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Article

A Fractional-Order Predator-Prey Model with Ratio-Dependent Functional Response and Linear Harvesting

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Abstract: We consider a model of predator-prey interaction at fractional-order where the predation obeys the ratio-dependent functional response and the prey is linearly harvested. For the proposed model, we show the existence, uniqueness, non-negativity as well as the boundedness of the solutions. Conditions for the existence of all possible equilibrium points and their stability criteria, both locally and globally, are also investigated. The local stability conditions are derived using the Magtinon's theorem, while the global stability is proven by formulating an appropriate Lyapunov function. The occurance of Hopf bifurcation around the interior point is also shown analytically. At the end, we implement the Predictor-Corrector scheme to perform some numerical simulations.

Keywords: fractional order differential equation, linear harvesting, stability analysis, Lyapunov function, Hopf bifurcation

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1. Introduction

One of interesting topics in ecological systems is the predator-prey model, which studies the dynamics of the populations as the extinction conditions of populations, and terms of its existence as the result of their interaction [1]. The famous predator-prey model was introduced by Rosenzweig and MacArthur [2] which was modified from the first predator-prey model by Lotka-Volterra [3]. This model is given by:

$$\begin{aligned} \frac{du}{dt} &= ru\left(1-\frac{u}{K}\right) - mp(u)v \\ \frac{dv}{dt} &= np(u)v - dv \end{aligned} \tag{1}$$

where $p(u) = \dfrac{u}{\omega + u}$ is the Michaelis-Menten functional response [2]. We respectively denote the population of prey and predator by u and v. The parameters r, K, m, n, ω, d are positive real numbers dan respectively denote r the prey intrinsic growth rate, the prey carrying capacity, the capturing rate of prey by predator, the conversion rate of predation into predator growth rate, the half saturation constant, and the predator death rate.

In modeling the interaction between predator and prey is important task in order to decide the spesific form of functional response [4], so the model is relevant to the expected ecological conditions. In the model (1), the predation rate depends on the functional response $p(u)$. Since the value of $p(u)$ is fluctuated by

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prey density and this functional response is called by "prey-dependence". Several researchers argue that the functional response not only prey-dependence, but also on the ratio of both populations [4–7], known also as "ratio-dependent" functional response. Such functional response is defined by $p(\frac{u}{v})$. Recently, Xiao and Cao [8] studied the interaction of prey and predator with a ratio-dependent functional response with linear harvesting for both prey and predator population:

$$\begin{aligned} \frac{du}{dt} &= ru\left(1-\frac{u}{K}\right)-\frac{muv}{u+\omega v}-k_1 u \\ \frac{dv}{dt} &= \frac{nuv}{u+\omega v}-dv-k_2 v. \end{aligned} \tag{2}$$

Using the following transformation

$$(u,v,t)\to\left(\frac{u}{K},\frac{\omega v}{K},rt\right)$$

model (2) can be simplified as

$$\begin{aligned} \frac{du}{dt} &= u(1-u)-\frac{auv}{u+v}-ku, \\ \frac{dv}{dt} &= \frac{buv}{u+v}-\delta v, \end{aligned} \tag{3}$$

where

$$a=\frac{m}{r},\ \ k=\frac{k_1}{r},\ \ b=\frac{n}{\omega},\ \ \delta=\frac{1}{r\omega}(d+k_2),\quad a,k,b,\delta>0.$$

Note that the prey and predator growth rates in the model (3) only depend on the current conditions. In fact, the growth rates of population having also depend on long-time memory. To include such memory effects, many researchers have applied fractional derivative in order to get fractional differential equations. There are various theories of fractional derivative in the literatures. Among many two of fractional derivatives are well known, namely, Riemann Liouville and Caputo. We consider here the Caputo fractional derivative since the classical initial values as in the differential equations of integer order can also be applied.

Definition 1. *[9] Suppose $\alpha > 0$. The fractional operator*

$$D_*^\alpha g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{g^{(n)}(s)}{(t-s)^{1+\alpha-n}}\,ds,$$

is called the Caputo fractional derivative of order α, where $n=\lceil\alpha\rceil$. Particularly, if $\alpha\in(0,1]$, then we have

$$D_*^\alpha g(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^t\frac{g'(s)}{(t-s)^{\alpha}}\,ds.$$

Note that the operator Caputo is nonlocal operator, i.e., includes the history from initial state to the current state. Therefore, the Caputo fractional derivative is often applied in modeling biological systems to describe the influence of memory effects, see e.g. [10–13], and [14]. Motivated by this fact, we replace the left hand side of system in (3) by the Caputo derivative and obtain the model of predator-prey interaction at fractional order as follows

$$\begin{aligned} D_*^\alpha u(t) &= u(1-u)-\frac{auv}{u+v}-ku, \\ D_*^\alpha v(t) &= \frac{buv}{u+v}-\delta v. \end{aligned} \tag{4}$$

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Now assume that the initial conditions are $u(0) = u_0 > 0$ and $v(0) = v_0 > 0$ where $\alpha \in (0, 1]$. In this study, further we consider $0 < k < 1$ as the harvesting parameter. By studying the current literature one can notice that the dynamical properties of system (4) has not been previously analyzed yet. Hence we aim to analyze the dynamics of system in (4). For this, we first introduce some basic concept of fractional differential equations.

2. Preliminaries

Theorem 1. *(See [15,16]). Consider an autonomous nonlinear fractional-order system*

$$D_*^\alpha \vec{u} = \vec{f}(\vec{u});\ \vec{u}(0) = \vec{u}_0;\ \alpha \in (0, 1].$$

A point $\vec{u}^*$ is called an equilibrium point of the system if it satisfies $\vec{f}(\vec{u}^*) = 0$. This equilibrium point is locally asymptotically stable if all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial f}{\partial u}$ evaluated at $\vec{u}^*$ satisfy $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$.

Lemma 1. *(See [17]). Let $u(t) \in C([0, +\infty))$. If $u(t)$ satisfies*

$$D_*^\alpha u(t) \leq -\lambda u(t) + \mu,\ \ u(0) = u_0 \in \mathbb{R},$$

where $\alpha \in (0, 1],\ \ \lambda, \mu \in \mathbb{R}$ and $\lambda \neq 0$, then

$$u(t) \leq \left(u_0 - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda t^\alpha] + \frac{\mu}{\lambda}.$$

Lemma 2. *[18] Let $u(t) \in C(\mathbb{R}_+)$ and its fractional derivatives of order α exist for any $\alpha \in (0, 1]$. Then, for any $t > 0$ we have*

$$D_*^\alpha \left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*}\right] \leq \left(1 - \frac{u^*}{u(t)}\right) D_*^\alpha u(t),\ \ u^* \in \mathbb{R}_+.$$

Lemma 3. *(See [19]). Consider a fractional order-system*

$$D_*^\alpha u(t) = f(t, u(t)),\ \ t > 0,\ u(0) \geq 0,\ \alpha \in (0, 1], \tag{5}$$

where $f : (0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \subseteq \mathbb{R}^n$. A unique solution of (5) on $(0, \infty) \times \Omega$ exists if $f(t, u(t))$ satisfies the locally lipschitz condition with respect to u.

Lemma 4. *(Generalized Lasalle Invariance Principle [20]). Suppose Ω is a bounded closed set and every solution of*

$$D_*^\alpha u(t) = f(u(t))$$

starts from a point in Ω and remains in Ω for all time. If $\exists\ V(u) : \Omega \to \mathbb{R}$ with continuous first partial derivatives satisfies

$$D_*^\alpha V|_{D_*^\alpha u(t) = f(u(t))} \leq 0.$$

Let $E := \left\{u | D_*^\alpha V|_{D_*^\alpha u(t) = f(u(t))} = 0\right\}$ and M be the largest invariant set of E. Then every solution $u(t)$ originating in Ω tends to M as $t \to \infty$.

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3. Main Results

3.1. Existence and uniqueness

In this section we investigate the existence and uniqueness of solution of the fractional order system (4) in the region $[0, \infty) \times \Omega_M$ where

$$\Omega_M = \left\{(u, v) \in \mathbb{R}^2 : \max\{|u|, |v|\} \leq \gamma\right\}$$

for sufficiently large γ. The existence of γ is guaranteed by the boundedness of the solution which will be shown later. We first denote $Y = (u, v)$ and $\bar{Y} = (\bar{u}.\bar{v})$, and then consider a mapping $F(Y) = (F_1(Y), F_2(Y))$ where

$$\begin{aligned} F_1(Y) &= u(1-u) - \frac{auv}{u+v} - ku \\ F_2(Y) &= \frac{buv}{u+v} - \delta v. \end{aligned}$$

For any $Y, \bar{Y} \in \Omega_M$, next we show that

$$\begin{aligned} \left\|F(Y) - F(\bar{Y})\right\| &= \left|F_1(Y) - F_1(\bar{Y})\right| + \left|F_2(Y) - F_2(\bar{Y})\right| \\ &= \left|u(1-u) - \frac{auv}{u+v} - ku - \bar{u}(1-\bar{u}) + \frac{a\bar{u}\bar{v}}{\bar{u}+\bar{v}} + k\bar{u}\right| + \\ &\quad \left|\frac{buv}{u+v} - \delta v - \frac{b\bar{u}\bar{v}}{\bar{u}+\bar{v}} + \delta\bar{v}\right| \\ &= \left|(1-k)(u-\bar{u}) - (u+\bar{u})(u-\bar{u}) - a\frac{u\bar{u}(v-\bar{v}) + v\bar{v}(u-\bar{u})}{(u+v)(\bar{u}+\bar{v})}\right| + \\ &\quad \left|b\frac{u\bar{u}(v-\bar{v}) + v\bar{v}(u-\bar{u})}{(u+v)(\bar{u}+\bar{v})} - \delta(v-\bar{v})\right| \\ &\leq (1-k)|u-\bar{u}| + 2\gamma|u-\bar{u}| + (a+b)\left|\frac{v\bar{v}}{(u+v)(\bar{u}+\bar{v})}\right||u-\bar{u}| + \\ &\quad (a+b)\left|\frac{u\bar{u}}{(u+v)(\bar{u}+\bar{v})}\right||v-\bar{v}| + \delta|v-\bar{v}| \\ &\leq (1-k+2\gamma+a+b)|u-\bar{u}| + (a+b+\delta)|v-\bar{v}| \\ &\leq L\left\|Y-\bar{Y}\right\|, \end{aligned}$$

where $L = \max\{1-k+2\gamma+a+b, a+b+\delta\}$. Hence, $F(Y)$ satisfies the Lipschitz condition. By Lemma 3, the fractional order system (4) with initial values $Y_0 = (u_0, v_0)$ where $u_0 \geq 0$ and $v_0 \geq 0$ has a unique solution $Y(t) = (u(t), v(t)) \in \Omega_M$. Thus, we establish the following existence and uniqueness of solution of system (4).

Theorem 2. *The fractional order predator-prey system (4) subject to any non-negative initial value (u_0, v_0) has a unique solution $(u(t), v(t)) \in \Omega_M$ for all $t > 0$.*

3.2. Boundedness and non-negativity

System (4) describes the interaction of prey population with predator population at fractional order and therefore solutions of this system must be bounded and non-negative. Let $\Omega_+ :=$

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$\{(u,v)|u \geq 0 \text{ and } v \geq 0\}$ denotes all non-negative real number in $\mathbb{R}^2$. The non-negativity and boundedness of solutions of system (4) are guaranteed by the following theorem.

Theorem 3. *All solutions of system (4) with $u_0 > 0$ and v_0 are uniformly bounded and non-negative.*

Proof. Assume that the initial values are $u_0 > 0$ and $v_0 > 0$ and define a function $w = u + \frac{a}{b}v$. From system (4), we obtain

$$\begin{aligned} D_*^\alpha w + \delta w &= u(1-u) - \frac{auv}{u+v} - ku + \frac{auv}{u+v} - \frac{a\delta}{b}v + \delta u + \frac{a\delta}{b}v \\ &= -u^2 + (1-k+\delta)u \\ &= -\left(u - \frac{1-k+\delta}{2}\right)^2 + \frac{(1-k+\delta)^2}{4} \\ &\leq \frac{(1-k+\delta)^2}{4}. \end{aligned}$$

Based on the comparison in theorem, we obtain, (see Lemma 1)

$$w(t) \leq \left(w(0) - \frac{(1-k+\delta)^2}{4\delta}\right) E_\alpha(-\delta t^\alpha) + \frac{(1-k+\delta)^2}{4\delta},$$

where E_α is the Mittag-Leffler function. Since

$$E_\alpha(-\delta t^\alpha) \longrightarrow 0 \;\; as \;\; t \longrightarrow \infty,$$

see [21, Lemma 5 and Collorary 6], we have

$$w(t) \leq \frac{(1-k+\delta)^2}{4\delta}, \;\; t \longrightarrow \infty.$$

Hence, all solutions of system (4) which start in R^2_+ are restricted to the region Ω_B where

$$\Omega_B = \left\{(u,v) \in \mathbb{R}^2_+ : u + \frac{a}{b}v \leq \frac{(1-k+\delta)^2}{4\delta} + \varepsilon, \varepsilon > 0\right\}. \tag{6}$$

Thus, all solutions of fractional order system (4) are uniformly bounded. Then in order to check the non-negativity solution, we first notice that

$$u + \frac{a}{b}v \leq \frac{(1-k+\delta)^2}{4\delta} \equiv \sigma. \tag{7}$$

Then by combining the first equation of system (4) with equation (7), we get

$$\begin{aligned} D_*^\alpha u &= u(1-u) - \frac{auv}{u+v} - ku \\ &\geq u(1-\sigma) - au - ku \\ &= (1-\sigma-a-k)u \\ &= \theta u, \end{aligned}$$

where $\theta = 1 - \sigma - a - k$. Using the comparison theorem (Lemma 1), we have

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$$u(t) \geq u_0 E_\alpha(\theta t^\alpha).$$

Since $E_\alpha(t) > 0$, for any $\alpha \in (0,1]$, we conclude that $u(t) > 0$ for any $t \geq 0$. According to the second equation in (4) we obtain

$$\begin{aligned} D_*^\alpha v &= \frac{buv}{u+v} - \delta v \\ &\geq -\delta v. \end{aligned}$$

Using the same previous argument, we get $v(t) \geq v_0 E_\alpha(-\delta t^\alpha)$ and therefore $v(t) > 0$ for any $t \geq 0$. Hence, the fractional order system (4) has always non-negative solutions. □

3.3. Local Stability

Based on Theorem (1), we can show that system (4) has three equilibrium points as follows:

1. The extinction point of both prey and predator population $E_0 = (0,0)$ which is always feasible.
2. The free predator point $E_1 = (k_0, 0)$ which also awalys exists. Here $k_0 = 1 - k$.
3. The interior point $E^* = (u^*, v^*)$ where $u^* = \frac{1}{b}(bk_0 - a(b-\delta))$ and $v^* = \frac{1}{\delta}(b-\delta)u^*$. Notice that E^* exists if $0 < (b-\delta) < \frac{b}{a}k_0$.

In the following we study the dynamics of system (4) around each of equilibrium point. For that, we linearize system (4) and get the following Jacobian matrix

$$J(E) = \begin{bmatrix} k_0 - 2u - \frac{av^2}{(u+v)^2} & -\frac{au^2}{(u+v)^2} \\ \frac{bv^2}{(u+v)^2} & \frac{bu^2}{(u+v)^2} - \delta \end{bmatrix}. \tag{8}$$

By evaluating this Jacobian matrix at each equilibrium points and applying theorem (1), we obtain the stability properties of E_0 and E_1 as follows.

Theorem 4. *For the fractional order system (4), the extinction of both population point (E_0) and the free predator point (E_1) have the following stability properties.*

1. *E_0 is a saddle point.*
2. *If $b < \delta$ then E_1 is locally asymptotically stable and it is a saddle if $b > \delta$.*

Proof.
1. The Jacobian matrix (8) evaluated at E_0 is

$$J(E_0) = \begin{bmatrix} k_0 & 0 \\ 0 & -\delta \end{bmatrix}.$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = k_0 > 0$ and $\lambda_2 = -\delta < 0$, and consequently we have $|\arg(\lambda_1)| = 0 < \alpha\pi/2$ and $|\arg(\lambda_2)| = \pi > \alpha\pi/2$ for $0 < \alpha < 1$. Hence E_0 is a saddle point.
2. If E_1 is substituted into the Jacobian matrix (8), then we have

$$J(E_1) = \begin{bmatrix} -k_0 & -a \\ 0 & b-\delta \end{bmatrix}.$$

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Obviously that $J(E_1)$ has eigenvalues $\lambda_1 = -k_0 < 0$ and $\lambda_2 = b - \delta$. We observe that $|\arg(\lambda_1)| = \pi > \alpha\pi/2$. If $b < \delta$ then $|\arg(\lambda_2)| = \pi > \alpha\pi/2$ and if $b > \delta$ then $|\arg(\lambda_2)| = 0 < \alpha\pi/2$. Therefore E_1 is asymtotically stable (locally) if $b < \delta$ and is a saddle point if $b > \delta$.
□

We now examine the stability of equilibrium E^*. The characteristics equation of the Jacobian matrix evaluated at E^* is given by

$$\lambda^2 - T\lambda + D = 0, \tag{9}$$

where $T = -\left(b^2k_0 + b^2(\delta - a) + \delta^2(a - b)\right)/b^2$ and $D = \left(b\delta k_0(b-\delta) - a\delta(b-\delta)^2\right)/b^2$. From the existence condition of E^* we notice that $D > 0$. The eigenvalues of $J(E^*)$ is

$$\lambda_{1,2} = \frac{T \pm \sqrt{\Delta}}{2}, \ \Delta = T^2 - 4D.$$

By analyzing these eigenvalues, the stability of E^* is stated in following theorem .

Theorem 5. *For the fractional order system (4), the interior point E^* is locally asymptotically stable if one of the following mutually exclusive conditions holds:*

1. $T < 0$ *and* $\Delta \geq 0$
2. $\Delta < 0$ *and* $\frac{\sqrt{|\Delta|}}{T} > \tan\left(\frac{\alpha\pi}{2}\right)$.

Proof.
1. Since $D > 0, T < 0$ and $\Delta \geq 0$, then $\lambda_{1,2} < 0$ and $\arg(\lambda_{1,2}) = \pi > \alpha\pi/2$. Therefore E^* is asymtotically stable.
2. Suppose $\Delta < 0$. If λ is an eigenvalue then its complex conjugate $(\bar{\lambda})$, is also an eigenvalue. We have that $\left|\frac{\lambda-\bar{\lambda}}{\lambda+\bar{\lambda}}\right| = \left|\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right| = \arg(\lambda) = \frac{\sqrt{|\Delta|}}{T}$. Using the Matignon's condition, see [15, Theorem 2], it is obvious that E^* is locally asymptotically stable if $\frac{\sqrt{|\Delta|}}{T} > \tan\left(\frac{\alpha\pi}{2}\right)$.

□

3.4. Hopf Bifurcation

For the following fractional-order commensurate system:

$$D_*^\alpha w = f(\mu, w), \ \ \alpha \in (0,1], \ w \in \mathbb{R}^2, \tag{10}$$

Abdelouahab et al. [22] stated that a Hopf bifurcation occurs around an equilibrium E at $\mu = \mu^*$ if the following conditions hold:

– The eigenvalues of the Jacobian matrix are a pair of complex-conjugate: $\lambda_{1,2}(\mu) = \zeta(\mu) \pm i\omega(\mu)$
– $p_{1,2}(\alpha, \mu^*) = 0$
– $\frac{\partial p_{1,2}}{\partial \mu}|_{\mu=\mu^*} \neq 0$,

where $p_j(\alpha, \mu) = \frac{\alpha\pi}{2} - |\arg(\lambda_j(\mu))|, j = 1, 2$.

The existence of a Hopf bifurcation in the system (4) is analyzed as follows. From the Theorem 5, we can derive the following theorem.

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Theorem 6. *Suppose $\Delta < 0$ and $T > 0$. The fractional model (4) undergoes a Hopf bifurcation at E^* when the fractional order α crosses the critical values*

$$\alpha^* = \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{|\Delta|}}{T}\right).$$

Proof. If $\Delta < 0$, $T > 0$ and $\alpha = \alpha^*$, then the characteristic equation of the Jacobian matrix at E^* has a pair of conjugate complex roots $\lambda_{1,2}$ located on the border of stability area $\arg(\lambda_{1,2}) = \frac{\alpha^*\pi}{2}$. If α changes around α^*, $\lambda_{1,2}$ pass through the stability margin and there occurs a Hopf bifurcation. □

3.5. Global asymptotic stability

Theorem 7. *Let $k_0 = 1 - k$. E_1 is globally asymptotically stable in the region* $\Omega_1 = \left\{(u,v)|u+v \geq \frac{bk_0}{\delta}\right\}$.

Proof. Define a Lyapunov function $\mathcal{U}(u,v) = \left(u - k_0 - k_0\ln\frac{u}{k_0} + \frac{a}{b}v\right)$. Using Lemma 2, we can show

$$\begin{aligned} D_*^\alpha\mathcal{U}(u,v) &\leq \frac{u-k_0}{u}D_*^\alpha u + \frac{a}{b}D_*^\alpha v \\ &= (u-k_0)\left[k_0 - u - a\frac{v}{u+v}\right] + \frac{a}{b}\left(b\frac{u}{u+v} - \delta\right)v \\ &= -(u-k_0)^2 + a\left[\frac{k_0}{u+v} - \frac{\delta}{b}\right]v. \end{aligned}$$

It is obvious that $D_*^\alpha\mathcal{U}(u,v) \leq 0, \forall(u,v) \in \Omega_1$. Furthermore, $D_*^\alpha\mathcal{U}(u,v) = 0$ implies that $u = k_0$ and $v = 0$. Hence, the only invariant set on which $D_*^\alpha\mathcal{U}(u,v) = 0$ is the singleton $\{E_1\}$. Using Lasalle invariance principle (see [20, Lemma 4.6]) we conclude that E_1 is globally asymptotically stable. □

Theorem 8. *E^* is globally asymptotically stable in* $\Omega_2 = \left\{(u,v)|\frac{v}{v^*} > \frac{u}{u^*} > 1\right\}$.

Proof. Consider a Lyapunov function

$$\mathcal{L}(u,v) = \left(u - u^* - u^*\ln\frac{u}{u^*}\right) + \frac{a}{b}\left(v - v^* - v^*\ln\frac{v}{v^*}\right).$$

Then based on Lemma 2, we show that

$$\begin{aligned} D_*^\alpha\mathcal{L}(u,v) &\leq \frac{u-u^*}{u}D_*^\alpha u(t) + \frac{a}{b}\left(\frac{v-v^*}{v}\right)D_*^\alpha v(t) \\ &= (u-u^*)\left(1 - u - a\frac{v}{(u+v)} - k\right) + \frac{a}{b}(v-v^*)\left(\frac{bu}{u+v} - \delta\right) \\ &= (u-u^*)\left(-u - a\frac{v}{(u+v)} + u^* + a\frac{v^*}{(u^*+v^*)}\right) + a(v-v^*)\left(\frac{u}{u+v} - \frac{u^*}{u^*+v^*}\right) \\ &= -(u-u^*)^2 + a\frac{(u-u^*)(uv^* - u^*v) + (v-v^*)(uv^* - u^*v)}{(u+v)(u^*+v^*)}. \end{aligned}$$

Hence, $D_*^\alpha\mathcal{L}(u,v) \leq 0$ for arbitrary $(u,v) \in \Omega_2$. Furthermore $D_*^\alpha\mathcal{L}(u,v) = 0$ implies that $u = u^*$ and $v = v^*$. Hence, the singleton $\{E^*\}$ is the only invariant set such that $D_*^\alpha\mathcal{L}(u,v) = 0$. Again, the Lasalle invariance principle (see [20, Lemma 4.6]) gives conclusion that E^* is globally asymptotically stable. □

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4. Numerical Simulations

To verify the previous analytical results, some numerical simulations of system (4) are performed. For that aim, we implement the predictor-corrector scheme developed by Diethelm [23] to solve our fractional-order model (4). For the first simulation, we use hypothetical parameter as in [8]: $a = 1.3$, $k = 0.25$, and $\delta = 0.4$. Based on Theorem 4, Theorem 5 and Theorem 6, we plot the bifurcation diagram in $(\alpha, b)-$plane as shown in Figure 1. In this figure, we can see three different regions. The yellow area represents the stable predator extinction point (E_1); the green area denotes the stable coexistence point (E^*); and the cyan area corresponds to the limit cycle oscillation. From this figure we see that for the case of $b = 0.3$ with $\alpha = 0.75$ or $\alpha = 0.9$, the predator extinction point $E_1 = (0.75, 0.0)$ is asymptotically stable. This behaviour is clearly seen from the phase-portraits shown in Figure 2, i.e. all solutions are convergent to E_1. From Theorem (6), we find that if $\Delta < 0$ and $T > 0$ then there occurs a Hopf bifurcation around E^* when α passes through the critical values α^*. The critical values of α in Figure 1 is shown by the line between green area and cyan area. This figure also shows that the Hopf bifurcation can also be driven by parameter b. To show the phenomenon of Hopf bifurcation, we solve system (4) with the same parameter values as before, except $b = 0.8$. From these parameter values, we get $\alpha^* = 0.94366$. Hence, $E^* = (0.1, 0.1)$ is asymptotically stable for $\alpha in(0, \alpha^*)$ and is unstable for $\alpha > \alpha^*$. The numerical solution depicted in Figure 3.$(a - b)$ shows that for $\alpha = 0.9 < \alpha^*$, the solution is convergent to E^*. On the other hand, for $\alpha = 0.95 > \alpha_*$, the solution is not convergent to any point, and it is converging to a periodic solution, see Figure 3.$(c - d)$. This shows that system (4) undergoes Hopf bifurcation.

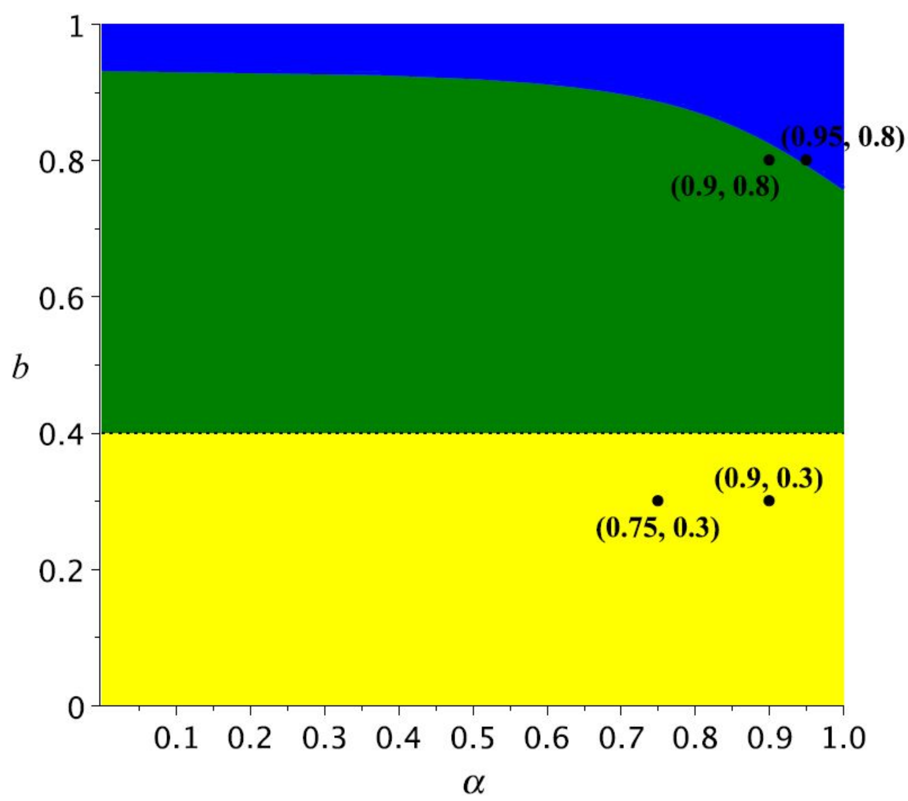

Figure 1. Bifurcation diagram in $(\alpha, b)-$plane for prey-predator system (4) with $a = 1.3, k = 0.25$ and $\delta = 0.4$.

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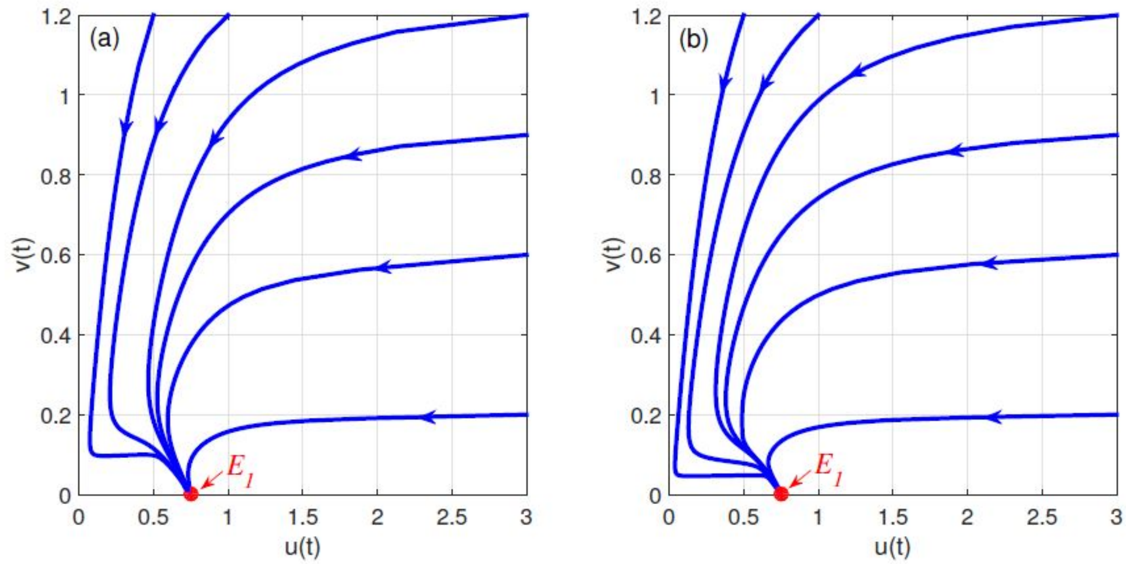

Figure 2. Phase-portraits of prey-predator system (4) with $a = 1.3, k = 0.25, \delta = 0.4$ and $b = 0.3$ for different order of fractional derivative: (a) $\alpha = 0.75$, and (b) $\alpha = 0.9$.

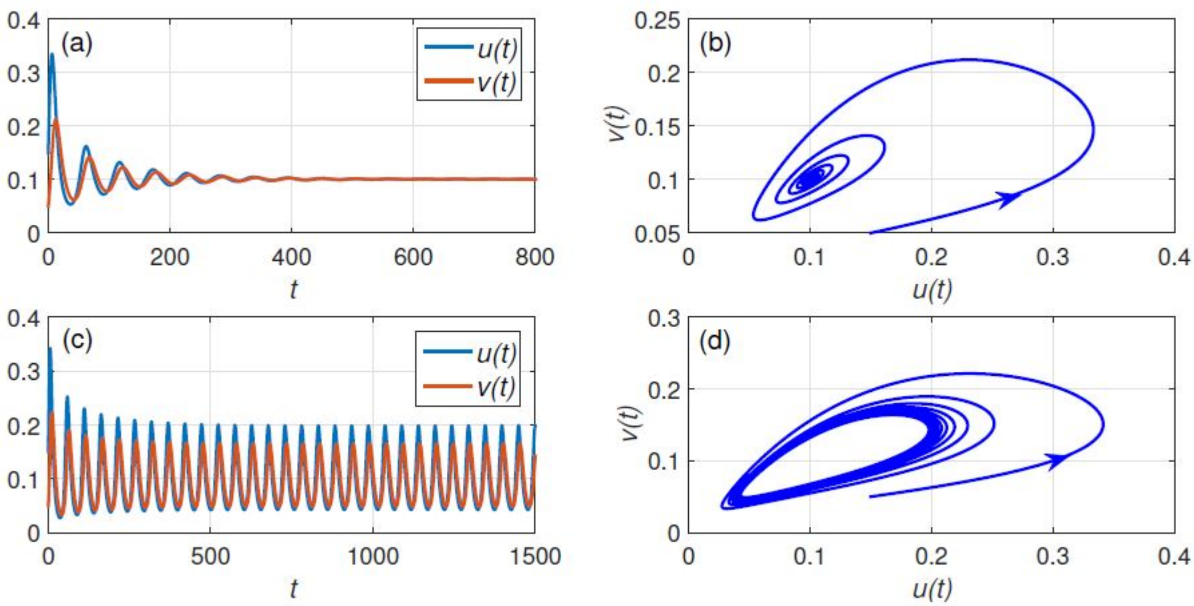

Figure 3. Numerical solutions of prey-predator population as function of time t and the phase-diagrams of system (4) with $a = 1.3, k = 0.25, \delta = 0.4, b = 0.8$ and different order of fractional derivative: (a-b) $\alpha = 0.9$, (c-d) $\alpha = 0.95$.

Next, we show the bifurcation diagram in $(\alpha, k)-$plane for system (4) with $a = 1.3, b = 0.8$, and $\delta = 0.4$, see Figure 4. Figure 4 shows that there are two different stability regions. As in the previous case, the green area represents the asymptotically stable area of coexistence point (E^*), while the cyan area represents the area of stable limit cycle. Thus, the line which separates the two areas corresponds to the Hopf bifurcation point. It is seen that smaller order of fractional derivative has a larger value of critical harvesting rate k^*. For example, Xiao and Cao [8] have shown that for the case of $\alpha = 1$, the critical value of harvesting rate is $k^* = 0.225$, see also Figure 4. If we reduce the value of α such that $\alpha = 0.9$ then the Hopf bifurcation point becomes $k^* = 0.26564$. Hence, for $\alpha = 0.9$ and $k = 0.25 < k^*$, the coexistence point E^* is asymptotically stable. This behavior can be seen in Figure 3.$(a - b)$. If we take $k = 0.3 > k^*$, then the solution converges to a periodic solution which shows that E^* is unstable; see Figure 5.

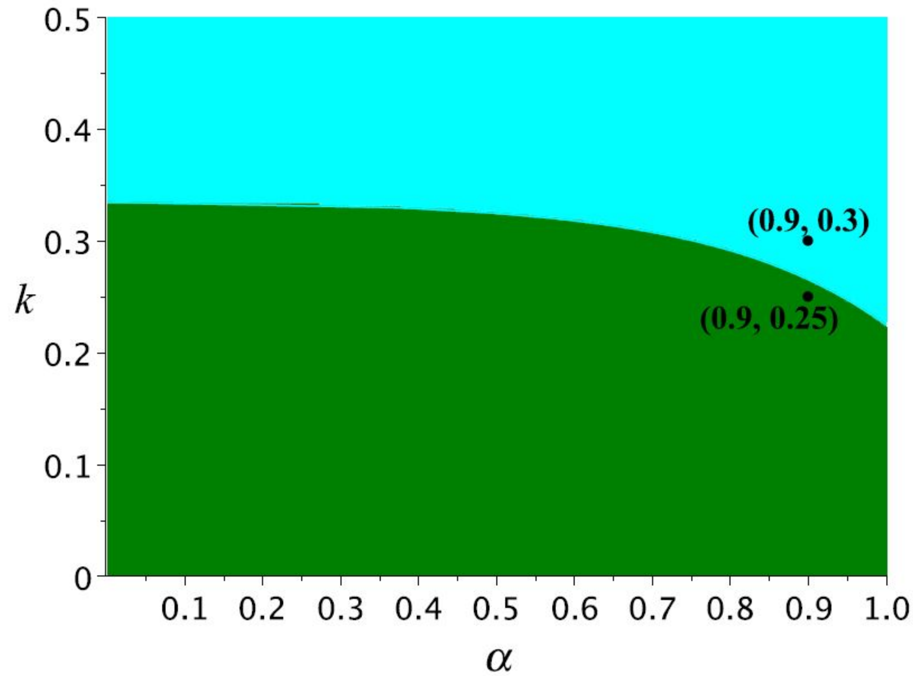

Figure 4. Bifurcation diagram in $(\alpha, k)-$plane for prey-predator (4) with $a = 1.3, b = 0.8$ and $\delta = 0.4$.

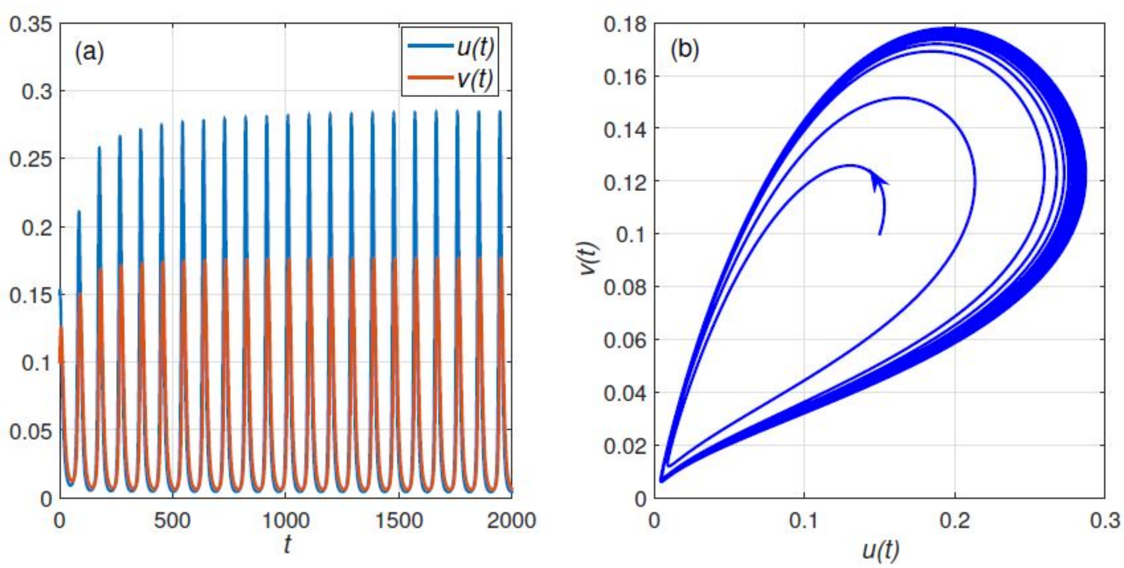

Figure 5. (a) Numerical solutions of prey-predator population as function of time t and the phase-diagrams of system (4) with $a = 1.3, b = 0.8, \delta = 0.4, k = 0.3$ and $\alpha = 0.9$.

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5. Conclusion

We have introduced and analyzed a fractional order ratio-dependent predator-prey model with linear harvesting. The existence, uniqueness, non-negativity as well as boundedness of solutions for the proposed model have been proven. Based on Matignon's Theorem, we have shown the local stability of all possible equilibrium points. Since the related Jacobian matrix has real number eigenvalues, the stability properties of the extinction point of both population and the free predator point are exactly the same as those of first order system (see [8]). However, it is not the case for the coexistence point as the eigenvalues of its Jacobian matrix might a be complex number. The global stability of the free predator point and the coexistence point were also studied by defining an appropriate Lyapunov function. Further, the existence of Hopf bifurcation driven by the order of fractional derivative (α) has also been established. From the bifurcation diagram, it is also shown that the Hopf bifurcation may be driven by parameter b or k. The dynamical properties of the proposed system have been confirmed by the numerical simulations.

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