

## Article

# Line Shapes in a Magnetic Field: Trajectory Modifications II: Full Collision-Time Statistics

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1 **Abstract:** In a previous paper[1], a variation of the Collision-time Statistics method was applied to  
2 identify the relevant perturbers for line broadening under the action of a constant magnetic field. As  
3 discussed, that version was simplified and inadequate for low magnetic field and/or large perturber  
4 mass (ions). The purpose of the present work is to augment the previous work, so that such cases can  
5 be handled efficiently. The results may also be used to construct analytic, i.e. impact/unified models  
6 under the usual assumptions in these models.

7 **Keywords:** Stark broadening; Magnetic field; spiralling trajectories; Collision-time statistics.

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## 8 1. Introduction

9 In a previous paper[1], a variation of the Collision-time Statistics method was applied to identify  
10 the relevant perturbers for line broadening under the action of a constant magnetic field. As discussed,  
11 that version was simplified and inefficient for low magnetic field and/or or large perturber mass  
12 (ions), because an unnecessarily large number of perturbers, the vast majority of which only contribute  
13 negligibly to broadening, were generated. The purpose of the present work is to augment the previous  
14 work, so that such cases can be handled efficiently. The main problem with the previous treatment  
15 was the case of large Larmor radii, larger than the effective interaction range (e.g. Debye length) and  
16 corresponding slow cyclotron frequencies, as in the case of large perturber mass and/or low magnetic  
17 field. In such cases the motion of the perturbers in the plane perpendicular to the magnetic field  
18 may be quite far from completing a full cycle. As a result in the simplified treatment of [1] a large  
19 number of ineffective perturbers would be included in that case. It is believed[2] that "as long as  
20 the gyro-radii of the electrons are much larger than the Debye sphere" the particle trajectories, or,  
21 equivalently, the dielectric function would be unaffected by the magnetic field, since the perpendicular  
22 motion involves presumably much smaller length scales than the parallel motion, although this belief  
23 remains to be quantified. It is shown here that this criterion is not quite correct and that the key  
24 parameter determining the adequacy of neglecting spiralling is the ratio of the cyclotron frequency to  
25 the width of the line in question.

## 26 2. Theoretical Formulation

27 As in [1] we assume that:

28 a. the distribution functions, e.g. the Maxwellian velocity distribution is *not* affected by the B-field.  
29 b. the shielding is also not affected, e.g. Debye screening

30 We also consider a neutral emitter in this work.

<sup>31</sup> 2.1. Collision-time Statistics

<sup>32</sup> To include *all* and only the relevant perturbers, we use a modification of the collision-time  
<sup>33</sup> statistics method of Hegerfeld and Kesting[3] with Seidel's improvement[4]- see Ref.[5] for details, as  
<sup>34</sup> discussed in [1].

As in [1], perturbers move in a helical path characterized by the parallel constant velocity  $v_z$ , where the magnetic field direction defines the z-axis (passing through the emitter), the perpendicular velocity with magnitude  $v_{\perp}$  and impact parameter  $\rho$ , which is the distance of the center of the spiral to the z-axis, i.e. the perpendicular motion in the x-y plane is a circular motion with the Larmor radius  $r_L = \frac{v_{\perp}}{\omega_L}$  around the center  $\rho$ , with  $\omega_L = |Q|B/m$  the cyclotron frequency and  $Q$  the perturber charge. For the impact parameter  $\rho$

$$\max(0, r_L - R_{max}) \leq \rho \leq R_{max} + r_L \quad (1)$$

<sup>35</sup> , i.e. the impact parameter lies in a disk or annulus depending on whether the range  $R_{max}$  of the  
<sup>36</sup> interaction, discussed below, is larger or smaller than  $r_L$ .

The relevant quantities for the helical trajectory  $\mathbf{R}(t)$  are as follows: The z-coordinate of the trajectory is

$$R_z(t) = v_z(t - t_i) = v_z t + z_0, \quad (2)$$

with

$$z_0 = -v_z t_i \quad (3)$$

<sup>37</sup> , with the times of closest approach  $t_i$  representing the time the perturber trajectory intersects the x-y  
<sup>38</sup> plane and being uniformly distributed.  $z_0$  thus represents how far from the x-y plane the perturber is  
<sup>39</sup> at  $t=0$ .

Hence

$$R_x(t) = \rho \cos \theta + r_L \cos(\omega_L t + \psi), R_y(t) = \rho \sin \theta + r_L \sin(\omega_L t + \psi) \quad (4)$$

<sup>40</sup> where  $\theta$  describes the position of the impact parameter vector in the x-y plane and is uniformly  
<sup>41</sup> distributed in  $(0, 2\pi)$ .  $\psi$  is an angle describing where on the circular trajectory projection the perturber  
<sup>42</sup> finds itself at  $t = 0$  and is also uniformly distributed in  $(0, 2\pi)$  and ultimately related to the time  
<sup>43</sup> the B-field was turned on. Each perturber is thus characterized by the vector  $(v_z, v_{\perp}, \rho, \theta, \psi, t_i)$  , or  
<sup>44</sup> equivalently  $z_0$  instead of  $t_i$ .

<sup>45</sup> As in [1] we consider as "relevant" perturbers those that come closer to the emitter than a distance  
<sup>46</sup>  $R_{max}$ , defined so that the interaction is negligible for distances larger than  $R_{max}$  during the time  
<sup>47</sup> interval of interest  $(0, \tau)$ . For a Debye interaction, we usually take  $R_{max} \approx 3\lambda_D$ , where  $\lambda_D$  denotes the  
<sup>48</sup> shielding(Debye) length. This is because the interaction becomes negligible ( $\leq 3\%$  for larger distances).

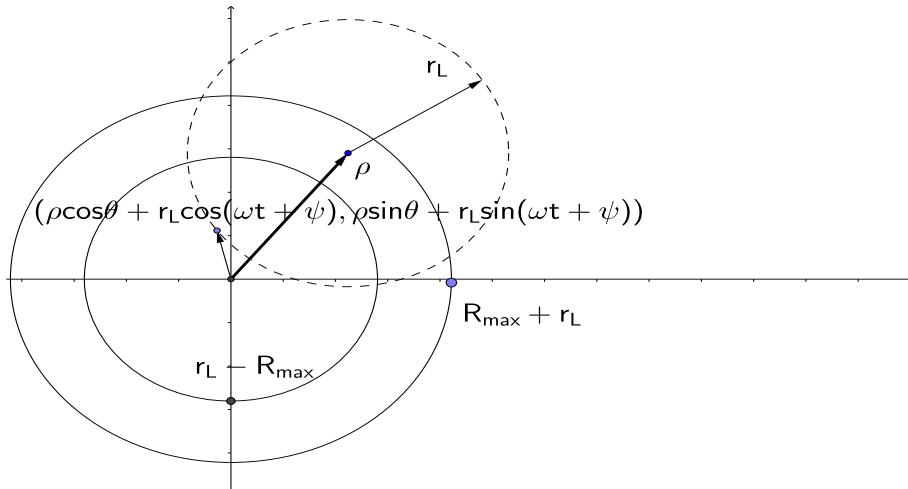
<sup>49</sup> Therefore for a perturber to be relevant the condition  $R(t) \leq R_{max}$  must hold for at least one time  
<sup>50</sup>  $t$  in  $(0, \tau)$ , where  $\tau$  is the time of interest, i.e. a time large enough that the Fourier transform of the  
<sup>51</sup> line profile  $C(t)$  has decayed to negligible levels, or an asymptotic form is identifiable.  $C(t)$  is a linear  
<sup>52</sup> combination of products of time evolution operators (U-matrices) of the upper and lower levels. These  
<sup>53</sup> time evolution operators -needed for times  $0 \leq t \leq \tau$ - are determined by solving the Schrödinger  
<sup>54</sup> equation in the Debye-shielded field  $V(t)$ . Therefore a particle will only be relevant if for at least one  
<sup>55</sup> time in the interval  $[0, \tau]$  it comes closer than  $R_{max}$  to the emitter(if not, then the perturbation produced  
<sup>56</sup> by that particle is negligible due to Debye screening), which means that for at least one time  $t$  in  $[0, \tau]$  :

$$(v_z t + z_0)^2 + (\rho \cos \theta + r_L \cos(\omega_L t + \psi))^2 + (\rho \sin \theta + r_L \sin(\omega_L t + \psi))^2 \leq R_{max}^2 \quad (5)$$

This reads:

$$R_{max}^2 - (v_z t - v_z t_i)^2 \geq \rho^2 + r_L^2 + 2\rho r_L \cos(\omega_L t + \psi - \theta) \quad (6)$$

<sup>57</sup> Thus we generate  $v_z, v_{\perp}, \rho, t_i$  and  $\theta$  as before, but also draw  $\psi$ , uniformly distributed in  $(0, 2\pi)$  as  
<sup>58</sup> illustrated in Fig. 1 and effectively only accept perturbers if, for at least one time in  $(0, \tau)$  Eq.6 is  
<sup>59</sup> satisfied. The *maximum* value of the LHS occurs for  $t = t_i$ . This in general imposes *restrictions* (i.e. not



**Figure 1.** x-y trajectory projections for  $r_L \geq R_{max}$ . Shown is the annular region between concentric circles with the origin (the emitter position) as center and radii  $r_L - R_{max}$  and  $r_L + R_{max}$ , respectively. For an impact parameter at a distance  $\rho$  from the center in the annular region, a circle with radius  $r_L$  (dashed) represents the projection of the perturber path in the x-y plane. Hence a point on that circle is  $(\rho \cos \theta + r_L \cos(\omega t + \psi), \rho \sin \theta + r_L \sin(\omega t + \psi))$ , with  $\psi$  the angle on the dashed circle. This must be no more than  $R_{max}$  away from the center, else this perturber does not contribute.

60 all  $\psi$  contribute for a given  $\theta$ ) on the values of  $\theta$  and  $\psi$  that a perturber can have and still contribute  
 61 effectively to broadening, specifically:

62 • a. If  $r_L < R_{max}$  and  $\rho \leq R_{max} - r_L$ ,  $R_{max} \geq \rho + r_L$  and Eq.6 is satisfied for  $t = t_i$  for any  $\theta, \psi$  and  
 63  $\omega_L t$ . Therefore in this case we have no restriction and  $\theta$  and  $\psi$  can independently take any value.  
 • b. In all other cases, Eq.6 results in the restriction:

$$\arccos\left(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}\right) \leq |\omega_L t + \psi - \theta| \quad (7)$$

64 Note that for Case b, the argument of the inverse cosine is absolutely  $\leq 1$ . Specifically:

63 1. If  $r_L \geq R_{max}$ , then

$$0 \geq \frac{R_{max}^2 - r_L^2 - \rho^2}{2\rho r_L} \geq -1 \quad (8)$$

The left of the inequality follows because  $R_{max} \leq r_L$  alone. The right part also follows since

$$r_L - R_{max} \leq \rho \leq r_L + R_{max} \Rightarrow -R_{max} \leq \rho - r_L \leq R_{max} \quad (9)$$

Hence

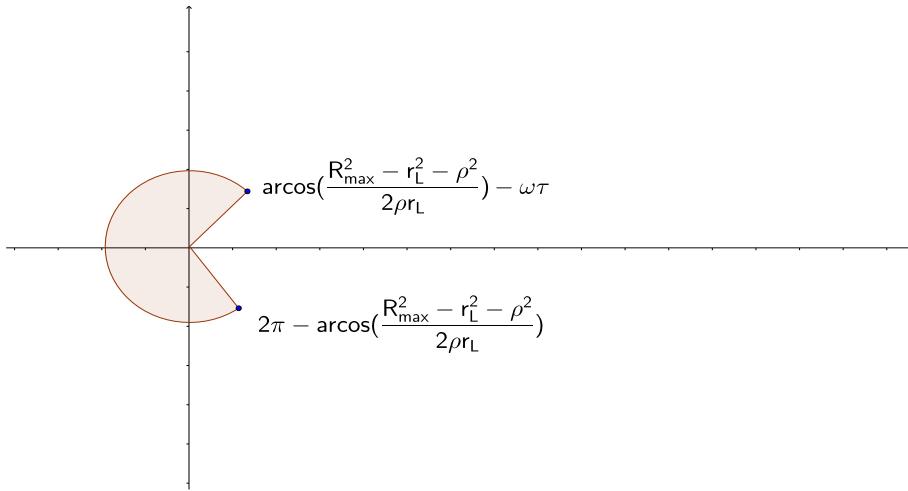
$$R_{max}^2 \geq \rho^2 + r_L^2 - 2\rho r_L \Rightarrow \frac{R_{max}^2 - r_L^2 - \rho^2}{2\rho r_L} \geq -1 \quad (10)$$

2. For  $r_L \leq R_{max}$ , Eq.9 is also valid since  $r_L - R_{max} \leq 0 \leq \rho$ , as is Eq.10. Since, as already discussed the case  $\rho \leq R_{max} - r_L$  imposes no restriction, we only consider here the case  $R_{max} - r_L \leq \rho \leq R_{max} + r_L$ , which implies  $\rho - r_L \leq R_{max} \leq \rho + r_L$ . It thus remains to show that

$$\frac{R_{max}^2 - r_L^2 - \rho^2}{2\rho r_L} \leq 1 \quad (11)$$

65 which follows from  $R_{max} \leq \rho + r_L$ .

66 Thus



**Figure 2.** The shaded area shows the difference  $\psi - \theta$  that satisfies Eq.6.

- 67 • a. For  $r_L \leq R_{max}$ ,  $\rho \leq r_L$ , Eq.6 is satisfied for  $t = t_i$  for any  $\theta, \psi$  and  $\omega_L t$  and there is no restriction  
68 on  $\psi, \theta$  and  $\omega_L t$ .
- 69 • b. In all other cases, the restriction imposed by Eq.7 applies and the argument of the inverse  
70 cosine is always absolutely  $\leq 1$ .

Hence the angle difference  $\psi - \theta$  must be in the shaded area shown in Fig. 2. So for at least one time  $t$  in  $(0, \tau)$ , the following must hold for the perturber with parameters  $\rho, r_L, \theta, \psi$  to contribute:

$$|\psi + \omega_L t - \theta| \geq \arccos\left(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}\right) \quad (12)$$

. i.e.

$$\psi + \omega_L t \geq \theta + \arccos\left(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}\right), \psi + \omega_L t \geq \theta \quad (13)$$

or

$$\theta - \arccos\left(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}\right) \geq \psi + \omega_L t, \theta \geq \psi + \omega_L t \quad (14)$$

The net result is that for each  $\theta$ , only a range of  $\psi$ ,

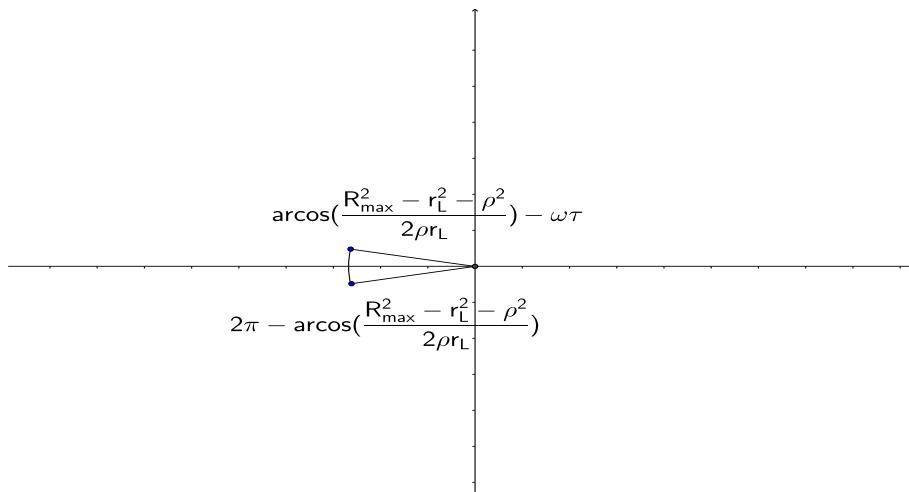
$$\Delta\psi \leq 2\left(\pi - \arccos\left(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}\right)\right) + \omega_L \tau \quad (15)$$

- 71 contributes( this means that a fraction  $\frac{\Delta\psi}{2\pi}$  contributes compared to the simplified case discussed in the  
72 previous work, i.e. the collision volume is smaller by  $\frac{\Delta\psi}{2\pi}$ ). If this is  $\ll 1$  we effectively have rectilinear  
73 trajectories for the time of interest. If this turns out to be larger than  $2\pi$ , we have a full revolution and  
74 we can use the simplified formulas discussed in [1]. As mentioned, we are mainly interested in the  
75 situation where  $R_{max} < r_L$  and  $\omega_L \tau < 2\pi$ , as this is the case of large  $r_L$ , but slow  $\omega_L$ , otherwise the  
76 relation between  $\psi$  and  $\theta$  is always satisfied for at least one  $t$  in  $(0, \tau)$ .

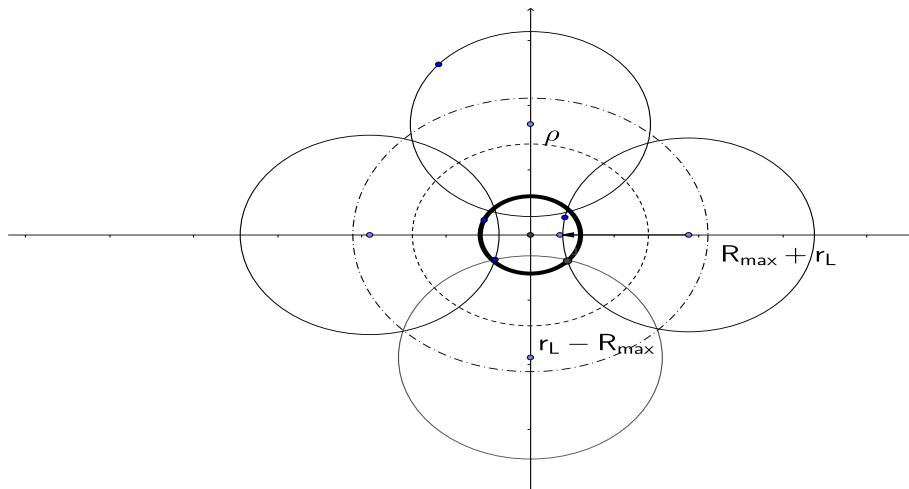
We can use the variable  $x = \rho - r_L$  with  $-R_{max} \leq x \leq R_{max}$  and write the argument of the inverse cosine as

$$\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L} = \frac{R_{max}^2 - x^2}{2r_L(x + r_L)} - 1 \quad (16)$$

- 77 Note that for low B (large  $r_L$ ) this tends to -1, hence the inverse cosine is close to  $\pi$ . This means that  
78 in this limit  $|\psi - \theta| \approx \pi$ , e.g. we get a  $\Delta\psi \approx 0$  (but note that in that limit we had divergencies in the  
79 relevant functions when computing the collision volume in [1]).



**Figure 3.** The shaded area shows the difference  $\psi - \theta$  that satisfies Eq.6.



**Figure 4.** Illustration of the relation between  $\theta$  and  $\psi$  for the case  $r_L \gg R_{max}$ . Impact parameters  $\rho$  lie between the dashed and dash-dotted circles with radii  $r_L - R_{max}$  and  $r_L + R_{max}$ . The part of the circular trajectory projections that are within  $R_{max}$  (bold circle) of the emitter (i.e. the center) are in the opposite direction of the impact parameter vector, e.g. to the north for the southern circular trajectory projection.

80 This situation is depicted in Fig. 3, which shows the typical situation for the phase space of the  
 81 quantity  $\psi - \theta$  that contributes. This is also illustrated in Fig. 4, which shows, for the same  $\rho$ , 4 different  
 82 angles  $\theta$ , which determine the centers of the spirals and the parts of the circular projections of these  
 83 spirals that are effective. For instance if the center of the spiral, i.e. the vector of the impact parameter  
 84 is to the right ( $\theta \approx 0$ ), then  $\psi \approx \pi$  (the leftmost of the circular trajectory projection) for  $r_L \gg R_{max}$ .  
 85 Similarly, if the impact parameter vector is to the south ( $\theta \approx 3\pi/2$ ), then the relevant  $\psi$  is to the north  
 86 of the circular trajectory projection, e.g.  $\psi \approx \pi/2$ .

87 In the limit  $B = 0$  (or infinite perturber mass),  $\arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L}) = \pi$  and we get from the  $\theta$  and  
 88  $\psi$  integrations a term  $\frac{\omega_L \tau}{2\pi}$ .

## 89 2.2. Collision Volume

The collision-time statistics method first computes the number of relevant particles, i.e. the density times the relevant volume, i.e. the above cylinder. This volume is as before[1], except that we also account for the polar angle  $\theta$ , describing the orientation of the impact parameter with respect to the x-axis and the angle  $\psi$  describing the position of the particle on the perpendicular x-y plane at time

**Table 1.**  $\frac{\Delta\psi(\rho)}{2\pi}$  vs. relevant parameters

$\omega_L \tau$	$\frac{r_L}{R_{max}}$	$\frac{\rho}{max(\rho_1, r_L)}$	$\frac{\Delta\psi(\rho)}{2\pi}$
$\geq 2\pi$			1
$< 2\pi$	$\leq 1$	$\leq 1$	1
$< 2\pi$	$\leq 1$	$> 1$	$(1 - \frac{\arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi}) + \frac{\omega_L \tau}{2\pi}$
$< 2\pi$	$> 1$		$(1 - \frac{\arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi}) + \frac{\omega_L \tau}{2\pi}$

t=0, i.e. we have the extra integrations  $\int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\psi}{2\pi}$ . The  $\psi$  integration simply returns a factor of 1 if  $\omega_L \tau \geq 2\pi$ , but it does so even under the weaker condition  $\omega_L \tau \geq 2 \arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L})$ , or

$$\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L} \geq \cos(\omega_L \tau / 2) \quad (17)$$

90 Otherwise it gives a factor of  $\frac{\Delta\psi}{2\pi}$  with  $\Delta\psi$  defined in Eq.15.

The nonnegative root of Eq.17 is

$$\rho_1 = -\cos(\omega_L \tau / 2) r_L + \sqrt{R_{max}^2 - \sin^2(\omega_L \tau / 2) r_L^2} \quad (18)$$

, i.e. the results of [1] are also valid for  $\rho \leq \rho_1$ , which in turn requires that

$$R_{max} > r_L \quad (19)$$

(else  $\rho_1 \leq 0$ ), which also guarantees the reality of  $\rho_1$ , i.e.

$$R_{max} \geq \sin(\omega_L \tau / 2) r_L \quad (20)$$

91 The  $\theta$  angular integration simply returns 1 in either case. As a result, the results of [1] need no  
92 modification for  $\omega_L \tau \geq 2\pi$ . Otherwise, the collision volume calculation runs as follows:

93 Hence for  $v_{\perp} < R_{max} \omega_L$ , i.e.  $r_L < R_{max}$ ,  $\Delta\psi(\rho) = 1$ . However, as already discussed, this is also  
94 valid (e.g. no restriction on  $\psi$  is required) also for  $\rho < r_L$ , hence  $\Delta\psi(\rho) = 1$  for  $\rho \leq max(r_L, \rho_1) = \rho_2$ .

95 For small  $R_{max}/r_L$  and  $\omega_L \tau, r_L$  is the maximum of the two. The collision volume reads:

$$\begin{aligned} V &= 2\pi \int_{-\infty}^{\infty} f(v_z) dv_z \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{max(0, \frac{v_{\perp}}{\omega_L} - R_{max})}^{R_{max} + \frac{v_{\perp}}{\omega_L}} \rho d\rho \Delta\psi(\rho) \int_{-\frac{\sqrt{R_{max}^2 - (\rho - r_L)^2}}{|v_z|}}^{\frac{\sqrt{R_{max}^2 - (\rho - r_L)^2}}{|v_z|}} dt_i \quad (21) \\ &= 2\pi \int_{-\infty}^{\infty} f(v_z) dv_z \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{max(0, \frac{v_{\perp}}{\omega_L} - R_{max})}^{R_{max} + \frac{v_{\perp}}{\omega_L}} (|v_z| \tau + 2\sqrt{R_{max}^2 - (\rho - r_L)^2}) \Delta\psi(\rho) \rho d\rho \\ &= 2\pi(C_1 + C_2) \end{aligned}$$

96 with  $f(v_z) = \sqrt{\frac{m}{2\pi kT}} e^{-mv_z^2/2kT}$  and  $f_2(v_{\perp}) = \frac{m}{kT} v_{\perp} e^{-mv_{\perp}^2/2kT}$  denoting a one and two-dimensional  
97 Maxwellian velocity distributions respectively and with  $\Delta\psi$  redefined as in Table 1:

98  $C_1$  and  $C_2$  are:

$$\begin{aligned}
 C_1 &= \tau \int_{-\infty}^{\infty} f(v_z) |v_z| dv_z \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\max(0, \frac{v_{\perp}}{\omega_L} - R_{\max})}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho d\rho \Delta\psi(\rho) \quad (22) \\
 &= \langle |v_z| \rangle \tau \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\max(0, \frac{v_{\perp}}{\omega_L} - R_{\max})}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho d\rho \Delta\psi(\rho) \\
 &= \langle |v_z| \rangle \tau \left[ \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_0^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho d\rho \Delta\psi(\rho) + \int_{R_{\max}\omega_L}^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\frac{v_{\perp}}{\omega_L} - R_{\max}}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho d\rho \Delta\psi(\rho) \right] \\
 &= \langle |v_z| \rangle \tau R_{\max}^2 (I_1 + I_2 + I_3)
 \end{aligned}$$

with  $\langle |v_z| \rangle = \int_{-\infty}^{\infty} |v| f(v) dv = \sqrt{\frac{2kT}{\pi m}}$ . The integrals  $I_1 - I_3$  are given explicitly below. However, we first define the dimensionless quantities:

$$s = \frac{v_{\perp}}{\omega_L R_{\max}} \quad (23)$$

and

$$r = \frac{\rho_2}{R_{\max}} = \max(-\cos(\omega_L \tau / 2)s + \sqrt{1 - \sin^2(\omega_L \tau / 2)s^2}, s) \quad (24)$$

and

$$q = R_{\max}\omega_L \sqrt{\frac{m}{2kT}} \quad (25)$$

99 (essentially the averaged inverse  $s R_{\max} / r_L$ ).

$$I_1 = R_{\max}^{-2} \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_0^{\rho_2} \rho d\rho = \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \frac{\rho_2^2(v_{\perp})}{2} \quad (26)$$

$$I_2 = R_{\max}^{-2} \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_{\rho_2}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho \left( \left( 1 - \frac{\arccos(\frac{R_{\max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi} \right) + \frac{\omega_L \tau}{2\pi} \right) d\rho \quad (27)$$

and

$$I_3 = R_{\max}^{-2} \int_{R_{\max}\omega_L}^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\frac{v_{\perp}}{\omega_L} - R_{\max}}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \rho \left( \left( 1 - \frac{\arccos(\frac{R_{\max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi} \right) + \frac{\omega_L \tau}{2\pi} \right) d\rho \quad (28)$$

100 while

$$\begin{aligned}
 C_2 &= 2 \int_{-\infty}^{\infty} f(v_z) dv_z \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\max(0, \frac{v_{\perp}}{\omega_L} - R_{\max})}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{\max}^2 - (\rho - r_L)^2} \rho d\rho \Delta\psi(\rho) \quad (29) \\
 &= 2 \int_0^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\max(0, \frac{v_{\perp}}{\omega_L} - R_{\max})}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{\max}^2 - (\rho - r_L)^2} \rho d\rho \Delta\psi(\rho) \\
 &= 2 \left[ \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_0^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{\max}^2 - (\rho - r_L)^2} \rho d\rho \Delta\psi(\rho) \right. \\
 &\quad \left. + \int_{R_{\max}\omega_L}^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\frac{v_{\perp}}{\omega_L} - R_{\max}}^{R_{\max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{\max}^2 - (\rho - r_L)^2} \rho d\rho \Delta\psi(\rho) \right] \\
 &= 2R_{\max}^3 (J_1 + J_2 + J_3)
 \end{aligned}$$

101 with

$$\begin{aligned}
 J_1 &= R_{\max}^{-3} \int_0^{R_{\max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_0^{\rho_2} \sqrt{R_{\max}^2 - (\rho - r_L)^2} \rho d\rho \quad (30) \\
 &= \int_0^{q^2} dx e^{-x} \left[ \frac{s}{2} (\arcsin(s) - \arcsin(s - r)) + \frac{(s^2 + 2)\sqrt{1 - s^2} - (s^2 + 2 + sr - 2r^2)\sqrt{1 - s^2 + 2sr - r^2}}{6} \right]
 \end{aligned}$$

102 with  $x = \frac{mv_{\perp}^2}{2kT}$  and  $s = \frac{x^{1/2}}{q}$ ,  $r$  of course functions of  $x$ .

$$J_2 = R_{max}^{-3} \int_0^{R_{max}\omega_L} f_2(v_{\perp}) dv_{\perp} \int_{\rho_2}^{R_{max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{max}^2 - (\rho - r_L)^2} \rho \left( 1 - \frac{\arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi} \right) + \frac{\omega_L \tau}{2\pi} d\rho \quad (31)$$

and

$$J_3 = R_{max}^{-3} \int_{R_{max}\omega_L}^{\infty} f_2(v_{\perp}) dv_{\perp} \int_{\frac{v_{\perp}}{\omega_L} - R_{max}}^{R_{max} + \frac{v_{\perp}}{\omega_L}} \sqrt{R_{max}^2 - (\rho - r_L)^2} \rho \left( 1 - \frac{\arccos(\frac{R_{max}^2 - \rho^2 - r_L^2}{2\rho r_L})}{\pi} \right) + \frac{\omega_L \tau}{2\pi} d\rho \quad (32)$$

103 Note that the *only* difference from the previous work[1] is the factor  $\Delta\psi(\rho)$  for  $\omega_L \tau < 2\pi$ . Also  
 104 note that in [1], the corresponding integrations to infinity, e.g. the equivalents of  $I_3$  and  $J_3$  diverged as  
 105  $q \rightarrow 0$ . This divergence has been eliminated here due to the  $\Delta\psi$  factor. This is shown in Appendix A,  
 106 which evaluates the  $I_3$  and  $J_3$  integrals.

107 The remaining contributions vanished in [1] as  $q \rightarrow 0$  and clearly continue to do so here.

108 As already mentioned in [1], the number of particles that are in this volume, and hence need to be  
 109 simulated, is simply the volume multiplied by the perturber density.

110 **2.3. Generating perturbers**

111 To generate perturbers we proceed as in [1], but also generate for each perturber an angle  
 112  $\theta$ , uniformly distributed in  $(0, 2\pi)$ . Once we have generated  $v_z, v_{\perp}, \rho, t_i$  and  $\theta$ , we also generate  $\psi$   
 113 uniformly distributed in  $(\theta + \arccos(\frac{R_{max}^2 - r_L^2 - \rho^2}{2\rho r_L}) - \omega_L \tau), \theta + 2\pi - \arccos(\frac{R_{max}^2 - r_L^2 - \rho^2}{2\rho r_L})$ .

114 In more detail, we first draw a random number uniformly distributed in  $(0, 1)$ . If this is smaller  
 115 than  $\frac{C_1}{C_1 + C_2}$ , then we generate  $|v_z|, v_{\perp}, \rho$  from the distribution  $P_1(|v_z|, v_{\perp}, \rho) = |v_z| f(v_z) f_2(v_{\perp}) \rho$  by  
 116 generating independently a  $v_z$  with the probability distribution  $|v_z| f(v_z)$ , a  $v_{\perp}$  with the probability  
 117 distribution  $f_2(v_{\perp})$  and a  $\rho$  with the probability density  $\rho d\rho$  in  $((\max(0, \frac{v_{\perp}}{\omega_L} - R_{max}), R_{max} + \frac{v_{\perp}}{\omega_L}))$ .

118 Otherwise we generate from the distribution  $P_2(|v_z|, v_{\perp}, \rho) = f(v_z) f_2(v_{\perp}) \rho \sqrt{R_{max}^2 - (\rho - r_L)^2}$ .  
 119 The generation of impact parameters was done by a rejection method, as straightforward inversion is  
 120 not possible.

121 Once  $v_z, v_{\perp}$  and  $\rho$  have been generated,  $t_i$  is selected as a uniformly distributed time in  
 122  $(-\sqrt{R_{max}^2 - (\rho - r_L)^2}, |v_z| \tau + \sqrt{R_{max}^2 - (\rho - r_L)^2})$ .  $\theta$  and  $\psi$  are also generated as discussed above.

123 **3. Conclusions**

124 The present work extends the simplified theory for spiralling motion in a constant magnetic field  
 125 presented in [1] which was typically efficient for electron perturbers to more cases of practical interest,  
 126 i.e. ion perturbers and/or weak magnetic fields. The results of [1] are seen to hold for  $\omega_L \tau \geq 2\pi$  and  
 127 are here extended to  $\omega_L \tau < 2\pi$ , i.e. a regime typical for ions or weak magnetic field, thus validating  
 128 the common wisdom that ion trajectories are usually unaffected by spiralling. In addition, this work  
 129 identifies relevant parameters (e.g.  $q, \omega_L \tau$ ) and criteria for using a straight line and also allows the  
 130 efficient treatment of spiraling ion trajectories if needed. The results of this work are also a useful  
 131 basis for approximate standard treatments, i.e. impact/unified theories, if (strong) collisions are  
 132 isolated/disentangled and further, to perturbative impact/unified treatments if these collisions may  
 133 be handled in perturbation theory.

134 With regard to the notion[2], discussed in the introduction, that the relevant quantity for  
 135 neglecting spiralling is that  $r_L \gg R_{max}$ , the present paper shows that although the idea is qualitatively  
 136 correct, i.e. one may indeed neglect spiralling for small  $B$  and/or large perturber mass, leading to  
 137 a large Larmor radius, the actual situation is more complex and described by Table 1. Specifically,  
 138 to neglect spiralling, it is necessary that  $\frac{\omega_L \tau}{2\pi} \ll 1$  or equivalently  $\frac{\omega_L}{HWHM} \ll 1$ , i.e. the ratio of the  
 139 cyclotron frequency to the width(HWHM) of the line is important. This is because if this ratio is small,  
 140 then the perturber motion does not cover a full revolution and if very small, the motion is essentially

141 unaffected by the magnetic field. However, if the line is very narrow ( $\tau$  is large) there may be enough  
 142 time to complete at least a sizeable portion of a revolution, even if the Larmor radius is much larger  
 143 than the Debye length.

144

145 1. S.Alexiou, Atoms 7, 52(2019).  
 146 2. S.Günter and A.Könenies, JQSRT 62, 425-431(1999).  
 147 3. C.G.Hegerfeld and V.Kesting, Phys.Rev.A37, 1488(1988).  
 148 4. J.Seidel, Verhandl.DPG 372 (1990); J.Seidel, in Spectral Line Shapes 6, L.Frommhold and J.W.Keto (editors),  
 149 AIP Conference Proceedings vol. 216, New York 1990.  
 150 5. S.Alexiou, High Energy Density Physics 9, 375(2013).

151 **Appendix A: Calculation of the  $I_3$  and  $J_3$  double integrals**

152 Note that the integrands are the same except that  $J_n$  involves an extra factor of  
 153  $\sqrt{R_{max}^2 - (\rho - r_L)^2}$ .

154 Note that these integrals are the *only* contributions for  $q = 0$  ( $B = 0$ ).

155 *A.1 The  $I_3$  integration*

156 For  $I_2$  and  $I_3$ , the non-trivial  $\rho$  integrals reduce to ( $x = \rho/R_{max}$ )

$$\int dx x \arccos\left(\frac{1-x^2-s^2}{2xs}\right) = \frac{1}{4}[2x^2 \arccos\left(\frac{1-x^2-s^2}{2xs}\right) + \sqrt{-s^4 + 2s^2(x^2+1) - (x^2-1)^2}] + \frac{1}{2} \arctan\left(\frac{1+s^2-x^2}{\sqrt{-s^4 + 2s^2(x^2+1) - (x^2-1)^2}}\right) \quad (33)$$

For both  $I_2$  and  $I_3$ , the upper limit is  $s+1$ , for which

$$\arccos\left(\frac{1-x^2-s^2}{2xs}\right) = \arccos(-1) = \pi \quad (34)$$

and

$$\sqrt{-s^4 + 2s^2(x^2+1) - (x^2-1)^2} = 0 \quad (35)$$

Hence the middle term vanishes, while the arctangent is

$$\arctan\left(\frac{1+s^2-x^2}{\sqrt{-s^4 + 2s^2(x^2+1) - (x^2-1)^2}}\right) = \arctan\left(\frac{-2s}{0^+}\right) = -\frac{\pi}{2} \quad (36)$$

If (as in  $I_3$ ) the lower limit is  $s-1$ , the argument of the square root is also 0,  $\arccos\left(\frac{1-x^2-s^2}{2xs}\right) = \arccos(-1) = \pi$  and

$$\arctan\left(\frac{1+s^2-x^2}{\sqrt{-s^4 + 2s^2(x^2+1) - (x^2-1)^2}}\right) = \arctan\left(\frac{2s}{0^+}\right) = \frac{\pi}{2} \quad (37)$$

157  $I_3$  is

$$\begin{aligned} I_3 &= \int_{R_{max}\omega_L}^{\infty} dv_{\perp} f_2(v_{\perp}) \int_{s-1}^{s+1} dx x \left(1 + \frac{\omega_L \tau}{2\pi} - \frac{\arccos\left(\frac{1-x^2-s^2}{2xs}\right)}{\pi}\right) \\ &= \int_{R_{max}\omega_L}^{\infty} dv_{\perp} f_2(v_{\perp}) \left(2s\left(1 + \frac{\omega_L \tau}{2\pi}\right) - \frac{4s-1}{2}\right) = \left[\frac{e^{-q^2}}{2} + \frac{\omega_L \tau}{2\pi} \frac{2}{q} \int_{q^2}^{\infty} dx x^{1/2} e^{-x}\right] \\ &= \left[\frac{e^{-q^2}}{2} + \frac{2\omega_L \tau}{2\pi} (e^{-q^2} + \pi^{1/2} \frac{1-erf(q)}{2q})\right] = \frac{e^{-q^2}}{2} \left(1 + \frac{2\omega_L \tau}{\pi}\right) + (1 - erf(q)) \frac{\langle|v_z|\rangle \tau}{2R_{max}} \end{aligned} \quad (38)$$

158 Note that unlike the simplified version[1],  $I_3$  does *not* diverge because the  $\omega_L \tau$  cancels the  $q^{-1}$ .

159 A.2 The  $J_3$  integration

We break the  $J_3$  integral as

$$J_3 = J_{31} + J_{32} \quad (39)$$

160 where using  $s = \frac{v_{\perp}}{R_{max}\omega_L}$ , we have:

$$\begin{aligned} J_{31} &= \frac{\omega_L\tau}{2\pi} \int_{R_{max}\omega_L}^{\infty} dv_{\perp} f_2(v_{\perp}) \int_{s-1}^{s+1} dx x \sqrt{1-(x-s)^2} = 2q^2 \frac{\omega_L\tau}{2\pi} \int_1^{\infty} se^{-s^2 q^2 \frac{s\pi}{2}} \\ &= \frac{\pi}{2} \frac{\omega_L\tau}{2\pi} [\sqrt{\pi} \frac{1-erf(q)}{2q} + e^{-q^2}] = \frac{\omega_L\tau e^{-q^2}}{4} + \frac{\tau}{8R_{max}} \sqrt{\frac{2\pi kT}{m}} (1 - erf(q)) \end{aligned} \quad (40)$$

, which does *not* diverge for  $\omega_L \rightarrow 0$  and

$$J_{32} = \int_{R_{max}\omega_L}^{\infty} dv_{\perp} f_2(v_{\perp}) \int_{s-1}^{s+1} dx \sqrt{1-(x-s)^2} x (1 - \frac{\arccos \frac{1-x^2-s^2}{2xs}}{\pi}) = 2q^2 \int_1^{\infty} se^{-s^2 q^2} N_3 \quad (41)$$

with

$$N_3 = \int_{s-1}^{s+1} dx \sqrt{1-(x-s)^2} x [1 - \frac{\arccos(-1 + \frac{1-(x-s)^2}{2xs})}{\pi}] \quad (42)$$

Using  $z = x - s$  this becomes:

$$N_3 = \int_{-1}^1 dz \sqrt{1-z^2} (s+z) [1 - \frac{\arccos(-1 + \frac{1-z^2}{2s(z+s)})}{\pi}] = N_{31} + N_{32} \quad (43)$$

with

$$N_{31} = \int_{-1}^1 dz \sqrt{1-z^2} z [1 - \frac{\arccos(-1 + \frac{1-z^2}{2s(z+s)})}{\pi}] \quad (44)$$

and

$$N_{32} = s \int_{-1}^1 dz \sqrt{1-z^2} [1 - \frac{\arccos(-1 + \frac{1-z^2}{2s(z+s)})}{\pi}] \quad (45)$$

If  $\omega_L \rightarrow 0, s \rightarrow \infty$ , and  $\frac{1-z^2}{2s(z+s)} \rightarrow 0$ , hence the argument of the inverse cosine is very nearly -1, and thus we have  $[1 - \frac{\arccos(-1 + \frac{1-(x-s)^2}{2xs})}{\pi}] \rightarrow 0$  which gives a vanishing contribution for  $N_{31}$ . However, for  $N_{32}$  this is multiplied by  $s \rightarrow \infty$  so the result is not immediately clear. Since Taylor-expanding the inverse cosine around -1 does not work due to the infinite derivative, we can use Frobenius's method or write

$$x = \frac{1-z^2}{2s(z+s)} \rightarrow 0 \quad (46)$$

and write

$$\arccos(-1+x) = \pi - \arccos(1-x) \quad (47)$$

Next, use L'Hopital's rule to evaluate as  $x \rightarrow 0$

$$\frac{\arccos(1-x)}{x^{1/2}} \rightarrow 2^{1/2} \quad (48)$$

with the final result that

$$s[1 - \pi^{-1} \arccos(-1 + \frac{1-z^2}{2s(z+s)})] \rightarrow \frac{s}{\pi} \sqrt{\frac{2(1-z^2)}{2s(z+s)}} = \pi^{-1} \sqrt{\frac{1-z^2}{1+z/s}} \rightarrow \pi^{-1} \sqrt{1-z^2} \quad (49)$$

161 Therefore we again have no divergence for small  $q$  (large  $s$ ); furthermore the above large  $s$  asymptotic  
162 result is useful numerically due to possible underflows of  $[1 - \pi^{-1} \arccos(-1 + \frac{1-z^2}{2s(z+s)})]$ .

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