 doi:10.20944/preprints201910.0017.v1

Peer-reviewed version available at *Symmetry* **2019**, *11*, 1390; doi:10.3390/sym11111390

Article

Application of Differential Transform to Multi-Term Fractional Differential Equations with Non-Commensurate Orders

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Abstract: The differential transformation, an approach based on Taylor's theorem, is proposed as convenient for finding exact or approximate solution to the initial value problem with multiple Caputo fractional derivatives of generally non-commensurate orders. The multi-term differential equation is first transformed into a multi-order system and then into a system of recurrence relations for coefficients of formal fractional power s eries. The order of the fractional power series is discussed in relation to orders of derivatives appearing in the original equation. Application of the algorithm to an initial value problem results in a reliable and expected outcome.

Keywords: fractional differential equation; non-commensurate orders; initial value problem; differential transform; fractional power series

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1. Introduction

To introduce the purpose of this paper - to give an answer to an open question - we need to place it in the context of analysis of multi-term fractional differential equations.

In Chapter 8 of the book [1] by Kai Diethelm, the author gives a thorough analysis to initial value problems (IVPs) for multi-term fractional differential equations with Caputo derivatives which in general may be non-commensurate.

The following quotation comes from Subchapter 6.5 of the same book:

> "The theorems above have given us a large amount of information about the smoothness properties of the solutions of fractional differential equations, and in particular about the exact behaviour of the solution as $x \to 0$, most notably the formal asymptotic expansion. [...] An aspect of special significance, for example in view of the development of numerical methods, is the question for the precise values of the constants in this expansion, and most importantly the question whether certain coefficients vanish. A suitable generalisation of the Taylor expansion technique for ordinary differential equations described in [88, Chapter I.8] could be useful in this context. Precise results in this connection seem to be unknown at the moment though."

The main aim of this paper is to answer the open question mentioned in the quotation above and give precise results about values of the constants in the formal asymptotic expansion of the solutions to IVPs for multi-term fractional differential equations with generally non-commensurate orders. An outline of the algorithm for IVP with two non-commensurate orders was introduced in [2].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 October 2019 doi:10.20944/preprints201910.0017.v1

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2. Results

2.1. Problem Statement

To avoid issues with limit expressions present in fractional initial conditions and use integer-order initial conditions which have clear practical meaning, we consider differential equations with Caputo fractional derivatives only. For the sake of clarity, we recall definition of the Caputo derivative.

Definition 1. *The fractional derivative of order λ in Caputo sense (see e.g. [3], [4]) is defined by*

$$ {}_{t_0}^{C}D_t^{\lambda} f(t) = \frac{1}{\Gamma(n-\lambda)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{1+\lambda-n}} ds. \tag{1} $$

where $n-1 \le \lambda < n$, $n \in \mathbb{N}$, $t > t_0$.

In this paper we consider a class of multi-term fractional differential equations with non-commensurate orders in the form

$$ {}_0^C D_t^{\lambda_k} y(t) = f(t, y(t), {}_0^C D_t^{\lambda_1} y(t), {}_0^C D_t^{\lambda_2} y(t), \ldots, {}_0^C D_t^{\lambda_{k-1}} y(t)) \tag{2} $$

with initial conditions

$$ y^{(i)}(0) = y_0^{(i)}, \quad i = 0, 1, \ldots, \lceil \lambda_k \rceil - 1, \tag{3} $$

where $\lambda_k > \lambda_{k-1} > \ldots > \lambda_1 > 0$, $\lambda_j - \lambda_{j-1} \le 1$ for all $j = 2, 3, \ldots, k$, $0 < \lambda_1 \le 1$, $\lceil \cdot \rceil$ is the ceiling function, f is an analytic function in some neighbourhood of $(0, y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(\lceil \lambda_k \rceil - 1)})$ and ${}_0^C D_t^{\beta}$ denotes the Caputo fractional derivative of order $\beta \in \mathbb{R}$. We assume that the orders of the equation (2) may in general be non-commensurate in the sense that $\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}$ for $i \neq j$, $i, j \in \{1, \ldots, k\}$.

2.2. Algorithm Description

Convenient approach to deal with equations of the type (2) is well described in monograph [1]. Two ways how to rewrite a multi-term fractional differential equation (2) into a multi-order fractional differential system are presented in Chapter 8 and equivalence theorems are proved, where multi-order system means a system of single-order equations. Single-order equations of both rational and irrational order are analysed in Chapter 6 of the same book. First we combine information in both chapters to find a formal solution in the form of power series convergent in a neighbourhood of the origin. Then we apply the fractional differential transform (FDT) to find recurrence formula for coefficients of the power series.

2.2.1. Multi-order System

Theorem 1. *Let $\lambda_k > \lambda_{k-1} > \ldots > \lambda_1 > 0$, $\lambda_j - \lambda_{j-1} \le 1$ for all $j = 2, 3, \ldots, k$, $0 < \lambda_1 \le 1$, λ_i and λ_j generally be non-commensurate for $i \neq j$. Consider the IVP (2), (3), and assume that f can be written in the form*

$$ f(t, y(t), {}_0^C D_t^{\lambda_1} y(t), \ldots, {}_0^C D_t^{\lambda_{k-1}} y(t)) = g(t, t^{\lambda_1}, \ldots, t^{\lambda_k}, y(t), {}_0^C D_t^{\lambda_1} y(t), \ldots, {}_0^C D_t^{\lambda_{k-1}} y(t)), \tag{4} $$

where g is analytic in a neighbourhood of $(0, 0, \ldots, 0, y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(\lceil \lambda_k \rceil - 1)})$. Then the solution y can be written in the form

$$ y(t) = \sum_{j_0, j_1, \ldots, j_k}^{\infty} \bar{y}_{(j_0, j_1, \ldots, j_k)} t^{j_0 + j_1 \beta_1 + \ldots + j_k \beta_k}, \tag{5} $$

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where each coefficient $\bar{y}_{(j_0,j_1,\ldots,j_k)}$ is uniquely determined in terms of the coefficients corresponding to smaller exponents and the exponents β_j, $j = 1,\ldots,k$ are defined as $\beta_1 := \lambda_1$, $\beta_j := \lambda_j - \lambda_{j-1}$, $j = 2, 3. \ldots, k$.

Proof of Theorem 1. First we need to rewrite equation (2) in the form of multi-order system, i.e., a system of single-order equations of different orders, generally non-commensurate. We follow the approach described in [1] (p. 176).

We start by constructing a finite sequence of orders of the single-order equations, denote it $\{\lambda_j\}_{j=1}^{k}$. Without loss of generality, we can assume that all integers between 0 and λ_k are members of the sequence too. Let us write $\beta_1 := \lambda_1$, $\beta_j := \lambda_j - \lambda_{j-1}$, $j = 2, 3. \ldots, k$ and observe that for all j, $0 < \beta_j \leq 1$. Then we may write $y_1 := y$ and $y_j := {}_0^C D_t^{\beta_{j-1}} y_{j-1}$, $j = 2, 3, \ldots, k$. Applying Theorem 3 now, we conclude that the solution to the IVP (2), (3) can be obtained from the solution to the system

$$
\begin{aligned}
{}_0^C D_t^{\beta_1} y_1(t) &= y_2(t),\\
{}_0^C D_t^{\beta_2} y_2(t) &= y_3(t),\\
\vdots &= \vdots\\
{}_0^C D_t^{\beta_{k-1}} y_{k-1}(t) &= y_k(t),\\
{}_0^C D_t^{\beta_k} y_k(t) &= f(t, y_1(t), y_2(t), \ldots, y_k(t))
\end{aligned} \tag{6}
$$

with the initial conditions

$$
y_j(0) = \begin{cases} y_0^{(0)} & \text{if } j = 1,\\ y_0^{(i)} & \text{if } \lambda_{j-1} = i \in \mathbb{N},\\ 0 & \text{else} \end{cases} \tag{7}
$$

by setting $y := y_1$.

Next step is to rewrite the system (6) into an equivalent system of Volterra-type integral equations using Theorem 4

$$
\begin{aligned}
y_1(t) &= y_0^{(0)} + \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1 - 1} y_2(s) \mathrm{d}s,\\
y_2(t) &= y_0^{(\lambda_1)} + \frac{1}{\Gamma(\beta_2)} \int_0^t (t-s)^{\beta_2 - 1} y_3(s) \mathrm{d}s,\\
\vdots &= \vdots\\
y_{k-1}(t) &= y_0^{(\lambda_{k-2})} + \frac{1}{\Gamma(\beta_{k-1})} \int_0^t (t-s)^{\beta_{k-1} - 1} y_k(s) \mathrm{d}s,\\
y_k(t) &= y_0^{(\lambda_{k-1})} + \frac{1}{\Gamma(\beta_k)} \int_0^t (t-s)^{\beta_k - 1} f(s, y_1(s), y_2(s), \ldots, y_k(s)) \mathrm{d}s.
\end{aligned} \tag{8}
$$

Now, since we assumed that the function f can be written in special form (4), we can apply Theorem 5 and Corollary 1 to each of the single-order equations in (6) with corresponding initial condition in (7), one by one, to obtain

$$
y(t) = \sum_{j_0, j_1, \ldots, j_k}^{\infty} \bar{y}_{(j_0, j_1, \ldots, j_k)} t^{j_0 + j_1 \beta_1 + \ldots + j_k \beta_k}, \tag{9}
$$

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where each coefficient $\bar{y}_{(j_0,j_1,\ldots,j_k)}$ is uniquely determined in terms of the coefficients corresponding to smaller exponents. The problem how to determine the coefficients will be subject to our study in the part 2.2.2. □

2.2.2. Implementation of the Differential Transform

Now we turn our attention back to the IVP (2), (3). Recall that we turned the problem into an equivalent IVP (6), (7) for a system of single-order fractional differential equations.

Applying the FDT tools developed in the Subsection 4.2, namely Theorem 6, we transform the system (6), (7) to the following system of recurrence relations

$$\begin{aligned} \frac{\Gamma(\alpha_1 j_1 + \beta_1 + 1)}{\Gamma(\alpha_1 j_1 + 1)} Y_{1,\alpha_1}\left(j_1 + \frac{\beta_1}{\alpha_1}\right) &= Y_{2,\alpha_1}(j_1), \\ \vdots &= \vdots \\ \frac{\Gamma(\alpha_k j_k + \beta_k + 1)}{\Gamma(\alpha_k j_k + 1)} Y_{k,\alpha_k}\left(j_k + \frac{\beta_k}{\alpha_k}\right) &= \mathcal{F}_{\alpha_k}(j_k, Y_{1,\alpha_k}(j_k), \ldots, Y_{k,\alpha_k}(j_k)), \end{aligned} \tag{10}$$

with transformed initial conditions

$$Y_j(0) = \begin{cases} y_0^{(0)} & \text{if } j = 1, \\ y_0^{(i)}/i! & \text{if } \lambda_{j-1} = i \in \mathbb{N}, \\ 0 & \text{else,} \end{cases} \tag{11}$$

where $\mathcal{F}_{\alpha_k}(j_k, Y_{1,\alpha_k}(j_k), \ldots, Y_{k,\alpha_k}(j_k))$ is the FDT of $f(t, y_1(t), y_2(t), \ldots, y_k(t))$ of order α_k and $0 < \alpha_1, \ldots, \alpha_k \leq 1$ are suitable real constants representing the order of the fractional power series (31). If we had commensurate orders only, we could take all $\alpha_1, \ldots, \alpha_k$ equal to the least common multiple of denominators of all orders of derivatives which appear in the equation. However, in the case of non-commensurate orders, we have to use different approach for the choice of $\alpha_1, \ldots, \alpha_k$. Specifically, the choice $\alpha_1 := \beta_1, \ldots, \alpha_k := \beta_k$ is one of the choices which lead to the solution of the given IVP in the form (5). The system (10) will then simplify to

$$\begin{aligned} \frac{\Gamma(\beta_1 j_1 + \beta_1 + 1)}{\Gamma(\beta_1 j_1 + 1)} Y_{1,\beta_1}(j_1 + 1) &= Y_{2,\beta_1}(j_1), \\ \vdots &= \vdots \\ \frac{\Gamma(\beta_k j_k + \beta_k + 1)}{\Gamma(\beta_k j_k + 1)} Y_{k,\beta_k}(j_k + 1) &= \mathcal{F}_{\beta_k}(j_k, Y_{1,\beta_k}(j_k), \ldots, Y_{k,\beta_k}(j_k)). \end{aligned} \tag{12}$$

As there are (generally non-commensurate) different orders of the FDT in (12), we need to find relation between coefficients with different orders.

Theorem 2. *Let a function f have the form $f(t) = {}^{C}_{t_0}D^{\beta}_{t} g(t)$, where we allow $\beta = 0$, and let $F_{\alpha_1}(k)$ and $F_{\alpha_2}(k)$ denote the FDT of f at t_0 of orders α_1 and α_2, respectively. Then*

$$F_{\alpha_2}(k) = F_{\alpha_1}\left(\frac{\alpha_2}{\alpha_1} k\right). \tag{13}$$

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Proof of Theorem 2. Application of Theorem 6 gives $F_{\alpha_1}(k) = \dfrac{\Gamma(\alpha_1 k+\beta+1)}{\Gamma(\alpha_1 k+1)} G_{\alpha_1}\left(k+\frac{\beta}{\alpha_1}\right)$ and $F_{\alpha_2}(k) = \dfrac{\Gamma(\alpha_2 k+\beta+1)}{\Gamma(\alpha_2 k+1)} G_{\alpha_2}\left(k+\frac{\beta}{\alpha_2}\right)$. Calculating G_{α_1} and G_{α_2} from the Definition 2 we get

$$
\begin{aligned}
G_{\alpha_1}\left(k+\frac{\beta}{\alpha_1}\right) &= \frac{1}{\Gamma(\alpha_1(k+\frac{\beta}{\alpha_1})+1)}\left[{}^C_{t_0}D_t^{\alpha_1(k+\frac{\beta}{\alpha_1})} g(t)\right]_{t=t_0} = \frac{1}{\Gamma(\alpha_1 k+\beta+1)}\left[{}^C_{t_0}D_t^{\alpha_1 k+\beta} g(t)\right]_{t=t_0},\\
G_{\alpha_2}\left(k+\frac{\beta}{\alpha_2}\right) &= \frac{1}{\Gamma(\alpha_2(k+\frac{\beta}{\alpha_2})+1)}\left[{}^C_{t_0}D_t^{\alpha_2(k+\frac{\beta}{\alpha_2})} g(t)\right]_{t=t_0} = \frac{1}{\Gamma(\alpha_2 k+\beta+1)}\left[{}^C_{t_0}D_t^{\alpha_2 k+\beta} g(t)\right]_{t=t_0}.
\end{aligned}
$$

Substituting $\dfrac{\alpha_2}{\alpha_1}k$ into F_{α_1} and combining all formulas brings us to the relation

$$
\begin{aligned}
F_{\alpha_1}\left(\frac{\alpha_2}{\alpha_1}k\right) &= \frac{\Gamma(\alpha_1(\frac{\alpha_2}{\alpha_1}k)+\beta+1)}{\Gamma(\alpha_1(\frac{\alpha_2}{\alpha_1}k)+1)} G_{\alpha_1}\left((\frac{\alpha_2}{\alpha_1}k)+\frac{\beta}{\alpha_1}\right) && (14)\\
&= \frac{\Gamma(\alpha_2 k+\beta+1)}{\Gamma(\alpha_2 k+1)} \frac{1}{\Gamma(\alpha_1(\frac{\alpha_2}{\alpha_1}k)+\beta+1)}\left[{}^C_{t_0}D_t^{\alpha_1(\frac{\alpha_2}{\alpha_1}k)+\beta} g(t)\right]_{t=t_0} && (15)\\
&= \frac{\Gamma(\alpha_2 k+\beta+1)}{\Gamma(\alpha_2 k+1)} \frac{1}{\Gamma(\alpha_2 k+\beta+1)}\left[{}^C_{t_0}D_t^{\alpha_2 k+\beta} g(t)\right]_{t=t_0} && (16)\\
&= \frac{\Gamma(\alpha_2 k+\beta+1)}{\Gamma(\alpha_2 k+1)} G_{\alpha_2}\left(k+\frac{\beta}{\alpha_2}\right) = F_{\alpha_2}(k). && (17)
\end{aligned}
$$

□

Theorem 2 allows us to solve the recurrence relations (11) with respect to $j_1, \ldots, j_k$ to find the sequences of coefficients $\{Y_{1,\beta_1}(j_1)\}, \ldots, \{Y_{k,\beta_k}(j_k)\}$. Applying IFDT (Definition 3) yields

$$
y(t) = y_1(t) = {}^C\mathcal{D}_{\beta_1}^{-1}\left\{\{Y_{1,\beta_1}(j_1)\}_{j_1=0}^{\infty}\right\}[0] = \sum_{j_1=0}^{\infty} Y_{1,\beta_1}(j_1) t^{\beta_1 j_1} = \sum_{j_1=0}^{\infty} Y_{1,\beta_1}(j_1) t^{\lambda_1 j_1}. \tag{18}
$$

Although is looks like there are only powers of order λ_1 in the solution, it is not the case. Recall that according to Definition 2 of FDT, indexes j_1 belong to a countable subset of $[0, \infty)$, not necessarily being integers. In fact there will be integer multiples of all powers $\lambda_1, \ldots, \lambda_k$ and some integer powers of t arising from the initial conditions in the solution, which is demonstrated in the following part 2.2.3.

2.2.3. Applications

Example 1. *Consider two-term fractional differential equation in the form*

$$
\Gamma\left(\frac{4}{3}\right) {}^C_0D_t^{\frac{1}{\sqrt{2}}} u(t) + \Gamma\left(\frac{1}{\sqrt{2}}+1\right) {}^C_0D_t^{\frac{1}{3}} u(t) = \Gamma\left(\frac{4}{3}+\frac{1}{\sqrt{2}}\right)\left(t^{\frac{1}{3}} + t^{\frac{1}{\sqrt{2}}}\right) \tag{19}
$$

with initial condition $u(0) = 0$. *Rewriting* (19) *into a two-order system we get*

$$
\begin{aligned}
{}^C_0D_t^{\frac{1}{3}} u_1(t) &= u_2, && (20)\\
{}^C_0D_t^{\frac{1}{\sqrt{2}}-\frac{1}{3}} u_2(t) &= \frac{1}{\Gamma\left(\frac{4}{3}\right)}\left(\Gamma\left(\frac{4}{3}+\frac{1}{\sqrt{2}}\right)\left(t^{\frac{1}{3}}+t^{\frac{1}{\sqrt{2}}}\right) - \Gamma\left(\frac{1}{\sqrt{2}}+1\right) u_2(t)\right), && (21)
\end{aligned}
$$

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with initial conditions $u_1(0) = 0$, $u_2(0) = 0$. *Following the algorithm, we choose* $\alpha_1 = \frac{1}{3}$ *and* $\alpha_2 = \frac{1}{\sqrt{2}} - \frac{1}{3}$. *Fractional differential transform of the system* (20), (21) *is*

$$\frac{\Gamma(\frac{1}{3}k + \frac{1}{3} + 1)}{\Gamma(\frac{1}{3}k + 1)} U_{1,\alpha_1}(k+1) = U_{2,\alpha_1}(k), \tag{22}$$

$$\frac{\Gamma(\alpha_2 k + \alpha_2 + 1)}{\Gamma(\alpha_2 k + 1)} U_{2,\alpha_2}(k+1) = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \left(\Gamma\left(\frac{4}{3} + \frac{1}{\sqrt{2}}\right) \left(\delta\left(k - \frac{1}{3\alpha_2}\right) + \delta(k - \frac{1}{\sqrt{2}\alpha_2})\right) - \Gamma\left(\frac{1}{\sqrt{2}} + 1\right) U_{2,\alpha_2}(k) \right), \tag{23}$$

with transformed initial conditions $U_{1,\alpha_1}(0) = 0$, $U_{2,\alpha_2}(0) = 0$. *As* (23) *does not depend on* U_1, *we can solve it first completely and then come back to* (22). *We can see that the first nonzero coefficient we get for* $k = \frac{1}{3\alpha_2}$: $U_{2,\alpha_2}\left(\frac{1}{\sqrt{2}\alpha_2}\right) = \frac{\Gamma\left(\frac{4}{3} + \frac{1}{\sqrt{2}}\right)}{\Gamma\left(\frac{1}{\sqrt{2}} + 1\right)}$. *The next possibility of nonzero coefficient we observe for* $k = \frac{1}{\sqrt{2}\alpha_2}$:

$$\Gamma\left(\frac{2}{\sqrt{2}} - \frac{1}{3} + 1\right) U_{2,\alpha_2}\left(\frac{2 - \frac{\sqrt{2}}{3}}{1 - \frac{\sqrt{2}}{3}}\right) = \frac{\Gamma\left(\frac{1}{\sqrt{2}} + 1\right)}{\Gamma\left(\frac{4}{3}\right)} \left(\Gamma\left(\frac{4}{3} + \frac{1}{\sqrt{2}}\right) - \Gamma\left(\frac{1}{\sqrt{2}} + 1\right) U_{2,\alpha_2}\left(\frac{1}{\sqrt{2}\alpha_2}\right)\right) = 0,$$

which means that all coefficients $U_{2,\alpha_2}(k)$ *are zero except* $U_{2,\alpha_2}\left(\frac{1}{\sqrt{2}\alpha_2}\right)$. *Finally we feed* $U_{2,\alpha_2}\left(\frac{1}{\sqrt{2}\alpha_2}\right)$ *back into* (22) *(with* $k = \frac{3}{\sqrt{2}}$*) and obtain* $U_{1,\frac{1}{3}}\left(\frac{3}{\sqrt{2}} + 1\right) = \frac{\Gamma\left(\frac{1}{\sqrt{2}} + 1\right)}{\Gamma\left(\frac{4}{3} + \frac{1}{\sqrt{2}}\right)} U_{2,\frac{1}{3}}\left(\frac{3}{\sqrt{2}}\right) = 1$. *Hence the unique solution* u *of the problem* (2), (3) *is*

$$u(t) = u_1(t) = t^{\frac{1}{3}\left(\frac{3}{\sqrt{2}} + 1\right)} = t^{\frac{1}{3} + \frac{1}{\sqrt{2}}}. \tag{24}$$

3. Discussion

In the paper we have proposed an algorithm how to obtain values of the constants in the formal asymptotic expansion of solution to IVP for multi-term fractional differential equation with generally non-commensurate orders. In particular, we have proceeded in the following sequence of steps:

1. Transformation of the multi-term equation to a multi-order differential system.
2. Description of an equivalent system of Volterra integral equations.
3. Finding the general form of the asymptotic expansion of the solution.
4. Transformation of the multi-order differential system to a system of recurrence relations (formal application of FDT).
5. Choice of convenient orders of FDT.
6. Description of relation between different orders of FDT.

The algorithm provides an answer to the open question raised in the monograph [1]. An obvious subject to discuss is the choice of convenient orders of FDT (step 5). We expect that there might be a different combination of orders used, with possibility to optimize the computational effort. Convergence properties should be studied to ensure that a computer implementation of the algorithm is reliable and efficient. As the orders are generally non-commensurate, i.e. irrational, software using symbolic computations might have an advantage against purely numerical software.

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Peer-reviewed version available at Symmetry 2019, 11, 1390; doi:10.3390/sym11111390

4. Methods

4.1. Equivalence and Smoothness Theorems

We recall a few results necessary for justification of correctness of the algorithm.

Theorem 3 (See [1], Theorem 8.9, p. 176). *Subject to the conditions specified in Subsection 2.1, the multi-term equation (2) with initial conditions (3) is equivalent to the system (6) with the initial conditions (7) in the following sense:*

1. *Whenever the function $y \in C^{\lceil \lambda_k \rceil}[0,T]$ is a solution of the IVP (2), (3), the vector-valued function $Y := (y_1, \ldots, y_k)^T$ with*
$$y_j(t) := \begin{cases} y(t) & \text{if } j = 1, \\ {}_0^C D_t^{\lambda_{j-1}} y(t) & \text{if } j = 2, \ldots, k, \end{cases} \tag{25}$$
is a solution of the IVP (6), (7).
2. *Whenever the vector-valued function $Y := (y_1, \ldots, y_k)^T$ is a solution of the IVP (6), (7), the function $y := y_1$ is a solution of the IVP (2), (3).*

Theorem 4 (See [1], Lemma 6.2, p. 86). *Let $0 < n$ and $m = \lceil n \rceil$. Moreover let $y_0^{(0)}, \ldots, y_0^{(m-1)} \in \mathbb{R}$, $K > 0$ and $h^* > 0$. Define $G := \{(x,y) \colon x \in [0,h^*], |y - \sum_{k=0}^{m-1} x^k y_0^{(k)}/k!| \leq K\}$, and let the function $f \colon G \to \mathbb{R}$ be continuous. The function $y \in C[0,h]$ for some $0 < h < h^*$ is a solution of the initial value problem*
$${}_0^C D_x^n y(x) = f(x, y(x)), \tag{26}$$
$$D^k y(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, m-1 \tag{27}$$
if and only if it is a solution of the nonlinear Volterra integral equation of the second kind
$$y(x) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{x^k}{k!} + \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t, y(t)) \mathrm{d}t. \tag{28}$$

Theorem 5 (See [1], Theorem 6.35, p. 124). *Let n be a positive irrational number. Consider the initial value problem (26), (27) and assume that f can be written in the form $f(x,y) = \bar{f}(x, x^n, y)$ where $\bar{f}$ is analytic in a neighbourhood of $(0, 0, y_0^{(0)})$. Then, there exists a uniquely determined analytic function $\bar{y} \colon (-r, r) \times (-r^n, r^n) \to \mathbb{R}$ with some $r > 0$ such that $y(x) = \bar{y}(x, x^n)$ for $x \in [0, r)$.*

Corollary 1 (See [1], Corollary 6.37, p. 125). *Under the assumptions of Theorem 5, y is of the form*
$$y(x) = \sum_{\mu,\nu=0}^{\infty} \bar{y}_{\mu\nu} x^{\mu + \nu n}. \tag{29}$$

4.2. Fractional Differential Transformation

An introduction of a generalisation of the differential transformation (DT) called the fractional differential transformation (FDT) is given in this subsection. For more details on DT and FDT, we recommend sound papers [5],[6], [7], [8], [9], [10].

Definition 2. *Fractional differential transformation of order $\alpha \in \mathbb{R}^+$ of a real function $u(t)$ at a point $t_0 \in \mathbb{R}$ in Caputo sense is ${}^C\mathcal{D}_\alpha\{u(t)\}[t_0] = \{U_\alpha(k)\}_{k=0}^\infty$, $k \in I \subset \mathbb{R}_0^+$ where I is a countable subset of $[0, \infty)$, and*

$U_\alpha(k)$, the fractional differential transformation of order α of the (αk)th derivative of function $u(t)$ at t_0, is defined as

$$U_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)} \left[{}^C_{t_0}D_t^{\alpha k} u(t)\right]_{t=t_0}, \tag{30}$$

provided that the original function $u(t)$ is analytic in some right neighbourhood of t_0.

Definition 3. *Inverse fractional differential transformation (IFDT) of $\{U_\alpha(k)\}_{k=0}^\infty$ is defined using a fractional power series as follows:*

$$u(t) = {}^C\mathcal{D}_\alpha^{-1}\left\{\{U_\alpha(k)\}_{k=0}^\infty\right\}[t_0] = \sum_{k=0}^\infty U_\alpha(k)(t-t_0)^{\alpha k}. \tag{31}$$

In applications, we will use some basic FDT formulas listed in [11]:

Theorem 6. *Assume that $\{F_\alpha(k)\}_{k=0}^\infty$, $\{G_\alpha(k)\}_{k=0}^\infty$ and $\{H_\alpha(k)\}_{k=0}^\infty$ are differential transformations of order α at t_0 of functions $f(t)$, $g(t)$ and $h(t)$, respectively. Further assume that $r > 0$, $\beta > 0$.*

$$\text{If } f(t) = (t-t_0)^r, \text{ then } F_\alpha(k) = \delta\left(k - \frac{r}{\alpha}\right), \text{ where } \delta(x-y) = \delta_{xy} \text{ is the Kronecker delta.} \tag{32}$$

$$\text{If } f(t) = g(t)h(t), \text{ then } F_\alpha(k) = \sum_{l=0}^{k} G_\alpha(l) H_\alpha(k-l). \tag{33}$$

$$\text{If } f(t) = \frac{g(t)}{(t-t_0)^r}, \text{ then } F_\alpha(k) = G_\alpha\left(k + \frac{r}{\alpha}\right), \text{ provided } G_\alpha(l) = 0 \text{ for } l < \frac{r}{\alpha}. \tag{34}$$

$$\text{If } f(t) = {}^C_{t_0}D_t^\beta g(t), \text{ then } F_\alpha(k) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} G_\alpha\left(k + \frac{\beta}{\alpha}\right). \tag{35}$$

Acknowledgments: The author was supported by the Czech Science Foundation under the project 19-23815S. The APC was funded by Open Access Fund of Brno University of Technology. This support is gratefully acknowledged.

Conflicts of Interest: The author declares no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Abbreviations

The following abbreviations are used in this manuscript:

IVP	Initial Value Problem
DT	Differential Transformation
FDT	Fractional Differential Transformation
IFDT	Inverse Fractional Differential Transformation

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