drawn, but the readers is invited to judge the cases from the presented situations. Radical conclusions by the writer would have finished the investigations.

The Yang–Baxter equation is also related to the theory of universal quantum gates, to quantum computers, big data, etc. The explosion of data generated by the digital age led to very skillful and powerful / quantum computers. So, apparently, in order to deal with big data, there are two choices: teaching computers to be more effective or building stronger / quantum computers. However, there exists a third option: developing smarter hybrid multi-agent systems (cf. [6]).

The Yang–Baxter equation has many applications in quantum groups and knot theory. Many Yang-Baxter operators lead to knot invariants (see [7]). We will consider a generalization of this equation, called the Modified Yang-Baxter equation, in the next section.

The third section deals with *Euler's formula*, $e^{ix} = \cos x + i \sin x$. Voted the most famous formula by students, *Euler's identity*, $e^{i\pi} = -1$, contains the transcendental numbers e and π and the imaginary number i. The Euler's formula is more general than the the Euler's identity. We have previously obtained an Euler's formula for hyperbolic functions. Now we will refer to Euler's formulas for dual numbers (which can be related to the Yang–Baxter equation).

There is also a section on non-associative structures. Mathematics was in the beginning associative and commutative, but it then became non-commutative, and afterwards it became non-associative also (see [8–13]). There are two important classes of non-associative structures: Lie structures and Jordan structures. Various Jordan structures play an important role in quantum group theory and in fundamental physical theories. In recent years, several attempts to unify non-associative structures have led to interesting results. The UJLA structures are not the only structures which realize such a unification. Associative algebras and Lie algebras can be unified at the level of Yang–Baxter structures.

We also present a section on the unification of means and their inequalities. Thus, for example, the *classical means inequalities* can be unified. More inequalities will follow from our approach.

The motivations for this work are multiple. Finding solutions to the coloured Yang–Baxter equation is a very important and difficult problem. After proposing the Modified Yang-Baxter equation in [2], the authors of [19] have sent us their work and we would like to thank them by giving at least a simple answer to their suggestion. Many of our results are related to concepts like *transcendence* ([14,15]) and *transdisciplinarity* ([16–18]), allowing us to step forward toward literature and philosophy. We also propose several open problems.

We will work over the field k, when it is not otherwise specified. The tensor products will be defined over k. As usually, we write $M_n(k)$ for the ring of all $n \times n$-matrices over the field k. In particular we write I for the identity matrix in $M_4(k)$ respectively I' for the identity matrix in $M_2(k)$.

2. Modified Yang-Baxter equation

For $A \in M_n(\mathbb{C})$ and $D \in M_n(\mathbb{C})$ a diagonal matrix, we proposed (see [2]) the problem of finding $X \in M_n(\mathbb{C})$ such that

$$AXA + XAX = D \tag{1}$$

Remark 2.1 *The equation (1) is a type of Yang-Baxter matrix equation if $D = O_n$ and $X = -Y$. It is related to the three matrix problem, and it can be interpreted as "a generalized eigenvalue problem".*

For $A \in M_2(\mathbb{C})$, a matrix with trace -1, and

$$D = -\begin{pmatrix} det(A) & 0 \\ 0 & det(A) \end{pmatrix} \tag{2}$$

(1) has a solution X= I'.

Remark 2.2 *We think that the methods of [19] will lead to solutions for the equation (1). For example, solving the following system*

$$AXA + CX = D\,, \quad C = XA \tag{3}$$

leads to solutions of (1).

3. Euler's formulas for dual numbers

Following our previous study ([1,2]), an Euler's formula for dual numbers (see [20]) could be the following formula: $1 + ax = e^{ax}$, where $a^2 = 0$. The applications of this formula could be of the following type. If we consider the complex valuated matrix ($c, d \in \mathbb{C}$):

$$J = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

then,

$$J^2 = 0_4\,, \qquad J^{12}J^{23} = J^{23}J^{12}\,,$$

So,

$$I + Jx = e^{xJ} \; = R(x) \tag{5}$$

and

$$(R \otimes I')(x) \circ (I' \otimes R)(x+y) \circ (R \otimes I')(y) \; = (I' \otimes R)(y) \circ (R \otimes I')(x+y) \circ (I' \otimes R)(x)\,. \tag{6}$$

(Formula (6) is verified because $xJ^{12} + (x+y)\,J^{23} + yJ^{12} = yJ^{23} + (x+y)\,J^{12} + xJ^{23}$.)

Remark 3.1 *Formula (5) can be interpreted in terms of coalgebras (see [2,21]). Thus, there exists a coalgebra over $\frac{k[X]}{X}^2 = k[a]$, where $a^2 = 0$, generated by two generators u and i, such that $\Delta(u) = u \otimes u$, $\Delta(i) = u \otimes i + i \otimes u$, $\varepsilon(u) = 1$ and $\varepsilon(i) = 0$. (5) leads to the subcoalgebra generated by $u + ai$.*

Remark 3.2 *In this case we can also consider another Euler's formula:*

$\cos x + a \sin x = \sum_{j \geq 0} (-1)^j \frac{x^{2j}}{(2j)!} e^{\frac{ax}{2j+1}}$.

Remark 3.3 *It would be interesting to study a unifying theory for complex numbers, dual numbers, and other cases (see, for example, [2]) in which $k = \mathbb{R}[a]$, with $a^2 \in \mathbb{R}$ (see, also, Remark 5 of [1]).*

4. Unification of non-associative structures and differential geometry

This section is related to a talk given at the 14-th International Workshop on Differential Geometry and Its Applications, UPG Ploiesti, 2019.

Definition 4.1 *For a k-space V, let $\eta : V \otimes V \to V, \ \ a \otimes b \mapsto ab$, be a linear map such that:*

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \tag{7}$$

$$(a^2b)a \ = \ a^2(ba), \quad (ab)a^2 \ = \ a(ba^2), \quad (ba^2)a \ = \ (ba)a^2, \quad a^2(ab) \ = \ a(a^2b), \tag{8}$$

$\forall\, a, b, c \in V$. *Then, (V, η) is called a* **UJLA structure**.

Theorem 4.2 *For (V, η) a UJLA structure, $D(x) = D_b(x) = bx - xb$ is a* **UJLA–derivation** *(i.e., $D(a^2a) = D(a^2)a + a^2D(a)\ \forall a \in V$).*

Proof. In formula (7) we take $c = a^2$: $\quad (ab)a^2 + (ba^2)a + (a^2a)b = a(ba^2) + b(a^2a) + a^2(ab)$.
It follows from (8) that $(ba^2)a + (a^2a)b = b(a^2a) + a^2(ab)$.
So, $(ba^2)a - a^2(ab) = b(a^2a) - (a^2a)b$; so, $b(a^2a) - (a^2a)b = (ba^2 - a^2b)a + a^2(ba - ab)$.
Thus, $D(a^2a) = D(a^2)a + a^2D(a)$.

Definition 4.3 *For the vector space V, let $d : V \to V$ and $\phi : V \otimes V \to V \otimes V$, be a linear map which satisfies:*

$$\phi^{12} \circ \phi^{23} \circ \phi^{12} = \phi^{23} \circ \phi^{12} \circ \phi^{23} \tag{9}$$

where $\phi^{12} = \phi \otimes I,\ \phi^{23} = I \otimes \phi, \ \ I : V \to V, \ \ a \mapsto a$.

Then, (V, d, ϕ) is called a **generalized derivation** *if $\phi \circ (d \otimes I + I \otimes d) = (d \otimes I + I \otimes d) \circ \phi$.*

Remark 4.4 *If A is an associtive algebra, $d : A \to A$ a derivation (so, $d(1_A)$=0), and $\phi : A \otimes A \to A \otimes A, \ \ a \otimes b \mapsto ab \otimes 1 + 1 \otimes ab - a \otimes b$, then (A, d, ϕ) is a generalized derivation.*

If C is a coalgebra, $d : C \to C$ a coderivation, and $\psi : C \otimes C \to C \otimes C, \ \ c \otimes d \mapsto \varepsilon(d)c_1 \otimes c_2 + \varepsilon(c)d_1 \otimes d_2 - c \otimes d$, then (C, d, ψ) is a generalized derivation.

Definition 4.5 *Let A is an associtive algebra, $d : A \to A$ a derivation, M an A-bimodule, and $D : M \to M$ with the property $D(am) = d(a)m + aD(m)$. Then, (A, d, M, D) is called a* **module derivation**.

Theorem 4.6 *([2]) In the above case, $A \times M$ becomes an algebra, and $\delta : A \times M \to A \times M, \ (a,\ m) \mapsto (a, D(m))$ is a derivation in this algebra.*

Translated into the "language" of Differential Geometry, the above theorem says that the Lie derivative is a derivation (i.e., $d(ab) = d(a)b + ad(b)$) on the product of the algebra of functions defined on the manifold M with the set of vector fields on M (see [22]).

Remark 4.7 *A dual for Theorem 4.6 (using coderivations and comodules) and the unification of module derivation and comodule derivation are work in progress.*

5. Unification of mean inequalities

We present a family of inequaties which unify and enhance the means inequalities.

Theorem 5.1 *For three real numbers* $a > 0,\ b > 0$ *and* $r > 0$ *the real function* $M_r : \mathbb{R} \to \mathbb{R},\quad M_r(x) = \left(\frac{a^x+b^x}{a^{x-r}+b^{x-r}}\right)^{\frac{1}{r}}$ *is an increasing function.*

Proof. One way to prove this theorem is by direct computations.
Alternatively, the proof is a particular case of the general case, which will be discussed below.

Remark 5.2 *The above theorem includes the classical means inequalities (the harmonic mean is less or equal than the geometric mean, which is less or equal than the artmetic mean) because* $M_1(0) \leq M_1(\frac{1}{2}) \leq M_1(1)$.

So, the means are unified and their inequalities are included in the property that M_1 *is an increasing function.*

Now, the question is what are the relations between $M_r(x)$ and $M_p(y)$. A tentative answer is given below.

Theorem 5.3 *The following inequality for real numbers, where* $a > 0,\ b > 0$, *and* $x \leq y$,

$$M_r(x) = \left(\frac{a^x+b^x}{a^{x-r}+b^{x-r}}\right)^{\frac{1}{r}} \leq \left(\frac{a^y+b^y}{a^{y-p}+b^{y-p}}\right)^{\frac{1}{p}} = M_p(y) \tag{10}$$

holds for the following additional conditions:
(i) $p = r$;
(ii) $p = \frac{1}{2},\ r = 1,\ x = y$;
(iii) $p = 1,\ r = \frac{1}{2}$ *and* $x + \frac{1}{2} = y$.

Proof. Straightforward.

Theorem 5.4 *For n pozitive real numbers* $a_j,\ 1 \leq j \leq n$, *and for* $r > 0$, *the real function* $M_r(x) = \left(\frac{\sum a_j^x}{\sum a_j^{x-r}}\right)^{\frac{1}{r}}$ *is an increasing function.*

Proof. One can show that the derivative of $M_r(x)$ is positive, for a fixed r.

We now present inequalities which follow directly from the above Theorem 5.3 (for $a > 0,\ b > 0$):

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \left(\frac{2}{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}\right)^2 \leq \sqrt{ab} \leq \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+b}{2} \leq a+b-\sqrt{ab} \leq \frac{a^2+b^2}{a+b}\,. \tag{11}$$

More demanding are the inequalities given below ($a > 0,\ b > 0$):

$$\frac{a+b}{2} \leq \left(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} \leq \sqrt{\frac{a^2+b^2}{2}} \leq a+b-\sqrt{ab}\,. \tag{12}$$

Remark 5.5 *Note also the following types of inequalities for means* ($a > 0,\ b > 0$)*:*

$$\sqrt{ab} \leq \frac{\frac{a+b}{2} + \sqrt{ab} + \frac{2}{\frac{1}{a}+\frac{1}{b}}}{3} \leq \frac{\frac{a+b}{2} + \frac{2}{\frac{1}{a}+\frac{1}{b}}}{2}\ . \tag{13}$$

Remark 5.6 *It is an open problem to prove that for* $p \geq r,\ \ M_r(x) \leq M_p(x)\ \forall x \in \mathbb{R}$.

Also we conjecture that $M_r(x) \leq M_p(y)\ \forall x - r \leq y - p$.

Future investigations will try to extend the conditions (i-iii) in the above Theorem 5.3, and to prove versions of the above inequalities with more variables.

Other increasing functions might be considered in order to unify the means inequalities.

According to Theorems 3.2 and 3.3 (for $\alpha = 1$ *and* $\beta = \frac{1}{2}$*) from* Axioms 2015, 4, 423–435*, the classical means are related to the set-theoretical Yang-Baxter equation. It follows easily that* $M_x(x)$ *is also related to the set-theoretical Yang-Baxter equation. In the general case, for* $M_r(x)$*, the question on this relationship this remains a reserch direction.*

References

1. Nichita, F.F. *Unification Theories: New Results and Examples*. Axioms 2019, 8, 60.
2. Nichita, F.F. *Unification Theories: Examples and Applications*. Axioms 2018, 7, 85.
3. Oner, T.; Senturk, I.; Oner, G. *An Independent Set of Axioms of MV-Algebras and Solutions of the Set-Theoretical Yang–Baxter Equation*. Axioms 2017, 6, 17.
4. Oner, T.; Katican, T. *On Solutions to the Set–Theoretical Yang–Baxter Equation in Wajsberg–Algebras*. Axioms 2018, 7, 6.
5. Nichita, F.F. (Ed.) *Hopf Algebras, Quantum Groups and Yang-Baxter Equations*, MDPI, 2019, ISBN 978-3-03897-324-9 (Pbk); ISBN 978-3-03897-325-6 (PDF).
6. Nichita, F.F. *Machine Learning, Quantum Computers and Hybrid Multi–Agent Systems*, preprint, private communication, Easter Michigan University, spring 2019.
7. Nichita, F.F. *Introduction to the Yang–Baxter Equation with Open Problems*. *Axioms* 2012, 1, 3337.
8. Nichita, F.F. *On Jordan algebras and unification theories*, REVUE ROUMAINE DE MATHMATIQUES PURES ET APPLIQUES (ROMANIAN JOURNAL OF PURE AND APPLIED MATHEMATICS) Selected papers from the 12th International Workshop on Differential Geometry and its Applications (23-26 September, 2015) TOME LXI NO 4 2016, 305-316.
9. Iordanescu, R.; Nichita, F.F.; Nichita I.M. *The Yang-Baxter Equation, (Quantum) Computers and Unifying Theories*, Axioms 2014, 3(4), 360-368.
10. Nichita, F.F. *Jordan Algebras, Jordan Coalgebras and Unification Theories*; Editions Universitaires Europeennes, 2017, ISBN: 978-3-639-62347-5.
11. Iordanescu, R. *The Associativity in Present Mathematics and Present Physics*, ON FORM AN PATERN, Ed: C. Vasilescu, M.L. Flonta, I. Craciun, Editura Academiei Romane, Bucuresti 2015, 67-76
12. Iordanescu, R. *Jordan structures in geometry and physics with an Appendix on Jordan structures in analysis*, Romanian Academy Press, 2003.

13. Iordanescu, R. *Romanian contributions to the study of Jordan Structures and their applications*, Mitteilungen des Humboldt–Club Rumanien, 8–9, Bukarest 2004–2005, 29–35.
14. Marcus, S.; Nichita, F.F. *On Transcendental Numbers: New Results and a Little History*. Axioms 2018, 7, 15.
15. Marcus, S. *Transcendence, as a Universal Paradigm*. *BALANCE, A Club of Rome Magazine*, 2015, pp. 50–70.
16. Nicolescu, B. *Manifesto of Transdisciplinarity*; State University of New York (SUNY) Press: New York, NY, USA, 2002.
17. Nicolescu, B. *Transdisciplinarity–Past, present and future*. In Moving Worldviews–Reshaping Sciences, Policies and Practices for Endogenous Sustainable Development, COMPAS ed.; Haverkort, B., Reijntjes, C., Ed.; 2006. pp. 142166.
18. Nichita, F.F. *On Models for Transdisciplinarity*. *Transdiscipl. J. Eng. Sci.* **2011**, *2011*, 4246.
19. Dehghan, M.; Shirilord, A. *Solving complex Sylvester matrix equation by accelerated double-step scale splitting (ADSS) method*, Engineering and Computers, 27 August 2019.
20. Behr, N.; Dattoli, G.; Lattanzi, A.; Licciardi, S. *Dual Numbers and Operational Umbral Methods*. Axioms 2019, 8, 77.
21. Majid, S. *A Quantum Groups Primer*, Cambridge University Press, 2002.
22. *Lie derivative*, From Wikipedia, the free encyclopedia, https://en.wikipedia.org/wiki/Lie_derivative