On The Dynamics of a System of Difference Equations $x_{n+1} = x_{n-1}y_n - 1$, $y_{n+1} = y_{n-1}z_n - 1$, $z_{n+1} = z_{n-1}x_n - 1$

Erkan Taşdemir[1*], Yüksel Soykan[2]
*Corresponding Author Mail: erkantasdemir@hotmail.com

[1]Kırklareli University,
Pınarhisar Vocational School of Higher Education,
39300, Kırklareli, TURKEY, erkan.tasdemir@klu.edu.tr

[2]Bülent Ecevit University,
Faculty of Art and Science,
Department of Mathematics,
67100, Zonguldak, TURKEY

October 29, 2019

Abstract

In this paper, we study the dynamics of following system of nonlinear difference equations $x_{n+1} = x_{n-1}y_n - 1$, $y_{n+1} = y_{n-1}z_n - 1$, $z_{n+1} = z_{n\ 1}x_n - 1$. Especially we investigate the periodicity, boundedness and stability of related system of difference equations.

Keywords: difference equations, dynamical systems, periodicity, stability, boundedness

1 Introduction

Over the last years difference equations and systems of difference equations have been huge attention by scientists and mathematicians. This attention is particularly related to applications in different fields of science especially ecology, economy, physics and so on. As long as they achieved more meaningful and impressive results and applications, this attention continues to increase at the high level. Several latest results can be found in the following papers:

In [15], Kent et al studied dynamics of difference equation

$$x_{n+1} = x_n x_{n-1} - 1.$$

Further, in [1], Liu et al and in [23], Wang et al obtained some significant results about related difference equation.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2019 doi:10.20944/preprints201910.0350.v1

In [24], Kurbanlı et al investigated positive solutions of system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

In [25], Kurbanlı studied the solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} + 1}.$$

Furthermore, there are many books and papers related to difference equations see [7] - [23].

In this paper, we investigate the dynamics of following system of nonlinear difference equations:

$$x_{n+1} = x_{n-1}y_n - 1,\ \ y_{n+1} = y_{n-1}z_n - 1,\ \ z_{n+1} = z_{n-1}x_n - 1, n = 0, 1, \cdots, \quad (1)$$

where the all initial conditions are real numbers. Especially, we study equilibrium points, stability of solutions, existence of periodic solutions and boundedness of solutions of related system.

Firstly, we give some definitions and theorems which are used during this study.

Let us introduce a six-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f_1\left(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}\right), \\ y_{n+1} &= f_2\left(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}\right), \\ z_{n+1} &= f_3\left(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}\right) \end{aligned} \quad (2)$$

$n = 0, 1, ...,$ where $f_1 : I_1^2 \times I_2^2 \times I_3^2 \to I_1$, $f_2 : I_1^2 \times I_2^2 \times I_3^2 \to I_2$ and $f_3 : I_1^2 \times I_2^2 \times I_3^2 \to I_3$ are continuously differentiable functions and I_1, I_2, I_3 are some intervals of real numbers. Moreover, a solution $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ of system (2) is uniquely determined by initial values $(x_i, y_i, z_i) \in I_1 \times I_2 \times I_3$ for $i \in \{-1, 0\}$.

Definition 1 *Along with the system (2), we consider the corresponding vector map*

$$F = \{f_1, x_n, x_{n-1}, f_2, y_n, y_{n-1}, f_3, z_n, z_{n-1}\}.$$

A point $(\bar{x}, \bar{y}, \bar{z})$ is called an equilibrium point of the system (2) if

$$\begin{aligned} \bar{x} &= f_1\left(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}\right), \\ \bar{y} &= f_2\left(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}\right), \\ \bar{z} &= f_3\left(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}\right). \end{aligned}$$

The point $(\bar{x}, \bar{y}, \bar{z})$ is also called a fixed point of the vector map F.

Definition 2 *Let $(\bar{x}, \bar{y}, \bar{z})$ be an equilibrium point of the system (2).*

 doi:10.20944/preprints201910.0350.v1

(i) *An equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *of system (2) is called stable if, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for every initial value* $(x_i, y_i, z_i) \in I_1 \times I_2 \times I_3$, *with*

$$\sum_{i=-1}^{0} |x_i - \bar{x}| < \delta, \sum_{i=-1}^{0} |y_i - \bar{y}| < \delta, \sum_{i=-1}^{0} |z_i - \bar{z}| < \delta$$

implying $|x_n - \bar{x}| < \varepsilon, |y_n - \bar{y}| < \varepsilon$ *and* $|z_n - \bar{z}| < \varepsilon$ *for* $n \in \mathbb{N}$.

(ii) *An equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *of system (2) is called unstable, if it is not stable.*

(iii) *An equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *of system (2) is called locally asymptotically stable if it is stable and if, in addition, there exists* $\gamma > 0$ *such that*

$$\sum_{i=-1}^{0} |x_i - \bar{x}| < \gamma, \sum_{i=-1}^{0} |y_i - \bar{y}| < \gamma, \sum_{i=-1}^{0} |z_i - \bar{z}| < \gamma,$$

and $(x_n, y_n, z_n) \to (\bar{x}, \bar{y}, \bar{z})$ *as* $n \to \infty$.

(iv) *An equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *of system (2) is called a global attractor if* $(x_n, y_n, z_n) \to (\bar{x}, \bar{y}, \bar{z})$ *as* $n \to \infty$.

(v) *An equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *of system (2) is called globally asymptotically stable if it is stable and a global attractor.*

Definition 3 *Let* $(\bar{x}, \bar{y}, \bar{z})$ *be an equilibrium point of the map* F *where* f_1, f_2 *and* f_3 *are continuously differentiable functions at* $(\bar{x}, \bar{y}, \bar{z})$. *The linearized system of system (2) about the equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$ *is*

$$X_{n+1} = F(X_n) = BX_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \\ z_n \\ z_{n-1} \end{pmatrix}$$

and B *is a Jacobian matrix of system (2) about the equilibrium point* $(\bar{x}, \bar{y}, \bar{z})$.

Definition 4 *Assume that* $X_{n+1} = F(X_n), n = 0, 1, \cdots,$ *is a system of difference equations such that* $\bar{X}$ *is a fixed point of* F. *If no eigenvalues of the Jacobian matrix* B *about* $\bar{X}$ *have absolute value equal to one, then* $\bar{X}$ *is called hyperbolic. Otherwise,* $\bar{X}$ *is said to be nonhyperbolic.*

Theorem 5 (Linearized Stability Theorem [26], p.11) *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, \cdots,$$

is a system of difference equations such that $\bar{X}$ *is a fixed point of* F.

(i) *If all eigenvalues of the Jacobian matrix B about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.*

(ii) *If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Definition 6 *A solution $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ of system (2) is bounded and persists if there exist constants M, N such that $M < N$ and*

$$M < x_n, y_n, z_n < N,\ n = -m, -m+1, \cdots .$$

Definition 7 *A positive solution $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ of system (2) is periodic with period p if*

$$x_{n+p} = x_n, y_{n+p} = y_p, z_{n+p} = z_n \text{ for all } n \geq -1.$$

2 Equilibrium Points of System (1)

This section, we find out the equilibrium points of system (1).

Theorem 8 *There are two equilibrium points of system (1) which are each elements of equilibrium points golden ratio or its conjugate. The equilibrium points of system (1) are:*

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1) = \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right), \tag{3}$$

$$(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right). \tag{4}$$

Note that all elements of the first equilibrium point equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$ which is golden ratio.

Proof. Let $x_n = \bar{x}$, $y_n = \bar{y}$ and $z_n = \bar{z}$ for all $n \geq -1$. Then, we obtain following system from system (1):

$$\bar{x} = \bar{x} \cdot \bar{y} - 1, \tag{5}$$

$$\bar{y} = \bar{y} \cdot \bar{z} - 1, \tag{6}$$

$$\bar{z} = \bar{z} \cdot \bar{x} - 1. \tag{7}$$

Therefore, we have easily from (5)-(7):

$$x = \frac{1+\sqrt{5}}{2},\ y = \frac{1+\sqrt{5}}{2},\ z = \frac{1+\sqrt{5}}{2},$$

$$x = \frac{1-\sqrt{5}}{2},\ y = \frac{1-\sqrt{5}}{2},\ z = \frac{1-\sqrt{5}}{2}.$$

So, the proof completed. ■

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2019 doi:10.20944/preprints201910.0350.v1

3 Existence of Periodic Solutions of System (1)

In this here, we study the periodic or non-periodic solutions of system (1). Moreover we obtain the initial values for the periodic solutions of system.

Theorem 9 *There are no two periodic solutions of system (1).*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a two periodic solution of system (1). Therefore, $x_{2n} = a$, $x_{2n-1} = b$, $y_{2n} = c$, $y_{2n-1} = d$, $z_{2n} = e$ and $z_{2n-1} = f$ for all $n \in \mathbb{N}_0$, $a, b, c, d, e, f \in \mathbb{R}$ such that $a \neq b, c \neq d$ and $e \neq f$. Hence, we have from system (1)

$$\begin{aligned}
x_{2n+1} &= x_{2n-1}y_{2n} - 1, \\
x_{2n} &= x_{2n-2}y_{2n-1} - 1, \\
y_{2n+1} &= y_{2n-1}z_{2n} - 1, \\
y_{2n} &= y_{2n-2}z_{2n-1} - 1, \\
z_{2n+1} &= z_{2n-1}x_{2n} - 1, \\
z_{2n} &= z_{2n-2}x_{2n-1} - 1.
\end{aligned}$$

Thus, we obtain the following equalities:

$$\begin{aligned}
b &= bc - 1, && (8) \\
a &= ad - 1, && (9) \\
d &= de - 1, && (10) \\
c &= cf - 1, && (11) \\
f &= fa - 1, && (12) \\
e &= eb - 1, && (13)
\end{aligned}$$

So, we have from (8)-(13),

$$\begin{aligned}
a &= b = c = d = e = f = \frac{1+\sqrt{5}}{2} = \bar{x}_1 = \bar{y}_1 = \bar{z}_1, \\
a &= b = c = d = e = f = \frac{1-\sqrt{5}}{2} = \bar{x}_2 = \bar{y}_2 = \bar{z}_2.
\end{aligned}$$

Since $a \neq b, c \neq d$ and $e \neq f$, this is a contradiction. The proof completed. ∎

Theorem 10 *System (1) has three periodic solutions with the initial values as*

$$\begin{aligned}
x_{-1} &= -1, x_0 = -1, y_{-1} = -1, y_0 = -1, z_{-1} = -1, z_0 = -1, && (14) \\
x_{-1} &= 0, x_0 = -1, y_{-1} = 0, y_0 = -1, z_{-1} = 0, z_0 = -1. && (15)
\end{aligned}$$

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a three periodic solution of system (1). Hence, $x_{-1} = a$, $x_0 = b$, $y_{-1} = c$, $y_0 = d$, $z_{-1} = e$ and $z_0 = f$ for all $n \in \mathbb{N}_0$,

$a, b, c, d, e, f \in \mathbb{R}$. Therefore, we obtain that:

$$\begin{aligned}
x_1 &= x_{-1}y_0 - 1 = ad - 1 \\
y_1 &= y_{-1}z_0 - 1 = cf - 1 \\
z_1 &= z_{-1}x_0 - 1 = ea - 1 \\
x_2 &= x_0y_1 - 1 = b(cf-1) - 1 = a \quad (16) \\
y_2 &= y_0z_1 - 1 = d(ea-1) - 1 = c \quad (17) \\
z_2 &= z_0x_1 - 1 = f(ad-1) - 1 = e \quad (18) \\
x_3 &= x_1y_2 - 1 = (ad-1)c - 1 = b \quad (19) \\
y_3 &= y_1z_2 - 1 = (cf-1)e - 1 = d \quad (20) \\
z_3 &= z_1x_2 - 1 = (ea-1)a - 1 = f \quad (21)
\end{aligned}$$

Thus, we have four cases from solutions of system of equations (16)-(21):

$$\begin{aligned}
a &= b = c = d = e = f = -1, \quad (22) \\
a &= 0, b = -1, c = 0, d = -1, e = 0, f = -1, \quad (23) \\
a &= b = c = d = e = f = \frac{1+\sqrt{5}}{2}, \quad (24) \\
a &= b = c = d = e = f = \frac{1-\sqrt{5}}{2}. \quad (25)
\end{aligned}$$

(22) and (23) are three periodic solutions but the other cases aren't periodic solutions. Because they are equilibrium solutions. The proof completed. ■

Remark 11 *From (14) and (15), three periodic cycle of system (1) is*

$$\{\cdots, (-1,-1,-1), (0,0,0), (-1,-1,-1), (-1,-1,-1), \cdots\}.$$

Proof. We take the initial values are $x_{-1} = 0, x_0 = -1, y_{-1} = 0, y_0 = -1, z_{-1} = 0, z_0 = -1$. Therefore, we obtain the followings:

$$\begin{aligned}
x_1 &= x_{-1}y_0 - 1 = 0 \cdot (-1) - 1 = -1, \\
y_1 &= y_{-1}z_0 - 1 = 0 \cdot (-1) - 1 = -1, \\
z_1 &= z_{-1}x_0 - 1 = 0 \cdot (-1) - 1 = -1, \\
x_2 &= x_0y_1 - 1 = (-1) \cdot (-1) - 1 = 0, \\
y_2 &= y_0z_1 - 1 = (-1) \cdot (-1) - 1 = 0, \\
z_2 &= z_0x_1 - 1 = (-1) \cdot (-1) - 1 = 0, \\
x_3 &= x_1y_2 - 1 = (-1) \cdot 0 - 1 = -1, \\
y_3 &= y_1z_2 - 1 = (-1) \cdot 0 - 1 = -1, \\
z_3 &= z_1x_2 - 1 = (-1) \cdot 0 - 1 = -1.
\end{aligned}$$

Hence, system (1) has three periodic cycle as:

$$\{\cdots, (-1,-1,-1), (0,0,0), (-1,-1,-1), (-1,-1,-1), \cdots\}.$$

■

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2019 doi:10.20944/preprints201910.0350.v1

4 Boundedness of System (1)

During this section we study the bounded or unbounded solutions of system (1).

Theorem 12 *Let $x_i, y_i, z_i \in (-1, 0)$ for $i \in \{-1, 0\}$, then the solutions of system (1) are such that $x_n, y_n, z_n \in (-1, 0)$ for $n \geq -1$.*

Proof. Let $x_i, y_i, z_i \in (-1, 0)$ for $i \in \{-1, 0\}$. Thus we obtain from System (1):

$$\begin{aligned} x_1 &= x_{-1}y_0 - 1 \in (-1, 0), \\ y_1 &= y_{-1}z_0 - 1 \in (-1, 0), \\ z_1 &= z_{-1}x_0 - 1 \in (-1, 0). \end{aligned}$$

Therefore, we have by induction

$$\begin{aligned} x_n &= x_{n-2}y_{n-1} - 1 \in (-1, 0), \\ y_n &= y_{n-2}z_{n-1} - 1 \in (-1, 0), \\ z_n &= z_{n-2}x_{n-1} - 1 \in (-1, 0) \end{aligned}$$

for $n \geq -1$. The proof is completed. ■

Theorem 13 *Let the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 < -1$. Then*

$$\begin{aligned} x_1, y_1, z_1 &> 0, \\ x_2, y_2, z_2 &< -1, \\ x_3, y_3, z_3 &< -1. \end{aligned}$$

Proof. Let the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 < -1$. We have from System (1):

$$x_1 = x_{-1}y_0 - 1 > 0.$$

Calculations of $y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ are similar to x_1, so we leave them to the readers. ■

Theorem 14 *Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a solution of system (1). Then,*

$$\begin{aligned} x_{n+3} - y_n &= (y_{n+2} + 1)(z_{n+1} + 1) - (x_{n+1} + 1)(y_n + 1), & (26) \\ y_{n+3} - z_n &= (z_{n+2} + 1)(x_{n+1} + 1) - (y_{n+1} + 1)(z_n + 1), & (27) \\ z_{n+3} - x_n &= (x_{n+2} + 1)(y_{n+1} + 1) - (z_{n+1} + 1)(x_n + 1). & (28) \end{aligned}$$

 doi:10.20944/preprints201910.0350.v1

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a solution of system (1). Hence, we have from system (1) and by some calculations:

$$\begin{aligned}
x_{n+3} - y_n &= (x_{n+1}y_{n+2} - 1) - y_n \\
&= x_{n+1}(y_n z_{n+1} - 1) - 1 - y_n \\
&= x_{n+1}y_n z_{n+1} - x_{n+1} - 1 - y_n \\
&= x_{n+1}y_n(z_{n-1}x_n - 1) - x_{n+1} - 1 - y_n \\
&= x_{n+1}y_n z_{n-1}x_n - x_{n+1}y_n - x_{n+1} - 1 - y_n \\
&= x_{n+1}y_n z_{n-1}x_n - x_{n+1}(y_n + 1) - (y_n + 1) \\
&= x_{n+1}y_n z_{n-1}x_n - z_{n-1}x_n + z_{n-1}x_n - (y_n + 1)(x_{n+1} + 1) \\
&= z_{n-1}x_n(x_{n+1}y_n - 1) + z_{n-1}x_n - (y_n + 1)(x_{n+1} + 1) \\
&= z_{n-1}x_n(y_{n+2} + 1) - (y_n + 1)(x_{n+1} + 1) \\
&= z_{n-1}x_n(y_{n+2} + 1) - (y_{n+2} + 1) + (y_{n+2} + 1) - (y_n + 1)(x_{n+1} + 1) \\
&= (y_{n+2} + 1)(z_{n-1}x_n - 1) + (y_{n+2} + 1) - (y_n + 1)(x_{n+1} + 1) \\
&= (y_{n+2} + 1)z_{n+1} + (y_{n+2} + 1) - (y_n + 1)(x_{n+1} + 1) \\
&= (y_{n+2} + 1)(z_{n+1} + 1) - (x_{n+1} + 1)(y_n + 1).
\end{aligned}$$

Because proofs of (27) and (28) are similar to (26), we leave them to the readers.
■

Theorem 15 *Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a solution of system (1). Let the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 < -1$. Then the following statements are true:*

(i)

$$\begin{aligned}
0 &< x_1 < z_4 < y_7 < \cdots < x_{9k+1} < z_{9k+4} < y_{9k+7} < \cdots, \\
0 &< y_1 < x_4 < z_7 < \cdots < y_{9k+1} < x_{9k+4} < z_{9k+7} < \cdots, \\
0 &< z_1 < y_4 < x_7 < \cdots < z_{9k+1} < y_{9k+4} < x_{9k+7} < \cdots, \\
-1 &> x_2 > z_5 > y_8 > \cdots > x_{9k+2} > z_{9k+5} > y_{9k+8} > \cdots, \\
-1 &> y_2 > x_5 > z_8 > \cdots > y_{9k+2} > x_{9k+5} > z_{9k+8} > \cdots, \\
-1 &> z_2 > y_5 > x_8 > \cdots > z_{9k+2} > y_{9k+5} > z_{9k+8} > \cdots, \\
-1 &> x_3 > z_6 > y_9 > \cdots > x_{9k+3} > z_{9k+6} > y_{9k+9} > \cdots, \\
-1 &> y_3 > x_6 > z_9 > \cdots > y_{9k+3} > x_{9k+6} > z_{9k+9} > \cdots, \\
-1 &> z_3 > y_6 > x_9 > \cdots > z_{9k+3} > y_{9k+6} > z_{9k+9} > \cdots.
\end{aligned}$$

(ii)

doi:10.20944/preprints201910.0350.v1

$$\begin{array}{lll}
\lim_{n\to\infty} x_{9n+1} = \infty, & \lim_{n\to\infty} x_{9n+4} = \infty, & \lim_{n\to\infty} x_{9n+7} = \infty, \\
\lim_{n\to\infty} y_{9n+1} = \infty, & \lim_{n\to\infty} y_{9n+4} = \infty, & \lim_{n\to\infty} y_{9n+7} = \infty, \\
\lim_{n\to\infty} z_{9n+1} = \infty, & \lim_{n\to\infty} z_{9n+4} = \infty, & \lim_{n\to\infty} z_{9n+7} = \infty, \\
\lim_{n\to\infty} x_{9n+2} = -\infty, & \lim_{n\to\infty} x_{9n+5} = -\infty, & \lim_{n\to\infty} x_{9n+8} = -\infty, \\
\lim_{n\to\infty} y_{9n+2} = -\infty, & \lim_{n\to\infty} y_{9n+5} = -\infty, & \lim_{n\to\infty} y_{9n+8} = -\infty, \\
\lim_{n\to\infty} z_{9n+2} = -\infty, & \lim_{n\to\infty} z_{9n+5} = -\infty, & \lim_{n\to\infty} z_{9n+8} = -\infty, \\
\lim_{n\to\infty} x_{9n+3} = -\infty, & \lim_{n\to\infty} x_{9n+6} = -\infty, & \lim_{n\to\infty} x_{9n+9} = -\infty, \\
\lim_{n\to\infty} y_{9n+3} = -\infty, & \lim_{n\to\infty} y_{9n+6} = -\infty, & \lim_{n\to\infty} y_{9n+9} = -\infty, \\
\lim_{n\to\infty} z_{9n+3} = -\infty, & \lim_{n\to\infty} z_{9n+6} = -\infty, & \lim_{n\to\infty} z_{9n+9} = -\infty.
\end{array}$$

Proof.

(i) Let $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ be a solution of system (1). Let the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 < -1$. Therefore, we know the followings from Theorem 13

$$\begin{aligned}
x_1, y_1, z_1 &> 0, \\
x_2, y_2, z_2 &< -1, \\
x_3, y_3, z_3 &< -1.
\end{aligned}$$

Firstly we consider (26) for $n = 1$. Thus we have

$$x_4 - y_1 = (y_3 + 1)(z_2 + 1) - (x_2 + 1)(y_1 + 1).$$

Since $y_3, z_2, x_2 < -1$ and $y_1 > 0$, we obtain that

$$x_4 - y_1 > 0 \Rightarrow x_4 > y_1.$$

So $x_4 > y_1 > 0$ and similarly $y_4 > z_1 > 0$ and $z_4 > x_1 > 0$.

Now we take $n = 2$ for (26), therefore we get

$$x_5 - y_2 = (y_4 + 1)(z_3 + 1) - (x_3 + 1)(y_2 + 1).$$

Because of $y_4 > 0$ and $x_3, z_3, y_2 < -1$, we have

$$x_5 - y_2 < 0 \Rightarrow x_5 < y_2.$$

Hence we obtain $x_5 < y_2 < -1$ and $y_5 < z_2 < -1$, $z_5 < x_2 < -1$ similarly.

Next we get (26) for $n = 3$. We attain that

$$\begin{aligned}
x_6 - y_3 &= (y_5 + 1)(z_4 + 1) - (x_4 + 1)(y_3 + 1) \\
&= (y_3 z_4 - 1 + 1)(z_4 + 1) - (x_2 y_3 - 1 + 1)(y_3 + 1) \\
&= y_3 z_4^2 + y_3 z_4 - x_2 y_3^2 - x_2 y_3
\end{aligned}$$

 doi:10.20944/preprints201910.0350.v1

From $y_3, x_2 < -1$, we have

$$\begin{aligned} x_6 - y_3 &< -z_4^2 + -z_4 + y_3^2 + y_3 \\ &= (y_3 - z_4)(y_3 + z_4 + 1) \end{aligned}$$

From $y_3 < -1$, we obtain

$$x_6 - y_3 < (y_3 - z_4) z_4 < 0.$$

So we get $x_6 < y_3 < -1$ and similarly $y_6 < z_3 < -1$ and $z_6 < x_3 < -1$.

Now we take (26) for $n = 4$. We obtain that

$$x_7 - y_4 = (y_6 + 1)(z_5 + 1) - (x_5 + 1)(y_4 + 1).$$

Since $y_6, z_5, x_5 < -1$ and $y_4 > 0$, we have $x_7 - y_4 > 0$ and $x_7 > y_4 > z_1 > 0$. Therefore we get $y_7 > z_4 > x_1 > 0$ and $z_7 > x_4 > y_1 > 0$.

We consider (26) for $n = 5$. We have that

$$x_8 - y_5 = (y_7 + 1)(z_6 + 1) - (x_6 + 1)(y_5 + 1).$$

Thus we obtain following

$$x_8 - y_5 < 0 \Rightarrow x_8 < y_5 < z_2 < -1.$$

from $y_7 > 0$ and $z_6, x_6, y_5 < -1$. Similarly we get $y_8 < z_5 < x_2 < -1$ and $z_8 < x_5 < y_2 < -1$.

We take (26) for $n = 6$. We have

$$x_9 - y_6 = (y_8 + 1)(z_7 + 1) - (x_7 + 1)(y_6 + 1) < 0$$

from $y_8, y_6 < -1$ and $z_7, x_7 > 0$. Thus we obtain $x_9 < y_6 < z_3 < -1$, $y_9 < z_6 < x_3 < -1$ and $z_9 < x_6 < y_3 < -1$.

Now we consider (26) for $n = 7$. Hence we obtain

$$x_{10} - y_7 = (y_9 + 1)(z_8 + 1) - (x_8 + 1)(y_7 + 1) > 0$$

from $y_9, z_8, x_8 < -1$ and $y_7 > 0$. So we have $x_{10} > y_7 > z_4 > x_1 > 0$, $y_{10} > z_7 > x_4 > y_1 > 0$ and $z_{10} > x_7 > y_4 > z_1 > 0$.

Finally we obtain the followings by induction

$$\begin{aligned}
0 &< x_1 < z_4 < y_7 < \cdots < x_{9k+1} < z_{9k+4} < y_{9k+7} < \cdots, \\
0 &< y_1 < x_4 < z_7 < \cdots < y_{9k+1} < x_{9k+4} < z_{9k+7} < \cdots, \\
0 &< z_1 < y_4 < x_7 < \cdots < z_{9k+1} < y_{9k+4} < x_{9k+7} < \cdots, \\
-1 &> x_2 > z_5 > y_8 > \cdots > x_{9k+2} > z_{9k+5} > y_{9k+8} > \cdots, \\
-1 &> y_2 > x_5 > z_8 > \cdots > y_{9k+2} > x_{9k+5} > z_{9k+8} > \cdots, \\
-1 &> z_2 > y_5 > x_8 > \cdots > z_{9k+2} > y_{9k+5} > z_{9k+8} > \cdots, \\
-1 &> x_3 > z_6 > y_9 > \cdots > x_{9k+3} > z_{9k+6} > y_{9k+9} > \cdots, \\
-1 &> y_3 > x_6 > z_9 > \cdots > y_{9k+3} > x_{9k+6} > z_{9k+9} > \cdots, \\
-1 &> z_3 > y_6 > x_9 > \cdots > z_{9k+3} > y_{9k+6} > z_{9k+9} > \cdots.
\end{aligned}$$

Therefore the proof completed as desired.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2019 doi:10.20944/preprints201910.0350.v1

(ii)

$$\begin{aligned} x_{9n+1} &= x_{9n-1}y_{9n}-1 \\ &= \left(x_{9n-3}y_{9n-2}-1\right)\left(y_{9n-2}z_{9n-1}-1\right)-1 \\ &= x_{9n-3}y_{9n-2}^{2}z_{9n-1}-x_{9n-3}y_{9n-2}-y_{9n-2}z_{9n-1} \end{aligned}$$

We have from $x_{9n-3}y_{9n-2}^{2}z_{9n-1}>0$ and $x_{9n-3}y_{9n-2}<0$,

$$x_{9n+1}>-y_{9n-2}z_{9n-1}.$$

From $z_{9n-1}<-1$, we obtain

$$\begin{aligned} x_{9n+1} &> y_{9n-2}=y_{9n-4}z_{9n-3}-1 \\ &= y_{9n-4}\left(z_{9n-5}x_{9n-4}-1\right)-1 \\ &= y_{9n-4}z_{9n-5}x_{9n-4}-y_{9n-4}-1. \end{aligned}$$

From $y_{9n-4}<-1$ and $y_{9n-4}x_{9n-4}>1$ we have

$$\begin{aligned} x_{9n+1} &> y_{9n-4}z_{9n-5}x_{9n-4}>z_{9n-5} \\ &= z_{9n-7}x_{9n-6}-1 \\ &= z_{9n-7}\left(x_{9n-8}y_{9n-7}-1\right)-1 \\ &= z_{9n-7}x_{9n-8}y_{9n-7}-z_{9n-7}-1. \end{aligned}$$

Thus we get from $z_{9n-7}<-1$ and $y_{9n-7}<-1$

$$x_{9n+1}>z_{9n-7}x_{9n-8}y_{9n-7}>x_{9n-8}.$$

So $\lim_{n\to\infty}x_{9n+1}=\infty$. Since the proof of the other cases are similar to this, we leave them to readers.

■

5 Stability of System (1)

Throughout this section we investigate the stability of system (1).

Now, we take into account the transformation to set up the linearized form of system (1):

$$\left(x_n,x_{n-1},y_n,y_{n-1},z_n,z_{n-1}\right)\to\left(f,f_1,g,g_1,h,h_1\right),$$

where

$$\begin{pmatrix} f=x_{n-1}y_n-1, \\ f_1=x_n, \\ g=y_{n-1}z_n-1, \\ g_1=y_n, \\ h=z_{n-1}x_n-1, \\ h_1=z_n. \end{pmatrix}.$$

Thus, we obtain the Jacobian matrix about the equilibrium point $(\bar{x}, \bar{y}, \bar{z})$:

$$B(\bar{x}, \bar{y}, \bar{z}) = \begin{pmatrix} 0 & \bar{y} & \bar{x} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{z} & \bar{y} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \bar{z} & 0 & 0 & 0 & 0 & \bar{x} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{29}$$

Theorem 16 *The equilibrium point* $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ *of system (1) is locally unstable.*

Proof. Linearized system of system (1) about the equilibrium point $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ is $X_{n+1} = B(\bar{x}, \bar{y}, \bar{z}) X_n$ where

$$X_n = ((x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}))^T$$

and

$$\begin{aligned} B(\bar{x}, \bar{y}, \bar{z}) &= B\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \\ &= \begin{pmatrix} 0 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1+\sqrt{5}}{2} & 0 & 0 & 0 & 0 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the characteristic equation of $B(\bar{x}, \bar{y}, \bar{z})$ about $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ is

$$\lambda^6 - \left(\frac{3+3\sqrt{5}}{2}\right)\lambda^4 + \left(2+\sqrt{5}\right)\lambda^3 + \left(\frac{9+3\sqrt{5}}{2}\right)\lambda^2 - 2 - \sqrt{5} = 0. \tag{30}$$

Then, six roots of (30) are

$$\begin{aligned} \lambda_1 &\approx -2.31651, \\ \lambda_2 &\approx -0.757501 - 0.456732i, \\ \lambda_3 &\approx -0.757501 + 0.456732i, \\ \lambda_4 &\approx 0.698478, \\ \lambda_5 &\approx 1.56652 - 0.944526i, \\ \lambda_6 &\approx 1.56652 + 0.944526i. \end{aligned}$$

Thus,

$$|\lambda_4| < |\lambda_2| = |\lambda_3| < 1 < |\lambda_5| = |\lambda_6| < |\lambda_1|.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2019 doi:10.20944/preprints201910.0350.v1

Hence, the first equilibrium point of system (1) is locally unstable from linearized stability theorem. ■

Theorem 17 *The equilibrium point* $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ *of system (1) is locally unstable.*

Proof. Linearized system of system (1) about the equilibrium point $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ is $X_{n+1} = B(\bar{x}, \bar{y}, \bar{z}) X_n$ where

$$X_n = ((x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}))^T$$

and

$$\begin{aligned} B(\bar{x}, \bar{y}, \bar{z}) &= B\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \\ &= \begin{pmatrix} 0 & \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1-\sqrt{5}}{2} & 0 & 0 & 0 & 0 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the characteristic equation of $B(\bar{x}, \bar{y}, \bar{z})$ about $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ is

$$\lambda^6 + \left(\frac{-3+3\sqrt{5}}{2}\right)\lambda^4 + \left(-2+\sqrt{5}\right)\lambda^3 + \left(\frac{9-3\sqrt{5}}{2}\right)\lambda^2 - 2 + \sqrt{5} = 0 \quad (31)$$

Hence, we have six roots of (31):

$$\begin{aligned} \lambda_1 &\approx -0.309017 - 0.722871i, \\ \lambda_2 &\approx -0.309017 + 0.722871i, \\ \lambda_3 &\approx 0.10393 - 0.549903i, \\ \lambda_4 &\approx 0.10393 + 0.549903i, \\ \lambda_5 &\approx 0.205087 - 1.08514i, \\ \lambda_6 &\approx 0.205087 + 1.08514i. \end{aligned}$$

From these we obtain that

$$|\lambda_3| = |\lambda_4| < |\lambda_1| = |\lambda_2| < 1 < |\lambda_5| = |\lambda_6|.$$

So, the second equilibrium point of system (1) is locally unstable from linearized stability theorem.

■

6 Conclusion

In this paper, we investigate the equilibrium points of system (1). Moreover we find out the periodic solutions of system (1) with three period. We also study the bounded or unbounded solutions of system (1). Finally, we analyze the stability of solutions of system (1) both the two equilibrium points.

References

[1] Liu K, Li P, Han F, Zhong W. Behavior of the Difference Equations x n+ 1= x n x n-1-1. J. Comput. Anal. Appl. 2017; 2017 22 (7), pp. 1361-1370.

[2] Okumuş İ, Soykan Y. On the Stability of a Nonlinear Difference Equation. Asian Journal of Mathematics and Computer Research 2017; 17 (2), pp. 88-110.

[3] Okumuş İ, Soykan Y. Some Technique To Show The Boundedness Of Rational Difference Equations. Journal of Progressive Research in Mathematics 2018; 13 (2), pp. 2246-2258.

[4] Okumuş İ, Soykan Y. Dynamical behavior of a system of three-dimensional nonlinear difference equations. Adv. Difference Equ. 2018; 2018:224, pp. 1-15.

[5] Papaschinopoulos G, Schinas CJ. On a system of two nonlinear difference equations. J. Math. Anal. Appl. 1998; 219 (2), pp. 415-426.

[6] Stevic S, Alghamdi MA, Maturi DA, Shahzad N. On the Periodicity of Some Classes of Systems of Nonlinear Difference Equations. Abstr. Appl. Anal. 2014; 2014, pp. 1-6.

[7] Agarwal RP, Wong PJ. Advanced topics in difference equations (Vol. 404). Springer Science & Business Media, 2013.

[8] Camouzis E, Ladas G. Dynamics of third order rational difference equations with open problems and conjectures, volume 5 of Advances in Discrete Mathematics and Applications, Chapman & Hall/CRC, Boca Raton, FL; 2008.

[9] Camouzis E, Papaschinopoulos G. Global asymptotic behavior of positive solutions on the system of rational difference equations xn+ 1= 1+ xnyn-m, yn+ 1= 1+ ynxn- m. Appl. Math. Lett. 2004; 17 (6): pp. 733-737.

[10] Din Q, Qureshi MN, Khan AQ. Dynamics of a fourth-order system of rational difference equations. Adv. Difference Equ. 2012; 2012:215, pp. 1-15.

[11] Elaydi S. An Introduction to Difference Equations. Springer-Verlag, 1996.

[12] Gelisken A, Kara M. Some general systems of rational difference equations. Journal of Difference Equations 2015; 2015, pp. 1-7.

[13] Göcen M, Güneysu M. The global attractivity of some rational difference equations. J. Comput. Anal. Appl. 2018; 25 (7), pp. 1233-1243.

[14] Göcen M, Cebeci A. On the Periodic Solutions of Some Systems of Higher Order Difference Equations. Rocky Mountain J. Math. 2018; 48 (3), pp. 845-858.

[15] Kent CM, Kosmala W, Radin MA, Stević S. Solutions of the difference equation $x_{n+1} = x_n x_{n-1} - 1$. Abstr. Appl. Anal. 2010; 2010, pp. 1-13.

[16] Kent CM, Kosmala W. On the Nature of Solutions of the Difference Equation $x_{n+1} = x_n x_{n-3} - 1$. International Journal of Nonlinear Analysis and Applications 2011; 2 (2), pp. 24-43.

[17] Kent CM, Kosmala W, Stević S. Long-term behavior of solutions of the difference equation $x_{n+1} = x_{n-1} x_{n-2} - 1$. Abstr. Appl. Anal. 2010; 2010, pp. 1-17.

[18] Kent CM, Kosmala W, Stević S. On the difference equation $x_{n+1} = x_n x_{n-2} - 1$. Abstr. Appl. Anal. 2011; 2011, pp. 1-15.

[19] Taşdemir E, Soykan Y. On the Periodicies of the Difference Equation $x_{n+1} = x_n x_{n-1} + \alpha$. Karaelmas Science and Engineering Journal 2016; 6 (2), pp. 329-333.

[20] Taşdemir E, Soykan Y. Long-Term Behavior of Solutions of the Non-Linear Difference Equation $x_{n+1} = x_{n-1} x_{n-3} - 1$. Gen. Math. Notes 2017; 38 (1), pp. 13-31.

[21] Taşdemir E, Soykan Y. Stability of Negative Equilibrium of a Non-Linear Difference Equation. J. Math. Sci. Adv. Appl. 2018; 49 (1), pp. 51-57.

[22] Taşdemir E, Soykan Y. Dynamical Analysis of a Non-Linear Difference Equation. J. Comput. Anal. Appl. 2019; 26 (2), pp. 288-301.

[23] Wang Y, Luo Y, Lu Z. Convergence of solutions of $x_{n+1} = x_n x_{n-1} - 1$. Appl. Math. E-Notes 2012; 12, pp. 153-157.

[24] Kurbanlı, A.S., Çinar, C., Yalçinkaya, İ. (2011). On the behavior of positive solutions of the system of rational difference equations xn+ 1= xn- 1ynxn- 1+ 1, yn+ 1= yn- 1xnyn- 1+ 1. Mathematical and Computer Modelling, 53(5-6), 1261-1267.

[25] Kurbanlı, A.S. On the Behavior of Solutions of the System of Rational Difference Equations: $x_{n+1} = x_{n-1}/y_n x_{n-1} + 1, y_{n+1} = y_{n-1}/x_n y_{n-1} + 1, z_{n+1} = z_{n-1}/y_n z_{n-1} + 1$, Discrete Dynamics in Nature and Society, vol. 2011, Article ID 932362, 12 pages, 2011.

[26] Kocic VL, Ladas G. Global behavior of nonlinear difference equations of higher order with applications. Vol. 256. Springer Science & Business Media, 1993.

[27] Kara M., Yazlık, Y., Solvability of a system of nonlinear difference equations of higher order*, Turk J Math, 43, 1533-1565, 2019.