilon) \tag{27}$$

$$\hat{v} = \varepsilon v_1 + o(\varepsilon) \tag{28}$$

$$\hat{p} = \varepsilon p_1 + o(\varepsilon) \tag{29}$$

$$\hat{\mathcal{P}} = \varepsilon \mathcal{P}_1 + o(\varepsilon) \tag{30}$$

$$\hat{\eta} = \varepsilon \eta_1 + o(\varepsilon) \tag{31}$$

$$\hat{V} = -\hat{y} + \varepsilon V_1 + o(\varepsilon) \tag{32}$$

The set of linear equations now becomes:

$$\frac{\partial^2 V_1}{\partial \hat{x}^2} + \frac{\partial^2 V_1}{\partial \hat{y}^2} = 0, \quad \hat{y} > 0, \tag{33}$$

$$\frac{\partial V_1}{\partial \hat{x}} - \frac{\partial \eta_1}{\partial \hat{x}} = 0, \quad \hat{y} = 0, \tag{34}$$

$$\frac{\partial u_1}{\partial \hat{t}} - \Omega y \frac{\partial u_1}{\partial \hat{x}} - \Omega v_1 = -\frac{\partial p_1}{\partial \hat{x}}, \quad -1 < \hat{y} < 0 \tag{35}$$

$$\frac{\partial v_1}{\partial \hat{t}} - \Omega y \frac{\partial v_1}{\partial \hat{x}} = -\frac{\partial p_1}{\partial y}, \quad -1 < \hat{y} < 0 \tag{36}$$

$$\frac{\partial^2 v_1}{\partial \hat{x}^2} + \frac{\partial^2 v_1}{\partial \hat{y}^2} = 0, \quad -1 < \hat{y} < 0 \tag{37}$$

$$\frac{\partial \eta_1}{\partial \hat{t}} = v_1, \quad \hat{y} = 0, \tag{38}$$

$$p_1 - B\eta_1 = \mathcal{P}_1 + E_b \frac{\partial V_1}{\partial \hat{y}} - \frac{\partial^2 \eta_1}{\partial \hat{x}^2} \tag{39}$$

3.1   General Dispersion Relation

To obtain a dispersion relation set $\mathcal{P}_1 = 0$ and write all perturbations in the form:

$$f(t, x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi(k), k, y) e^{i(kx - \xi t)} dk \tag{40}$$

Solving the equation for v_1 and V_1 shows that:

$$v_1 = \alpha \sinh k(y+1), \quad V_1 = \beta e^{-|k|y} \tag{41}$$

Using equations (34) and (38) yields β in terms of α:

$$\beta = -\frac{i}{\xi} \alpha \sinh k \tag{42}$$

Only p_1 at the surface is required, so it is possible to set $y = 0$ in equation (35) to find:

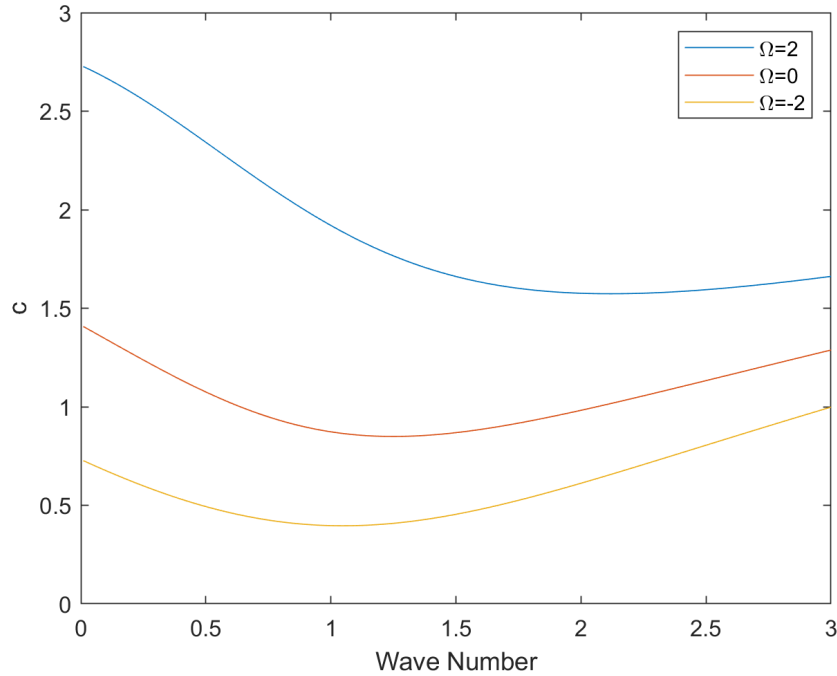$$\hat{p}_1 = \frac{\alpha i}{k} (\xi \cosh k - \Omega \sinh k) \tag{43}$$

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Figure 2: *Linear dispersion relation with constant vorticity for various values of parameter* Ω.

Inserting everything into the linearised Young-Laplace equations shows:

$$\xi^2 - \xi\Omega\tanh k - Bk\tanh k + E_b k|k|\tanh k - k^3\tanh k = 0 \qquad (44)$$

The phase velocity, c is given by $c = \xi/k$ and solving for the phase velocity shows that:

$$c = \frac{\Omega\tanh k}{2k} \pm \frac{1}{2k}\sqrt{\Omega^2\tanh^2 k + 4(Bk - E_b k|k| + k^3)\tanh k} \qquad (45)$$

It can be seen that setting $\Omega = 0$ reduces to the dispersion relation in (insert reference here). It was noted in (insert ref again) that in order to have a linear wave profile the parameters had to satisfy the inequality $4B \geqslant E_b^2$, however with the inclusion of positive vorticity this is no longer the case.

As can be seen in figure 2, there is a minimum which is positive. For various choices of B and E_b, there is a positive minimum for a wide range of vorticity(figure 3). By differentiation of equation (45), one can show:

$$\left.\frac{\mathrm{d}c}{\mathrm{d}k}\right|_{k=0} = -\frac{E_b}{\sqrt{\Omega^2 + 4B}} < 0 \qquad (46)$$

on the branch for which $c(0) > 0$. One can also show that for large k, $c(k) \sim \sqrt{k}$ as $k \to \infty$ which shows the existence of a minimum in the dispersion relation. The beginning point
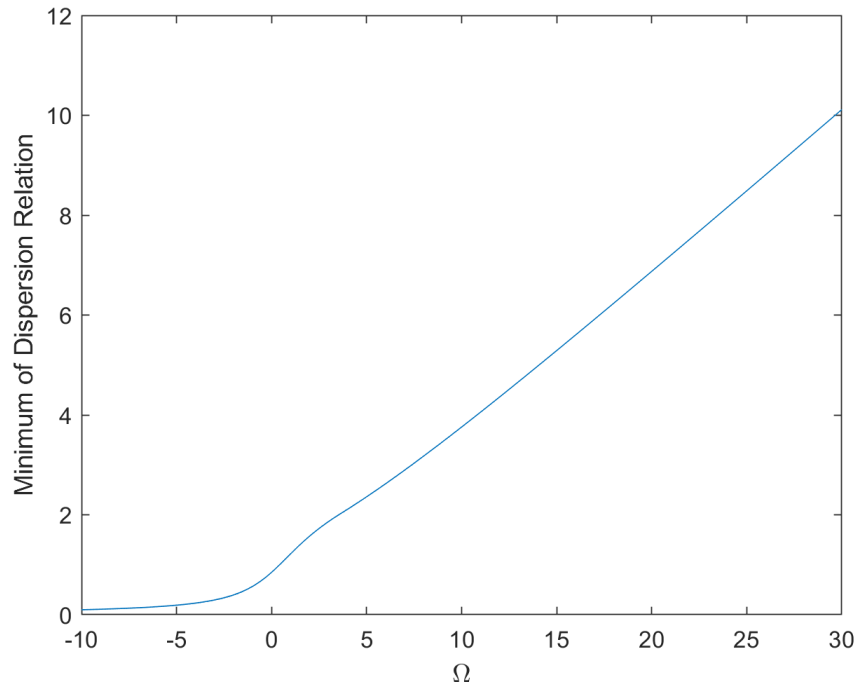
Figure 3: *Dispersion relations for various values of the vorticity parameter Ω, with $B = 0.1$ and $E_b = \sqrt{0.2}$.*

at $k = 0$ can be seen to satisfy the equation:

$$c^2 - \Omega c - 4B = 0 \Rightarrow c = \frac{\Omega \pm \sqrt{\Omega^2 + 4B}}{2} \tag{47}$$

3.2   Free Surface Profiles

Consider a moving pressure distribution moving with non-dimensional speed U. Then a frame of reference moving with speed U is selected and all time derivative terms may be dropped. The horizontal velocity component is expanded as:

$$\hat{u} = U - \Omega y + \varepsilon u_1 + o(\varepsilon) \tag{48}$$

The equations which are changed are then:

$$(U - \Omega y)\frac{\partial u_1}{\partial x} - \Omega v_1 = -\frac{\partial p_1}{\partial x} \tag{49}$$

$$(U - \Omega y)\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y} \tag{50}$$

$$U\frac{\partial \eta_1}{\partial x} = \eta_1 \tag{51}$$

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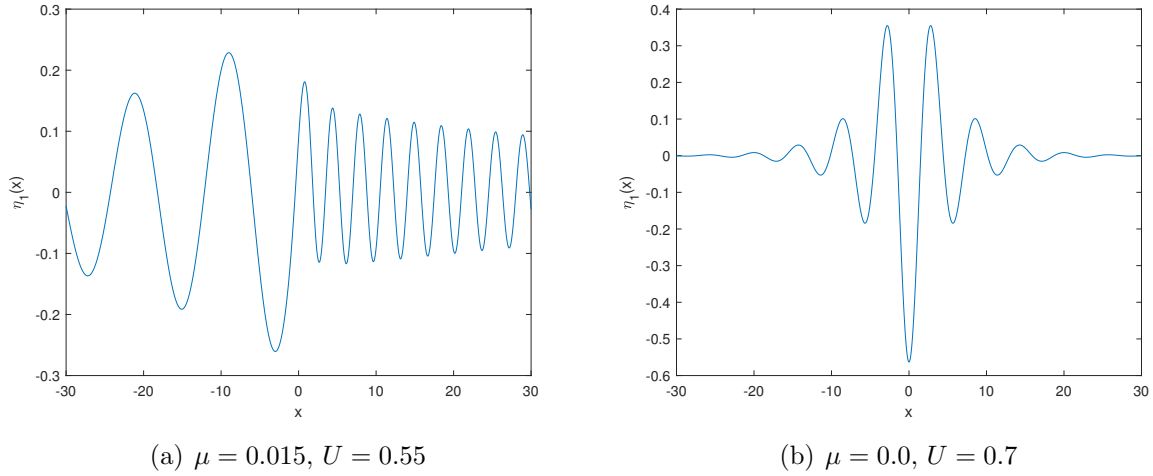(a) $\mu = 0.015$, $U = 0.55$ (b) $\mu = 0.0$, $U = 0.7$

Figure 4: *Free surface profiles under a moving pressure distribution.*

The method of derivation is very similar to that of the derivation of the dispersion relation will be omitted. The perturbations will be expressed at:

$$f(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(k,y)e^{ikx}dk. \tag{52}$$

The free surface is given by:

$$\hat{\eta} = \frac{\hat{\mathcal{P}}_1 \tanh k}{k\hat{U}^2 + (-B + \Omega\hat{U} + E_b|k| - k^2)\tanh k} \tag{53}$$

Figures 4(a) and 4(b) use $B = E_b = 2$ and $\Omega = 1$.

4   The Weakly Nonlinear Free Surface

To obtain a weakly nonlinear model of the phenomena, scale according to:

$$x = \lambda\hat{x}, \quad y^{(1)} = h\hat{y}, \quad y^{(2)} = \lambda\hat{y}, \quad t = \frac{\lambda}{c_0}\hat{t}, \quad u = c_0\hat{u}, \quad v = \frac{hc_0}{\lambda}\hat{v} \tag{54}$$

$$\eta = a\hat{\eta}, \quad V = \lambda E_0\hat{V}, \quad P = P_a - \rho gy - \frac{\epsilon_d E_0^2}{2} + \rho gh\hat{p} \tag{55}$$

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where $c_0 = \sqrt{gh}$. The scaled equations are then:

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} = 0$$

$$\frac{\partial \hat{V}}{\partial x} + \alpha \sqrt{\beta} \frac{\partial \hat{\eta}}{\partial \hat{x}} \frac{\partial \hat{V}}{\partial \hat{y}} = 0$$

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{p}}{\partial \hat{x}}$$

$$\beta \left(\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}}$$

$$\beta \frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\partial \hat{u}}{\partial \hat{y}} = \Omega$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0$$

$$\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{\eta}}{\partial \hat{x}} = \frac{\hat{v}}{\varepsilon}$$

Where $\Omega = h\omega/c_0$. The Young-Laplace equation becomes:

$$\hat{p} - \alpha\hat{\eta} - \frac{F_E^2}{2} = \alpha\hat{\mathcal{P}} - \frac{F_E^2}{1 + \alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2}(\alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2\hat{T}_{11} - 2\alpha\sqrt{\beta}\partial_{\hat{x}}\hat{\eta}\hat{T}_{12} + \hat{T}_{22}) -$$

$$- B\alpha\beta\frac{\partial_{\hat{x}}^2\hat{\eta}}{(1 + \alpha^2\beta(\partial_{\hat{x}}\hat{\eta})^2)^{\frac{3}{2}}} \quad (56)$$

Where:

$$F_E^2 = \frac{\epsilon_d E_0^2}{\rho g h}, \quad B = \frac{\sigma}{\rho g h^2} \quad (57)$$

The term F_E, is the ratio of a velocity to c_0 which shows that there is a natural velocity occurring which is given by, $U = \sqrt{\epsilon_d E_0^2/\rho}$, for this reason F_E should be referred to as the electric Froude number. The next step is to make the transformation:

$$(\hat{p}, \hat{u}, \hat{v}) = (\alpha\bar{p}, -\Omega y + \alpha\bar{u}, \alpha\bar{v}) \quad (58)$$

The KdV scaling is $\alpha = \beta = \varepsilon$. The speed of propagation, c of the (linear) waves is unknown at this point; thus the following co-ordinate transformation is used:

$$X = \hat{x} - c\hat{t}, \quad T = \varepsilon\hat{t} \quad (59)$$

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Dropping bars and hats the equations then become:

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad y > \varepsilon\eta \tag{60}$$

$$\frac{\partial V}{\partial X} + \varepsilon^{\frac{3}{2}}\frac{\partial \eta}{\partial X}\frac{\partial V}{\partial y} = 0, \quad y = \varepsilon\eta \tag{61}$$

$$-c\frac{\partial u}{\partial X} + \varepsilon\frac{\partial u}{\partial T} + \varepsilon u\frac{\partial u}{\partial X} + \varepsilon v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial X}, \quad -1 < y < \varepsilon\eta \tag{62}$$

$$\varepsilon\left(-c\frac{\partial v}{\partial X} + \varepsilon\frac{\partial v}{\partial T} + \varepsilon u\frac{\partial v}{\partial X} + \varepsilon v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}, \quad -1 < y < \varepsilon\eta \tag{63}$$

$$-c\frac{\partial \eta}{\partial X} + \varepsilon\frac{\partial \eta}{\partial T} + \varepsilon u\frac{\partial \eta}{\partial X} = v, \quad y = \varepsilon\eta \tag{64}$$

$$\varepsilon\frac{\partial v}{\partial X} - \frac{\partial u}{\partial y} = 0, \quad -1 < y < \varepsilon\eta \tag{65}$$

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial y} = 0, \quad -1 < y < \varepsilon\eta \tag{66}$$

$$p - \eta - \frac{F_E^2}{2\varepsilon} = \mathcal{P} - \frac{1}{\varepsilon}\frac{F_E^2}{1 + \varepsilon^3(\partial_X\eta)^2}(\varepsilon^2(\partial_X\eta)^2 T_{11} - 2\varepsilon^{\frac{3}{2}}\partial_X\eta T_{12} - T_{22}) -$$

$$- B\varepsilon\frac{\partial_X^2\eta}{(1 + \varepsilon^3(\partial_X\eta)^2)^{\frac{3}{2}}}, \quad y = \varepsilon\eta \tag{67}$$

It can be notes that the combination of equations (65) and (66) can be combined into:

$$\varepsilon\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad -1 < y < \varepsilon\eta \tag{68}$$

Expand according to:

$$u(T, X, y) = -\frac{\Omega y}{\varepsilon} + u_0(T, X, y) + \varepsilon u_1(T, X, y) + o(\varepsilon) \quad y > \varepsilon\eta \tag{69}$$

$$v(T, X, y) = v_0(T, X, y) + \varepsilon v_1(T, X, y) + o(\varepsilon) \tag{70}$$

$$p = p_0(T, X, y) + \varepsilon p_1(T, X, y) + o(\varepsilon) \tag{71}$$

$$\mathcal{P} = \varepsilon\mathcal{P}_1 + o(\varepsilon) \tag{72}$$

$$V(T, X, y) = -y + \varepsilon^{\frac{3}{2}}V_1(T, X, y) + o(\varepsilon^{\frac{3}{2}}) \tag{73}$$

$$\eta = \eta_0(T, X) + \varepsilon\eta_1(T, X) + o(\varepsilon) \tag{74}$$

The $O(1)$ equations are:

$$\partial_y p_0 = 0, \quad -(c + \Omega y)\partial_X u_0 - \Omega v_0 = -\partial_X p_0, \quad \partial_y^2 v_0 = 0, \quad \partial_X u_0 + \partial_y v_0 \tag{75}$$

With boundary conditions:

$$p_0 = \eta_0, \quad -c\partial_X\eta_0 = v_0, \quad \text{on} \quad y = 0 \tag{76}$$

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The equation $\partial_y^2 v_0 = 0$ has solution:

$$v_0 = (y+1)A_0(T,X) \Rightarrow \partial_X u_0 = -A_0(T,X) \tag{77}$$

Setting $y = 0$ shows that $A_0 = -c\partial_X \eta_0$ and so $u_0 = c\eta_0$. The equation $\partial_y p_0 = 0$ shows that $p_0 = \eta_0$. The equation $-(c + \Omega y)\partial_X u_0 - \Omega v_0 = -\partial_X p_0$ can be evaluated at $y = 0$. using the previous solutions yields:

$$-c^2\partial_X \eta_0 + \Omega c \partial_X \eta_0 = -\partial_X \eta_0 \tag{78}$$

which gives the following expression for c,

$$c^2 - \Omega c - 1 = 0 \Rightarrow c = \frac{\Omega \pm \sqrt{\Omega^2 + 4}}{2} \tag{79}$$

The usual way to obtain this expression is to evaluate the Burns condition, which in this case is evaluating the integral:

$$\int_{-1}^{0} \frac{dz}{(c + \Omega z)^2} = 1 \tag{80}$$

The method presented here bypasses the evaluation of increasingly complicated integrals with simple substitution. The next order equations are:

$$\frac{\partial^2 V_1}{\partial X^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 \tag{81}$$

$$V_1 - \eta_0 = 0 \tag{82}$$

$$-c\frac{\partial u_1}{\partial X} - \frac{\partial u_0}{\partial T} + u_0\frac{\partial u_0}{\partial X} - \Omega y\frac{\partial u_1}{\partial X} - \Omega v_1 = -\frac{\partial p_1}{\partial X} \tag{83}$$

$$(c + \Omega y)\frac{\partial v_0}{\partial X} = -\frac{\partial p_1}{\partial y} \tag{84}$$

$$\frac{\partial^2 v_0}{\partial X^2} + \frac{\partial^2 v_1}{\partial y^2} = 0 \tag{85}$$

$$-c\frac{\partial \eta_1}{\partial X} + \frac{\partial \eta_0}{\partial T} + (2c - \Omega)\eta_0\frac{\partial \eta_0}{\partial X} = v_1 \tag{86}$$

$$p_1 - \eta_1 = \mathcal{P}_1 - \hat{F}_E^2\frac{\partial V_1}{\partial y} - B\frac{\partial^2 \eta_0}{\partial X^2} \tag{87}$$

Where the scaling on F_E has been made, $F_E = \hat{F}_E \varepsilon^{1/4}$ to keep the electric term in the Young-Laplace equation. To progress, one finds the expression for p_1 by integrating (84) and using (87) to obtain:

$$p_1 = c\left(\frac{(c-\Omega)}{2}[(y+1)^2 - 1] - \frac{\Omega}{3}[(y+1)^3 - 1]\right)\frac{\partial^2 \eta_0}{\partial X^2} + \eta_1 + \mathcal{P}_1 - \hat{F}_E^2\frac{\partial V_1}{\partial y} - B\frac{\partial^2 \eta_0}{\partial X^2} \tag{88}$$

To obtain an equation, all one requires in $v_1(T, X, 0)$. Solving equation (85) for v_1 shows that:

$$v_1 = \frac{c}{6}(y+1)^3\frac{\partial^3 \eta_0}{\partial X^3} + A_1(T,X)(y+1) \tag{89}$$

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Setting $y = 0$ in equation (83) allows $A_1(T, X)$ to be computed:

$$(c - \Omega)A_1(T, X) = \left[-B + \frac{\Omega c}{6} - \frac{c^2}{2}\right]\frac{\partial^3 \eta_0}{\partial X^3} + \frac{\partial u_0}{\partial T} - u_0\frac{\partial u_0}{\partial X} - \frac{\partial \eta_1}{\partial X} - \frac{\partial \mathcal{P}_1}{\partial X} + \hat{F}_E^2\frac{\partial^2 V_1}{\partial X \partial y} \tag{90}$$

Showing that:

$$A_1(T, X) = c\left[B + \frac{\Omega c}{6} - \frac{c^2}{2}\right]\frac{\partial^3 \eta_0}{\partial X^3} + c\frac{\partial u_0}{\partial T} - cu_0\frac{\partial u_0}{\partial X} - c\frac{\partial \eta_1}{\partial X} - c\frac{\partial \mathcal{P}_1}{\partial X} + c\hat{F}_E^2\frac{\partial^2 V_1}{\partial X \partial y} \tag{91}$$

So this gives $v_1(T, X, 0)$ to be:

$$v_1(T, X, 0) = \frac{c}{6}\frac{\partial^3 \eta_0}{\partial X^3} + c\left[B + \frac{\Omega c}{6} - \frac{c^2}{2}\right]\frac{\partial^3 \eta_0}{\partial X^3} - c\frac{\partial u_0}{\partial T} - cu_0\frac{\partial u_0}{\partial X} - c\frac{\partial \eta_1}{\partial X} -$$
$$- c\frac{\partial \mathcal{P}_1}{\partial X} + c\hat{F}_E^2\frac{\partial^2 V_1}{\partial X \partial y} \tag{92}$$

To compute V_1, use the Fourier representation to obtain:

$$\frac{\partial^2 \hat{V}_1}{\partial y^2} - k^2\hat{V}_1 = 0, \quad \hat{V}_1(t, k, 0) = \hat{\eta}_0 \tag{93}$$

to obtain:

$$V_1 = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}_0 e^{-|k|y}e^{ikX}dk \tag{94}$$

Then:

$$\begin{aligned}
\frac{\partial v_1}{\partial y}(T, X, 0) &= \frac{1}{2\pi}\int_{\mathbb{R}} -|k|\hat{\eta}_0 e^{ikX}dk \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} (ik)(i\,\mathrm{sgn}(k))\hat{\eta}_0 e^{ikX}dk \\
&= \mathscr{H}(\widehat{\partial_X \eta_0})
\end{aligned}$$

Where $\mathscr{H}(\cdot)$ denotes the Hilbert transform. From the properties of Hilbert transforms:

$$\frac{\partial^2 V_1}{\partial X \partial y} = \mathscr{H}(\partial_X^2 \eta_0) \tag{95}$$

Inserting v_1 into equation (86) yields the equation of interest:

$$(1 + c^2)\frac{\partial \eta_0}{\partial T} + (2c + c^3 - \Omega)\eta_0\frac{\partial \eta_0}{\partial X} + c\left[\frac{c^2}{3} - B\right]\frac{\partial^3 \eta_0}{\partial X^3} - c\hat{F}_E^2\mathscr{H}\left(\frac{\partial^2 \eta_0}{\partial X^2}\right) + c\frac{\partial \mathcal{P}}{\partial X} = 0 \tag{96}$$

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A quick check by setting $c = 1$ and $\Omega = 0$ shows that it reduces to the original equation. Putting the dimensions back in shows that:

$$(1 + c^2) \left(c \frac{\partial \eta}{\partial x} + \frac{1}{c_0} \frac{\partial \eta}{\partial t} \right) + \frac{2c + c^3 - \Omega}{h} \eta \frac{\partial \eta}{\partial x} + h^2 c \left[\frac{c^2}{3} - B \right] \frac{\partial^3 \eta}{\partial x^3}$$
$$- ch F_E^2 \mathscr{H} \left(\frac{\partial^2 \eta}{\partial x^2} \right) + \frac{1}{\rho g} \frac{\partial \tilde{p}}{\partial x} = 0 \quad (97)$$

where $\tilde{p} = \varepsilon^2 p$, the scaled pressure. Now travelling wave profiles are examined by $\eta = \eta(x - Vt)$ to get the equation:

$$(1 + c^2)(c - F)\eta + \frac{2c + c^3 - \Omega}{h} \eta^2 + h^2 c \left[\frac{c^2}{3} - B \right] \eta'' - hc F_E^2 \mathscr{H}(\eta') + \frac{\tilde{p}}{\rho g} = 0 \quad (98)$$

5   Numerical methods and results

In the sequel we consider the free wave propagation only, *i.e.* $\tilde{p} \equiv 0$. In order to generate numerically the solitary wave solutions to Equation (98), we employ the classical Fourier-type pseudo-spectral discretization on a sufficiently large domain, where the solution decays below the machine accuracy to annihilate the effect of implied periodic boundary conditions. In other words, we solve formally a periodic BVP, but the repeated value is actually zero in agreement with the decaying properties of solitary waves. The discrete problem for spectral coefficients is solved using the classical Petviashvili iteration as it was described in [5] for the classical Benjamin equation. To implement the Petviashvili scheme, Equation (98) is written in the following form:

$$\mathcal{L}\eta \;=\; \mathcal{N}(\eta), \quad (99)$$

where the linear $\mathcal{L}$ and nonlinear $\mathcal{N}(\cdot)$ operators are defined as

$$\mathcal{L}\eta \;:=\; (1 + c^2)(c - F)\eta \;+\; h^2 c \left[\frac{c^2}{3} - B \right] \eta'' \;-\; h\, c\, F_E^2 \mathscr{H}(\eta'),$$

$$\mathcal{N}(\eta) \;:=\; \frac{\Omega - 2c - c^3}{h} \eta^2\,.$$

Then, the Petviashvili iteration for Equation (99) reads:

$$\eta^{(n+1)} \;=\; \gamma_n^2\, \mathcal{L}^{-1} \circ \mathcal{N}(\eta^{(n)}),$$

where we took into account the fact that the mapping $\mathcal{N}(\cdot)$ is a homogeneous function of degree two of its argument. Finally, the stabilizing factor γ_n is defined as

$$\gamma_n \;:=\; \frac{\int_{\mathbb{R}} \eta_n \cdot \mathcal{L}\eta_n \, \mathrm{d}x}{\int_{\mathbb{R}} \eta_n \cdot \mathcal{N}(\eta_n) \, \mathrm{d}x}\,.$$

For more details on the Petviashvili iteration we refer to [1, 5].

Table 1: *Dimensional parameters used in numerical computations.*

Parameter name	Value
Froude number, F	0.5
Electric Froude number, F_E	$0.5 \ldots 1.2$
Vorticity strength, Ω	1.0
Bond number, B	0.4
Fluid layer depth, d	1.0
Gravity acceleration, g	1.0
Celerity, c	$\frac{1}{2}\left(\Omega - \sqrt{4 + \Omega^2}\right) \approx -0.62$
Number of Fourier modes, N	1024
Domain half-length, L	10.0
Error tolerance, ϵ	5×10^{-15}

5.1 Numerical results

Equation (98) was discretized in space using the standard Fourier-type pseudo-spectral method. Namely, we used $N = 1024$ modes in our computations. Since we are looking at localized travelling waves, the computational domain was taken to be $T = [-10, 10]$ assuming periodic boundary conditions ensured by the choice of basis functions. The Petviashvili iterations were stopped when the L_∞ norm of the difference of two successive iterations became smaller than $\epsilon = 5 \times 10^{-15}$. As the initial guess we always took a localized bump of negative polarity. The convergence of the method was marginally dependent on our choice, which only influenced the total number of iterations. In any case, from the end user perspective, the computations lasted virtually instantaneously. We noticed that the number of iterations increased with the electric Froude number F_E. The computation of an oscillating travelling wave for $F_E = 1.27$ took 117 Petviashvili iterations, while for $F_E = 0.5$ the method needed only 59 iterations to converge. The dimensionless physical and numerical parameters used in this computation are reported in Table 1.

First of all, we would like to mention that for some values of parameters we were able to compute also the periodic travelling wave solutions, which was not our initial goal. One such solution is reported in Figure 5. So, from now on we focus on localized (in space) travelling wave solutions. Several examples of such structures are shown in Figure 6, where we gradually increased the electric Froude number F_E by keeping all other values constant (see Table 1).

6 Conclusions and perspectives

In the present manuscript we considered the propagation of free surface electrohydrodynamic waves in the presence of non-zero, but constant vorticity distribution. The problem was analyzed from the linear and weakly nonlinear points of view. The linear analysis allowed us to get rid of Burns's condition. The weakly nonlinear approach allowed us to
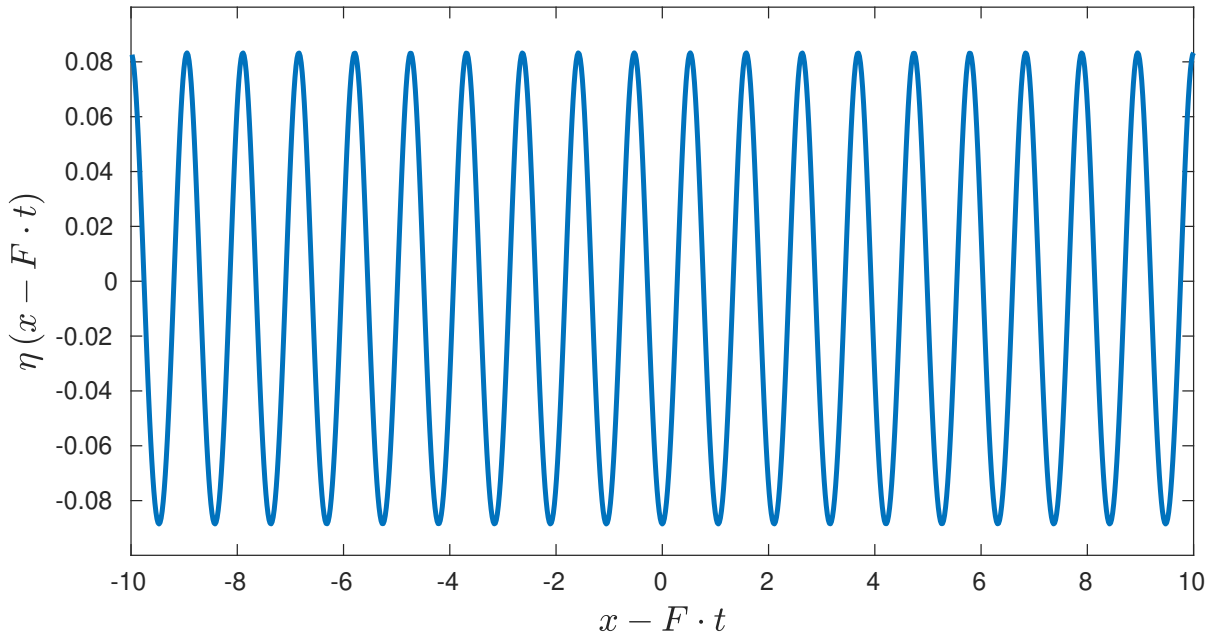
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Figure 5: *Fully converged travelling periodic wave solution to Equation* (98) *with* $B = 0.1$ *and* $F_E = 0.5$. *All other parameters are reported in Table* 1.

(a) $F_E = 0.5$

(b) $F_E = 0.9$

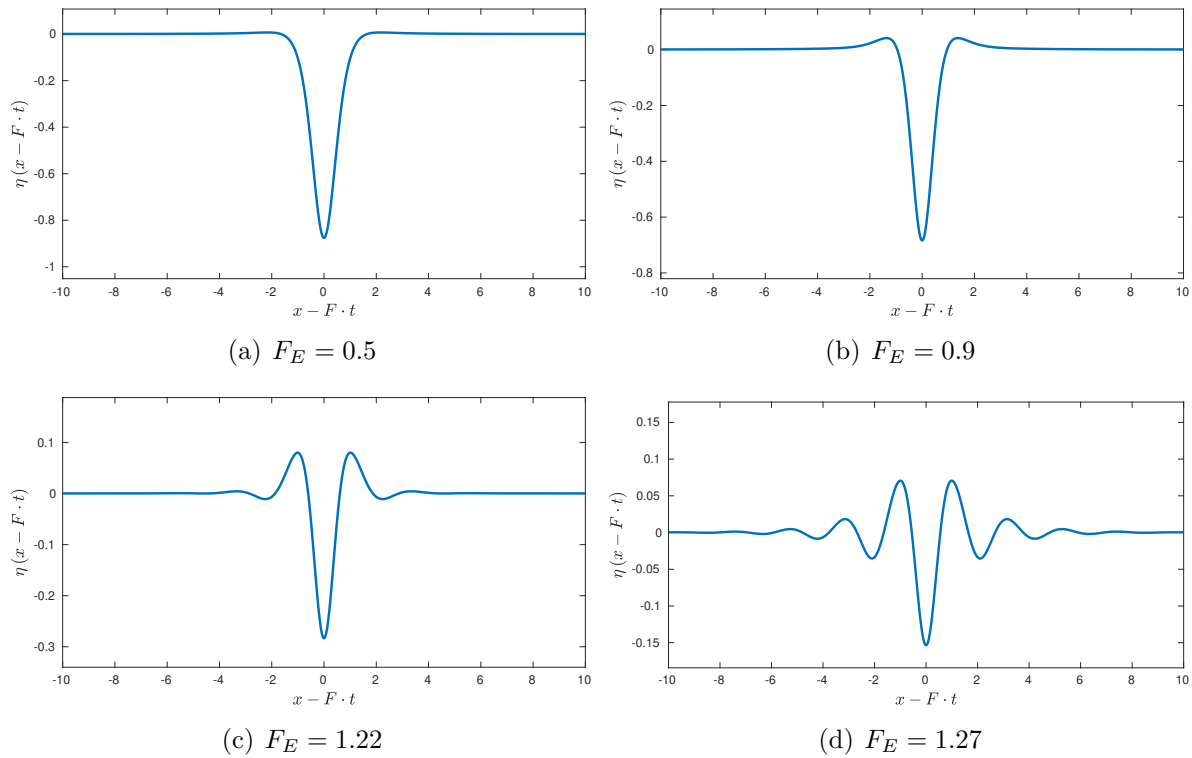(c) $F_E = 1.22$

(d) $F_E = 1.27$

Figure 6: *Computed localized solitary waves to Equation* (98) *for different values of the electric Froude number* F_E. *All other parameters are reported in Table* 1.

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compute solitary wave solutions by solving numerically the non-local ODE which describes them. The non-local effects are described by a linear term involving the Hilbert transform of the free surface excursion derivative η'. It turns out that the dynamics of weakly non-linear electrohydrodynamic waves is described by a generalized Benjamin equation, which appears clearly in this context for the first time, to our knowledge. So far, it appeared as a model equation for internal capillary-gravity waves. In our study it serves to predict the shape of coherent structures in electrohydrodynamic flows with constant vorticity.

Concerning the perspectives, in future works we would like to consider more general vorticity distributions. Another promising direction consists in considering the three-dimensional wave propagation problem. Finally, the unsteady simulations have to be performed to understand better the dynamics of solutions discussed hereinabove. We suspect also that the derived Benjamin-type Equation (98) possesses also other types of travelling wave solutions such as multi-pulsed solitary waves which were computed in [5] in the context of internal waves.

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