

Article

# The Entropic Dynamics approach to Quantum Mechanics

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**Abstract:** Entropic Dynamics (ED) is a framework in which Quantum Mechanics is derived as an application of entropic methods of inference. In ED the dynamics of the probability distribution is driven by entropy subject to constraints that are codified into a quantity later identified as the phase of the wave function. The central challenge is to specify how those constraints are themselves updated. In this paper we review and extend the ED framework in several directions. A new version of ED is introduced in which particles follow smooth differentiable Brownian trajectories (as opposed to non-differentiable Brownian paths). To construct ED we make use of the fact that the space of probabilities and phases has a natural symplectic structure (i.e., it is a phase space with Hamiltonian flows and Poisson brackets). Then, using an argument based on information geometry, a metric structure is introduced. It is shown that the ED that preserves the symplectic and metric structures – which is a Hamilton-Killing flow in phase space – is the linear Schrödinger equation. These developments allow us to discuss why wave functions are complex and the connections between the superposition principle, the single-valuedness of wave functions, and the quantization of electric charges. Finally, it is observed that Hilbert spaces are not necessary ingredients in this construction. They are a clever but merely optional trick that turns out to be convenient for practical calculations.

**Keywords:** quantum mechanics; entropic dynamics; symplectic geometry; information geometry

## 1. Introduction

Quantum mechanics has been commonly regarded as a generalization of classical mechanics with an added element of indeterminism. The standard quantization recipe starts with a description in terms of the system's classical coordinates and momenta  $\{q, p\}$  and then proceeds by applying a series of more or less ad hoc rules that replace the classical  $\{q, p\}$  by self-adjoint linear operators  $\{\hat{q}, \hat{p}\}$  acting on some complex Hilbert space [1]. The Hilbert space structure is given priority while the probabilistic structure is relegated to the less fundamental status of providing phenomenological rules for how to handle those mysterious physical processes called measurements. The result is a dichotomy between two separate and irreconcilable modes of wave function evolution: one is the linear and deterministic Schrödinger evolution and the other is the discontinuous and stochastic wave function collapse [2][3]. To put it bluntly, the dynamical and the probabilistic aspects of quantum theory are incompatible with each other. And furthermore, the dichotomy spreads to the interpretation of the quantum state itself.<sup>1</sup> It obscures the issue of whether the wave function describes the *ontic* state of the system or whether it describes an *epistemic* state about the system.<sup>2</sup>

<sup>1</sup> Excellent reviews with extended references to the literature are given in e.g. [4]-[7].

<sup>2</sup> Since the terms 'ontic' and 'epistemic' are not yet of widespread use outside the community of Foundations of QM a clarification might be useful. A concept is referred as 'ontic' when it describes something that is supposed to be real, to exist

32 In the Entropic Dynamics (ED) approach these problems are resolved by placing the probabilistic  
 33 aspects of QM at the forefront while the Hilbert space structure is relegated to the secondary role of a  
 34 convenient calculational tool [8]-[10]. ED tackles QM as an example of entropic inference, a framework  
 35 designed to handle insufficient information.<sup>3</sup> The starting point is to specify the subject matter, the  
 36 ontology — are we talking about the positions of particles or the configurations of fields? Once this  
 37 decision is made our inferences about these variables are driven by entropy subject to information  
 38 expressed by constraints. The main effort is directed towards choosing those constraints since it is  
 39 through them that the “physics” is introduced.

40 From the ED perspective many of the questions that seemed so urgent in other approaches are  
 41 successfully evaded. For example, when quantum theory is regarded as an extension of classical  
 42 mechanics any deviations from causality demand an explanation. In contrast, in the entropic approach  
 43 uncertainty and probabilities are the norm. Indeterminism is just the inevitable consequence of  
 44 incomplete information and no deeper explanation is needed. Instead, it is the certainty and  
 45 determinism of the classical limit that require explanations. Another example of a question that  
 46 has consumed an enormous effort is the problem of deriving the Born rule from a fundamental Hilbert  
 47 space structure. In the ED approach this question does not arise and the burden of explanation runs in  
 48 the opposite direction: how do objects such as wave functions involving complex numbers emerge  
 49 in a purely probabilistic framework? Yet a third example concerns the interpretation of the wave  
 50 function itself. ED offers an uncompromising and radically epistemic view of the wave function  $\Psi$ .  
 51 This turns out to be extremely restrictive: in a fully epistemic interpretation there is no logical room  
 52 for “quantum” probabilities obeying alternative rules of inference. Not only is the probability  $|\Psi|^2$   
 53 interpreted as a state of knowledge but, in addition, the epistemic significance of the phase of the  
 54 wave function must be clarified and made explicit. Furthermore, it is also required that all *updates* of  
 55  $\Psi$ , which include *both* its unitary time evolution *and* the wave function collapse during measurement,  
 56 must be obtained as a consequence of entropic and Bayesian updating rules.<sup>4</sup>

57 There is a large literature on reconstructions of quantum mechanics (see *e.g.*, [23]-[28] and  
 58 references therein) and there are several approaches based on information theory (see *e.g.*, [29]-[41]).  
 59 What distinguishes ED is a strict adherence to Bayesian and entropic methods and also a central  
 60 concern with the nature of time. The issue here is that any discussion of dynamics must inevitably  
 61 include a notion of time but the rules for inference do not mention time — they are totally atemporal.  
 62 One can make inferences about the past just as well as about the present or the future. This means  
 63 that any model of dynamics based on inference must also include assumptions about time, and those  
 64 assumptions must be explicitly stated. In ED “entropic” time is a book-keeping device *designed* to keep  
 65 track of changes. The construction of entropic time involves several ingredients. One must introduce  
 66 the notion of an ‘instant’; one must show that these instants are suitably ordered; and finally one must  
 67 define a convenient measure of the duration or interval between the successive instants. It turns out  
 68 that an arrow of time is generated automatically and entropic time is intrinsically directional.

69 This paper contains a review of previous work on ED and extends the formalism in several new  
 70 directions. In [8]-[10] the Schrödinger equation was derived as a peculiar non-dissipative diffusion in

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out there independently of any observer. A concept is referred as ‘epistemic’ when it is related to the state of knowledge, opinion, or belief of an agent, albeit an ideally rational agent. Examples of epistemic quantities are probabilities and entropies. An important point is that the distinction ontic/epistemic is not the same as the distinction objective/subjective. For example, probabilities are fully epistemic — they are tools for reasoning with incomplete information — but they can lie anywhere in the spectrum from being completely subjective (two different agents can have different beliefs) to being completely objective. In QM, for example, probabilities are epistemic and objective. Indeed, at the non-relativistic level anyone who computes probabilities that disagree with QM will be led to experimental predictions that are demonstrably wrong. We will say that the wave function  $\Psi$ , which is fully epistemic and objective, represents a “physical” state when it represents information about an actual “physical” situation.

<sup>3</sup> The principle of maximum entropy as a method for inference can be traced to the pioneering work of E. T. Jaynes [11]-[13]. For a pedagogical overview including more modern developments see [14][15].

<sup>4</sup> There exist many different Bayesian interpretations of probability. In section 13 we comment on how ED differs from the frameworks known as Quantum Bayesianism [16]-[18] and its closely related descendant QBism [19][20].

71 which the particles perform an irregular Brownian motion that resembles the Einstein-Smoluchowski  
 72 (ES) process [42]. The trajectories are continuous and non-differentiable so their velocity is undefined.  
 73 Since the expected length of the path between any two points is infinite this would be a very peculiar  
 74 motion indeed. Here we exhibit a new form of ED in which the Brownian motion resembles the much  
 75 smoother Ornstein-Uhlenbeck (OU) process [42]. The trajectories have finite expected lengths; they  
 76 are continuous and differentiable. On the other hand, although the velocities are well defined and  
 77 continuous, they are not differentiable.<sup>5</sup>

78 We had also shown that the irregular Brownian motion at the “microscopic” or sub-quantum level  
 79 was not unique. One can enhance or suppress the fluctuations while still obtaining the same emergent  
 80 Schrödinger behavior at the “macroscopic” or quantum level [43][44]. A similar phenomenon is also  
 81 found in the smoother ED developed here. In both the ES and the OU cases the special limiting case in  
 82 which fluctuations are totally suppressed turns out to be of particular interest because the particles  
 83 evolve deterministically along the smooth lines of probability flow. This means that ED includes the  
 84 Bohmian or causal form of quantum mechanics [45]-[47] as a limiting case.

85 ED consists in the entropic updating of probabilities through information supplied by constraints.  
 86 The main concern is how these constraints are chosen including, in particular, how the constraints  
 87 themselves are updated. In [48] an effective criterion was found by adapting Nelson’s seminal insight  
 88 that QM is a non-dissipative diffusion [49]. This amounts to updating constraints in such a way that  
 89 a certain energy functional is conserved. Unfortunately this criterion, while fully satisfactory in a  
 90 non-relativistic setting, fails in curved space-times where the concept of a globally conserved energy  
 91 may not exist.

92 The second contribution in this paper is a geometric framework for updating constraints that  
 93 does not rely on the notion of a conserved energy. Our framework draws inspiration from two sources:  
 94 one is the fact that QM has a rich geometrical structure [50]-[58]. The authors of [50]-[56] faced the task  
 95 of unveiling geometric structures that although well hidden are already present in the standard QM  
 96 framework. Our goal runs in the opposite direction: we impose these natural geometric structures as  
 97 the foundation upon which we reconstruct the QM formalism.

98 The other source of inspiration is the connection between QM and information geometry [14]-[60]  
 99 that was originally suggested in the work of Wootters [29]. This connection has been explored  
 100 in the context of quantum statistical inference [61], in the operational description of quantum  
 101 measurements [32][34], and in the reconstruction of QM [38][39]. Our previous presentation in  
 102 [10] has been considerably streamlined by recognizing the central importance of symmetry principles  
 103 when implemented in conjunction with concepts of information geometry.

104 In ED the degrees of freedom are the probability densities  $\rho(x)$  and certain “phase” fields  $\Phi(x)$   
 105 that represent the constraints that control the flow of probabilities. Thus we are concerned not just with  
 106 the “configuration” space of probabilities  $\{\rho\}$  but with the larger space of probabilities and phases  
 107  $\{\rho, \Phi\}$ . The latter has a natural symplectic structure, that is,  $\{\rho, \Phi\}$  is a phase space. Imposing a  
 108 dynamics that preserves this symplectic structure leads to Hamiltonian flows, Poisson brackets, and so  
 109 much of the canonical formalism associated with mechanics. To single out the particular Hamiltonian  
 110 flow that reproduces QM we extend the information geometry of the configuration space  $\{\rho\}$  to the  
 111 full phase space. This is achieved by imposing a symmetry that is natural in a probabilistic setting: we  
 112 extend the well-known spherically symmetric information geometry of the space  $\{\rho\}$  to the full phase  
 113 space  $\{\rho, \Phi\}$ . This construction yields a derivation of the Fubini-Study metric. A welcome by-product

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<sup>5</sup> In both the ES and the OU processes, which were originally meant to model the actual physical Brownian motion, friction and dissipation play essential roles. In contrast, ED is non-dissipative. ED formally resembles Nelson’s stochastic mechanics [23] but the conceptual differences are significant. Nelson’s mechanics attempted an ontic interpretation of QM as an ES process driven by real stochastic classical forces while ED is a purely epistemic model that does not appeal to an underlying classical mechanics.

114 is that the joint presence of a symplectic and a metric structure leads to a complex structure. This is the  
 115 reason why QM involves complex numbers.

116 The dynamics that preserves the metric structure is a Killing flow. We propose that the  
 117 desired geometric criterion for updating constraints is a dynamics that preserves both the symplectic  
 118 and the metric structures. Thus, in the final step of our reconstruction of QM we show that the  
 119 Hamiltonians that generate Hamiltonian-Killing flows lead to an entropic dynamics described by the  
 120 linear Schrödinger equation.

121 We conclude with some comments exploring various aspects of the ED formalism. We show that,  
 122 despite the arrow of entropic time, the resulting ED is symmetric under time reversal. We discuss the  
 123 connections between linearity, the superposition principle, the single-valuedness of wave functions,  
 124 and the quantization of charge. We also discuss the classical limit and the Bohmian limit in which  
 125 fluctuations are suppressed and particles follow deterministic trajectories. Finally, we discuss the  
 126 introduction of Hilbert spaces. We argue that, while strictly unnecessary in principle, Hilbert spaces  
 127 are extremely convenient for calculational purposes.

128 This paper focuses on the derivation of the Schrödinger equation but the ED approach has been  
 129 applied to a variety of other topics in quantum theory. These include: the quantum measurement  
 130 problem [21][22]; momentum and uncertainty relations [62][44],<sup>6</sup> the Bohmian limit [43][44] and the  
 131 classical limit [66]; extensions to curved spaces [67]; to relativistic fields [68][69]; and the ED of spin  
 132 [71].

## 133 2. The ED of short steps

134 We deal with  $N$  particles living in a flat 3-dimensional space  $\mathbf{X}$  with metric  $\delta_{ab}$ . For  $N$  particles  
 135 the configuration space is  $\mathbf{X}_N = \mathbf{X} \times \dots \times \mathbf{X}$ . We assume that the particles have definite positions  $x_n^a$   
 136 and it is their unknown values that we wish to infer.<sup>7</sup> (The index  $n = 1 \dots N$  denotes the particle and  
 137  $a = 1, 2, 3$  the spatial coordinates.)

138 In ED positions play a very special role: they define the ontic state of the system. This is in  
 139 contradiction with the standard Copenhagen notion that quantum particles acquire definite positions  
 140 only as a result of a measurement. For example, in the ED description of the double slit experiment the  
 141 particle definitely goes through one slit or the other but one might not know which. The wave function,  
 142 on the other hand, is a purely epistemic notion and, as it turns out, all other quantities, such as energy  
 143 or momentum, are epistemic too. They do not reflect properties of the particles but properties of the  
 144 wave function [21]-[62].

145 Having identified the microstates  $x \in \mathbf{X}_N$  we tackle the dynamics. The main dynamical  
 146 assumption is that the particles follow trajectories that are continuous. This represents an enormous  
 147 simplification because it implies that a generic motion can be analyzed as the accumulation of many  
 148 infinitesimally short steps. Therefore, the first task is to find the transition probability  $P(x'|x)$  for a  
 149 short step from an initial  $x$  to an unknown neighboring  $x'$  and only later we will determine how such  
 150 short steps accumulate to yield a finite displacement.

The probability  $P(x'|x)$  is found by maximizing the entropy

$$S[P, Q] = - \int dx' P(x'|x) \log \frac{P(x'|x)}{Q(x'|x)} \quad (1)$$

151 relative to the joint prior  $Q(x'|x)$  subject to constraints given below. (In multidimensional integrals  
 152 such as (1) the notation  $dx'$  stands for  $d^{3N}x'$ .)

<sup>6</sup> These are the well-known uncertainty relations due to Heiseberg and to Schrödinger. The entropic uncertainty relations proposed by Deutsch [63]-[65] have not yet been explored within the context of ED.

<sup>7</sup> In this work ED is a model for the quantum mechanics of particles. The same framework can be deployed to construct models for the quantum mechanics of fields, in which case it is the fields that are ontic and have well-defined albeit unknown values [68][69]

**The prior** — The choice of prior  $Q(x'|x)$  must reflect the state of knowledge that is common to all short steps. (It is through the constraints that the information that is specific to any particular short step will be supplied.) We adopt a prior that carries the information that the particles take infinitesimally short steps and also reflects the translational and rotational invariance of the Euclidean space  $\mathbf{X}$  but is otherwise uninformative. In particular, the prior expresses total ignorance about any correlations. Such a prior can itself be derived from the principle of maximum entropy. Indeed, maximize

$$S[Q] = - \int dx' Q(x'|x) \log \frac{Q(x'|x)}{\mu(x')}, \quad (2)$$

relative to the uniform measure  $\mu(x')$ ,<sup>8</sup> subject to normalization, and subject to the  $N$  independent constraints

$$\langle \delta_{ab} \Delta x_n^a \Delta x_n^b \rangle = \kappa_n, \quad (n = 1 \dots N), \quad (3)$$

where  $\kappa_n$  are small constants and  $\Delta x_n^a = x_n'^a - x_n^a$ . The result is a product of Gaussians,

$$Q(x'|x) \propto \exp -\frac{1}{2} \sum_n \alpha_n \delta_{ab} \Delta x_n^a \Delta x_n^b, \quad (4)$$

153 where, in order to reflect translational invariance and possibly non-identical particles, the Lagrange  
154 multipliers  $\alpha_n$  are independent of  $x$  but may depend on the index  $n$ . Eventually we will let  $\alpha_n \rightarrow \infty$  in  
155 order to implement infinitesimally short steps. Next we specify the constraints that are specific to each  
156 particular short step.

**The drift potential constraint** — In Newtonian dynamics one does not need to explain why a particle perseveres in its motion in a straight line; what demands an explanation — that is, a force — is why the particle deviates from inertial motion. In ED one does not require an explanation for why the particles move; what requires an explanation is how the motion can be both directional and highly correlated. This physical information is introduced through one constraint that acts simultaneously on all particles. The constraint involves a function  $\phi(x) = \phi(x_1 \dots x_N)$  on configuration space  $\mathbf{X}_N$  that we call the “drift” potential. We impose that the displacements  $\Delta x_n^a$  are such that the expected change of the drift potential  $\langle \Delta \phi \rangle$  is constrained to be

$$\langle \Delta \phi \rangle = \sum_{n=1}^N \langle \Delta x_n^a \rangle \frac{\partial \phi}{\partial x_n^a} = \kappa'(x), \quad (5)$$

157 where  $\kappa'(x)$  is another small but for now unspecified function. As we shall later see this information is  
158 already sufficient to construct an interesting ED. However, to reproduce the particular dynamics that  
159 describes quantum systems we must further require that the potential  $\phi(x)$  be a multi-valued function  
160 with the topological properties of an angle —  $\phi$  and  $\phi + 2\pi$  represent the same angle.<sup>9</sup>

161 The physical origin of the drift potential  $\phi(x)$  is at this point unknown so how can one justify its  
162 introduction? The idea is that identifying the relevant constraints can represent significant progress  
163 even when their physical origin remains unexplained. Indeed, with the *single* assumption of a  
164 constraint involving a drift potential we will explain and coordinate *several* features of quantum  
165 mechanics such as entanglement, the existence of complex and symplectic structures, the actual form  
166 of the Hamiltonian, and the linearity of the Schrödinger equation.

**The gauge constraints** — The single constraint (5) already leads to a rich entropic dynamics but by imposing additional constraints we can construct even more realistic models. To incorporate the

<sup>8</sup> In Cartesian coordinates  $\mu = \text{const}$  and may be ignored.

<sup>9</sup> The angular nature of the drift potential is explained when the ED framework is extended to particles with spin [71].

effect of an external electromagnetic field we impose that for each particle  $n$  the expected displacement  $\langle \Delta x_n^a \rangle$  will satisfy

$$\langle \Delta x_n^a \rangle A_a(x_n) = \kappa_n'' \quad \text{for } n = 1 \dots N, \quad (6)$$

167 where the electromagnetic vector potential  $A_a(x_n)$  is a field that lives in the 3-dimensional physical  
168 space ( $x_n \in \mathbf{X}$ ). The strength of the coupling is given by the values of the  $\kappa_n''$ . These quantities could be  
169 specified directly but, as is often the case in entropic inference, it is much more convenient to specify  
170 them indirectly in terms of the corresponding Lagrange multipliers.

171 **The transition probability** — An important feature of the ED model can already be discerned.  
172 The central object of the discussion so far, the transition probability  $P(x'|x)$ , codifies information  
173 supplied through the prior and the constraints. This information does not depend on the positions of  
174 the particles prior to the initial position  $x$ . This implies that ED will take the form of a Markov process.

175 The distribution  $P(x'|x)$  that maximizes the entropy  $\mathcal{S}[P, Q]$  in (1) relative to (4) and subject to (5),  
176 (6), and normalization is

$$P(x'|x) = \frac{1}{Z} \exp - \sum_n \left( \frac{\alpha_n}{2} \delta_{ab} \Delta x_n^a \Delta x_n^b - \alpha' [\partial_{na} \phi - \beta_n A_a(x_n)] \Delta x_n^a \right) \quad (7)$$

177 where  $\alpha'$  and  $\beta_n$  are Lagrange multipliers. This is conveniently written as

$$P(x'|x) = \frac{1}{Z} \exp - \sum_n \frac{\alpha_n}{2} \delta_{ab} (\Delta x_n^a - \Delta \bar{x}_n^a) (\Delta x_n^b - \Delta \bar{x}_n^b), \quad (8)$$

with a suitably modified normalization and

$$\Delta \bar{x}_n^a = \frac{\alpha'}{\alpha_n} [\partial_{na} \phi - \beta_n A_a(x_n)] = \langle \Delta x_n^a \rangle. \quad (9)$$

A generic displacement is expressed as a drift plus a fluctuation,

$$\Delta x_n^a = \langle \Delta x_n^a \rangle + \Delta w_n^a, \quad (10)$$

where

$$\langle \Delta w_n^a \rangle = 0, \quad \text{and} \quad \langle \Delta w_n^a \Delta w_{n'}^b \rangle = \frac{1}{\alpha_n} \delta_{nn'} \delta^{ab}, \quad (11)$$

178 The fact that the constraints (5) and (6) are not independent — both involve the same  
179 displacements  $\langle \Delta x_n^a \rangle$  — has turned out to be significant. We can already see in (7) and (9) that it  
180 leads to a gauge symmetry. As we shall later see the vector potential  $A_a$  will be interpreted as the  
181 corresponding gauge connection field and the multipliers  $\beta_n$  will be related to the electric charges  
182 through  $\beta_n = q_n / \hbar c$ .

### 183 3. Entropic time

184 The task of iterating the short steps described by the transition probability (8) to predict motion  
185 over finite distances leads us to introduce a book-keeping parameter  $t$ , to be called time, in order to  
186 keep track of the accumulation of short steps. The construction of time involves three ingredients: **(a)**  
187 we must specify what we mean by an ‘instant’; **(b)** these instants must be ordered; and finally, **(c)** one  
188 must specify the interval  $\Delta t$  between successive instants — one must define ‘duration’.

189 Since the foundation for any theory of time is the theory of change, that is, the dynamics, the  
190 notion of time constructed below will reflect the inferential nature of entropic dynamics. Such a  
191 construction we will call “entropic” time [8]. Later we will return to the question of whether and how  
192 this “entropic” time is related to the “physical” time that is measured by clocks.

193 3.1. Time as an ordered sequence of instants

ED consists of a succession of short steps. Consider, for example, the  $i$ th step which takes the system from  $x = x_{i-1}$  to  $x' = x_i$ . Integrating the joint probability,  $P(x_i, x_{i-1})$ , over  $x_{i-1}$  gives

$$P(x_i) = \int dx_{i-1} P(x_i, x_{i-1}) = \int dx_{i-1} P(x_i|x_{i-1})P(x_{i-1}). \quad (12)$$

No physical assumptions were involved in deriving this equation; it follows directly from the laws of probability. To establish the connection to time and dynamics we will make the physical assumption that if  $P(x_{i-1})$  is interpreted as the probability of different values of  $x_{i-1}$  at one “instant” labelled  $t$ , then we will interpret  $P(x_i)$  as the probability of values of  $x_i$  at the next “instant” labelled  $t'$ . More explicitly, if we write  $P(x_{i-1}) = \rho_t(x)$  and  $P(x_i) = \rho_{t'}(x')$  then we have

$$\rho_{t'}(x') = \int dx P(x'|x)\rho_t(x). \quad (13)$$

194 This equation defines the notion of “instant”: if the distribution  $\rho_t(x)$  refers to one instant  $t$ , then the  
 195 distribution  $\rho_{t'}(x')$  generated by  $P(x'|x)$  defines what we mean by the “next” instant  $t'$ . Iterating this  
 196 process defines the dynamics.

197 This construction of time is intimately related to information and inference. An instant is an  
 198 informational state that is complete in the sense that it is specified by the information — codified into  
 199 the distributions  $\rho_t(x)$  and  $P(x'|x)$  — that is sufficient for predicting the next instant. Thus, *the present*  
 200 *is defined through a sufficient amount of information such that, given the present, the future is independent of*  
 201 *the past.*

202 In the ED framework the notions of instant and of simultaneity are intimately related to the  
 203 distribution  $\rho_t(x)$ . To see how this comes about consider a single particle at the point  $\vec{x} = (x^1, x^2, x^3)$ .  
 204 It is implicit in the notation that  $x^1$ ,  $x^2$ , and  $x^3$  occur simultaneously. When we describe a system  
 205 of  $N$  particles by a single point  $x = (\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$  in  $3N$ -dimensional configuration space it is also  
 206 implicitly assumed that all the  $3N$  coordinate values refer to the same instant; they are simultaneous.  
 207 The very idea of a *point* in configuration space assumes simultaneity. And furthermore, whether we  
 208 deal with one particle or many, a distribution such as  $\rho_t(x)$  is meant to describe our uncertainty about  
 209 the possible configurations  $x$  of the system at the given instant. Thus, a probability distribution  $\rho_t(x)$   
 210 provides a criterion of simultaneity.<sup>10</sup>

211 3.2. The arrow of entropic time

The notion of time constructed according to eq.(13) is intrinsically directional. There is an absolute sense in which  $\rho_t(x)$  is prior and  $\rho_{t'}(x')$  is posterior. Indeed, the same rules of probability that led us to (13) can also lead us to the time-reversed evolution,

$$\rho_t(x) = \int dx' P(x|x')\rho_{t'}(x'). \quad (14)$$

Note, however, that there is a temporal asymmetry: while the distribution  $P(x'|x)$ , eq.(7), is a Gaussian derived using the maximum entropy method, its time-reversed version  $P(x|x')$  is related to  $P(x'|x)$  by Bayes' theorem,

$$P(x|x') = \frac{\rho_t(x)}{\rho_{t'}(x')} P(x'|x), \quad (15)$$

212 which in general will not be Gaussian.

<sup>10</sup> In a relativistic theory there is more freedom in the choice of instants and this translates into a greater flexibility with the notion of simultaneity. Conversely, the requirement of consistency among the different notions of simultaneity severely limits the allowed forms of relativistic ED [69].

213 The puzzle of the arrow of time (see *e.g.* [72][73]) arises from the difficulty in deriving a temporal  
 214 asymmetry from underlying laws of nature that are symmetric. The ED approach offers a fresh  
 215 perspective on this topic because it does not assume any underlying laws of nature — whether they be  
 216 symmetric or not. The asymmetry is the inevitable consequence of constructing time in a dynamics  
 217 driven by entropic inference.

218 From the ED point of view the challenge does not consist in explaining the arrow of time —  
 219 *entropic time itself only flows forward* — but rather in explaining how it comes about that despite the  
 220 arrow of time some laws of physics, such as the Schrödinger equation, turn out to be time reversible.  
 221 We will revisit this topic in section 9.

### 222 3.3. Duration and the sub-quantum motion

223 We have argued that the concept of time is intimately connected to the associated dynamics but at  
 224 this point neither the transition probability  $P(x'|x)$  that specifies the dynamics nor the corresponding  
 225 entropic time have been fully defined yet. It remains to specify how the multipliers  $\alpha_n$  and  $\alpha'$  are  
 226 related to the interval  $\Delta t$  between successive instants.

The basic criterion for this choice is convenience: *duration is defined so that motion looks simple*. The description of motion is simplest when it reflects the symmetry of translations in space and time. In a flat space-time this leads to an entropic time that resembles Newtonian time in that it flows “equally everywhere and everywhen.” Referring to eqs.(9) and (11) we choose  $\alpha'$  and  $\alpha_n$  to be independent of  $x$  and  $t$ , and we choose the ratio  $\alpha'/\alpha_n \propto \Delta t$  so that there is a well defined drift velocity. For future convenience the proportionality constants will be expressed in terms of some particle-specific constants  $m_n$ ,

$$\frac{\alpha'}{\alpha_n} = \frac{\hbar}{m_n} \Delta t, \quad (16)$$

227 where  $\hbar$  is an overall constant that fixes the units of the  $m_n$ s relative to the units of time. As we shall  
 228 later see, the constants  $m_n$  will eventually be identified with the particle masses while the constant  
 229  $\hbar$  will be identified as Planck’s constant. Having specified the ratio  $\alpha'/\alpha_n$  it remains to specify  $\alpha_n$   
 230 (or  $\alpha'$ ). It turns out that the choice is not unique. There is a variety of motions at the sub-quantum  
 231 “microscopic” level that lead to the same quantum mechanics at the “macroscopic” level.

In previous work [8]-[10] we chose  $\alpha_n$  proportional to  $1/\Delta t$ . This led to an ED in which the particles follow the highly irregular non-differentiable Brownian trajectories characteristic of an Einstein-Smoluchowski process. The first new contribution of this paper is to explore the consequences of choosing  $\alpha_n \propto 1/\Delta t^3$ . We write the proportionality constant as  $\hbar/\eta$  so that

$$\alpha_n = \frac{m_n}{\eta \Delta t^3}, \quad (17)$$

232 where a new constant  $\eta$  is introduced.

It is convenient to introduce a notation tailored to configuration space. Let  $x^A = x_n^a$ ,  $\partial_A = \partial/\partial x_n^a$ , and  $\delta_{AB} = \delta_{nn'}\delta_{ab}$ , where the upper case indices  $A, B, \dots$  label both the particles  $n, n', \dots$  and their coordinates  $a, b, \dots$ . Then the transition probability (8) becomes

$$P(x'|x) = \frac{1}{Z} \exp \left[ -\frac{1}{2\eta\Delta t} m_{AB} \left( \frac{\Delta x^A}{\Delta t} - v^A \right) \left( \frac{\Delta x^B}{\Delta t} - v^B \right) \right], \quad (18)$$

where we used (9) to define the drift velocity,

$$v^A = \frac{\langle \Delta x^A \rangle}{\Delta t} = m^{AB} [\partial_B \Phi - \bar{A}_B]. \quad (19)$$



The drift potential is rescaled into a new variable

$$\Phi = \hbar\phi \quad (20)$$

which will be called *the phase*. We also introduced the “mass” tensor and its inverse,

$$m_{AB} = m_n \delta_{AB} = m_n \delta_{nn'} \delta_{ab} \quad \text{and} \quad m^{AB} = \frac{1}{m_n} \delta^{AB}, \quad (21)$$

and  $\bar{A}_A$  is a field in configuration space with components,

$$\bar{A}_A(x) = \hbar\beta_n A_a(x_n), \quad (22)$$

A generic displacement is then written as a drift plus a fluctuation,

$$\Delta x^A = v^A \Delta t + \Delta w^A. \quad (23)$$

and the fluctuations  $\Delta w^A$  are given by

$$\langle \Delta w^A \rangle = 0 \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \eta m^{AB} \Delta t^3, \quad (24)$$

or

$$\left\langle \left( \frac{\Delta x^A}{\Delta t} - v^A \right) \left( \frac{\Delta x^B}{\Delta t} - v^B \right) \right\rangle = \eta m^{AB} \Delta t. \quad (25)$$

It is noteworthy that  $\langle \Delta x^A \rangle \sim O(\Delta t)$  and  $\Delta w^A \sim O(\Delta t^{3/2})$ . This means that for short steps the fluctuations are negligible and the dynamics is dominated by the drift. The particles follow trajectories that are indeterministic but differentiable. Since  $\Delta w^A \sim O(\Delta t^{3/2})$  the limit

$$V^A = \lim_{\Delta t \rightarrow 0} \frac{\Delta x^A}{\Delta t} = v^A. \quad (26)$$

233 is well defined. In words: the actual velocities of the particles coincide with the expected or drift  
 234 velocities. From eq.(19) we see that these velocities are continuous functions. The question of whether  
 235 the velocities themselves are differentiable or not is trickier.

Consider two successive displacements  $\Delta x = x' - x$  followed by  $\Delta x' = x'' - x'$ . The velocities are

$$V^A = \frac{\Delta x^A}{\Delta t} \quad \text{and} \quad V'^A = \frac{\Delta x'^A}{\Delta t} \quad (27)$$

The change in velocity is given by a Langevin equation,

$$\Delta V^A = \langle \Delta V^A \rangle_{x''x'} + \Delta U^A \quad (28)$$

where  $\langle \cdot \rangle_{x''x'}$  denotes taking the expectations over  $x''$  using  $P(x''|x')$ , and then over  $x'$  using  $P(x'|x)$ , and  $\Delta U^A$  is a fluctuation. It is straightforward to show that

$$\langle \Delta V^A \rangle_{x''x'} = (\partial_t v^A + v^B \partial_B v^A) \Delta t, \quad (29)$$

so that the expected acceleration is given by the convective derivative of the velocity field along itself,

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta V^A \rangle}{\Delta t} = (\partial_t + v^B \partial_B) v^A. \quad (30)$$

236 One can also show that

$$\langle \Delta U^A \rangle_{x''x'} = 0, \quad \text{and} \quad \langle \Delta U^A \Delta U^B \rangle_{x''x'} = 2\eta m^{AB} \Delta t, \quad (31)$$

237 which means that  $\Delta U$  is a Wiener process and we deal with a Brownian motion of the  
238 Ornstein-Uhlenbeck type.

239 We conclude this section with some general remarks.

240 **On the nature of clocks** — In Newtonian mechanics time is defined to simplify the dynamics.  
241 The prototype of a clock is a free particle which moves equal distances in equal times. In ED time  
242 is also defined to simplify the dynamics of free particles (for sufficiently short times all particles are  
243 free) and the prototype of a clock is a free particle too: as we see in (23) *the particle's mean displacement*  
244 *increases by equal amounts in equal times.*

245 **On the nature of mass** — In standard quantum mechanics, “what is mass?” and “why quantum  
246 fluctuations?” are two independent mysteries. In ED the mystery is somewhat alleviated: as we see in  
247 eq.(25) mass and fluctuations are two sides of the same coin. *Mass is an inverse measure of the velocity*  
248 *fluctuations.*

249 **The information metric of configuration space** — In addition to defining the dynamics the  
250 transition probability eq.(18) serves to define the geometry of the  $N$ -particle configuration space,  $\mathbf{X}_N$ .  
251 Since the physical single particle space  $\mathbf{X}$  is described by the Euclidean metric  $\delta_{ab}$  we can expect that  
252 the  $N$ -particle configuration space,  $\mathbf{X}_N = \mathbf{X} \times \dots \times \mathbf{X}$ , will also be flat, but for non-identical particles a  
253 question might be raised about the relative scales or weights associated to each  $\mathbf{X}$  factor. Information  
254 geometry provides the answer.

The fact that to each point  $x \in \mathbf{X}_N$  there corresponds a probability distribution  $P(x'|x)$  means that  
to the space  $\mathbf{X}_N$  we can associate a statistical manifold the geometry of which (up to an overall scale  
factor) is uniquely determined by the information metric [14][59],

$$\gamma_{AB} = \int dx' P(x'|x) \frac{\partial \log P(x'|x)}{\partial x^A} \frac{\partial \log P(x'|x)}{\partial x^B}. \quad (32)$$

Substituting eqs.(18) into (32) yields

$$\gamma_{AB} = \frac{1}{\eta \Delta t^3} m_{AB}. \quad (33)$$

255 The divergence as  $\Delta t \rightarrow 0$  arises because the information metric measures statistical distinguishability.  
256 As  $\Delta t \rightarrow 0$  the distributions  $P(x'|x)$  and  $P(x'|x + \Delta x)$  become more sharply peaked and increasingly  
257 easier to distinguish so that  $\gamma_{AB} \rightarrow \infty$ . Thus, up to a scale factor the metric of configuration space is  
258 basically the mass tensor.

259 The practice of describing a many-particle system as a single point in an abstract configuration  
260 space goes back to the work of H. Hertz in 1894 [75]. Historically the choice of the mass tensor as the  
261 metric of configuration space has been regarded as being convenient but of no particular significance.  
262 We can now see that the choice is not just a merely useful convention: up to an overall scale the metric  
263 follows uniquely from information geometry. Furthermore, it suggests the intriguing possibility of a  
264 deeper connection between kinetic energy and information geometry.

265 **Invariance under gauge transformations** — The fact that constraints (5) and (6) are not  
266 independent — they are both linear in the same displacements  $\langle \Delta x_n^a \rangle$  — leads to a gauge symmetry.

267 This is evident in eq.(7) where  $\phi$  and  $A_a$  appear in the combination  $\partial_{na}\phi - \beta_n A_a$  which is invariant  
 268 under the gauge transformations,

$$A_a(x_n) \rightarrow A'_a(x_n) = A_a(x_n) + \partial_a \chi(x_n), \quad (34)$$

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \sum_n \beta_n \chi(x_n). \quad (35)$$

These transformations are local in 3d-space. Introducing

$$\bar{\chi}(x) = \sum_n \hbar \beta_n \chi(x_n), \quad (36)$$

269 they can be written in the  $N$ -particle configuration space,

$$\bar{A}_A(x) \rightarrow \bar{A}'_A(x) = \bar{A}_A(x) + \partial_A \bar{\chi}(x), \quad (37)$$

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \bar{\chi}(x). \quad (38)$$

270 **Interpretation:** The drift potential  $\phi(x) = \phi(\vec{x}_1, \vec{x}_2, \dots)$  is assumed to be an “angle” —  $\phi(x)$  and  
 271  $\phi(x) + 2\pi$  are meant to describe the same angle. The angle at  $\vec{x}_1$  depends on the values of all the other  
 272 positions  $\vec{x}_2, \vec{x}_3, \dots$ , and the angle at  $\vec{x}_2$  depends on the values of all the other positions  $\vec{x}_1, \vec{x}_3, \dots$ , and  
 273 so on. The fact that the origins from which these angles are measured can be redefined by different  
 274 amounts at different places gives rise to a local gauge symmetry. In order to compare angles at different  
 275 locations one introduces a *connection* field, the vector potential  $A_a(\vec{x})$ . It defines which origin at  $\vec{x} + \Delta\vec{x}$   
 276 is the “same” as the origin at  $\vec{x}$ . This is implemented by imposing that as we change origins and  $\Phi(x)$   
 277 changes to  $\Phi + \bar{\chi}$  then the connection transforms as  $A_a \rightarrow A_a + \partial_a \chi$  so that the quantity  $\partial_A \Phi - \bar{A}_A$   
 278 remains invariant.

**A fractional Brownian motion?** — The choices  $\alpha_n \propto 1/\Delta t$  and  $\alpha_n \propto 1/\Delta t^3$  lead to Einstein-Smoluchowski and Ornstein-Uhlenbeck processes respectively. For definiteness throughout the rest of this paper we will assume that the sub-quantum motion is an OU process but more general fractional Brownian motions [74] are possible. Consider

$$\alpha_n = \frac{m_n}{\eta \Delta t^\gamma}, \quad (39)$$

where  $\gamma$  is some positive parameter. The corresponding transition probability (8),

$$P(x'|x) = \frac{1}{Z} \exp \left[ -\frac{1}{2\eta \Delta t^\gamma} m_{AB} \left( \Delta x^A - v^A \Delta t \right) \left( \Delta x^B - v^B \Delta t \right) \right], \quad (40)$$

leads to fluctuations such that

$$\langle \Delta w^A \rangle = 0 \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \eta m^{AB} \Delta t^\gamma, \quad (41)$$

or

$$\left\langle \left( \frac{\Delta x^A}{\Delta t} - v^A \right) \left( \frac{\Delta x^B}{\Delta t} - v^B \right) \right\rangle = \eta m^{AB} \Delta t^{\gamma-2}. \quad (42)$$

279 We will not pursue this topic further except to note that since  $\langle \Delta x^A \rangle \sim O(\Delta t)$  and  $\Delta w^A \sim O(\Delta t^{\gamma/2})$  for  
 280  $\gamma < 2$  the sub-quantum motion is dominated by fluctuations and the trajectories are non-differentiable,  
 281 while for  $\gamma > 2$  the drift dominates and velocities are well defined.

#### 282 4. The evolution equation in differential form

Entropic dynamics is generated by iterating eq.(13): given the information that defines one instant, the integral eq.(13) is used to construct the next instant. As so often in physics it is more convenient to rewrite the equation of evolution in differential form. The result is

$$\partial_t \rho = -\partial_A (v^A \rho) , \quad (43)$$

283 where  $v^A$  is given by (19). Before we proceed to its derivation we note that eq.(43) is a consequence  
 284 of the fact that the particles follow continuous paths. Accordingly, we will follow standard practice  
 285 and call it *the continuity equation*. Also note that in the OU process considered here ( $\gamma = 3$ ) the current  
 286 velocity — the velocity with which the probability flows in configuration space — coincides with the  
 287 drift velocity (19) and with the actual velocities of the particles (26).<sup>11</sup>

The derivation of (43) is obtained following a technique that is well known in diffusion theory [76]. (For an alternative derivation see [77].) The result of building up a *finite* change from an initial time  $t_0$  to a later time  $t$  leads to the distribution

$$\rho(x, t) = \int dx_0 P(x, t|x_0, t_0) \rho(x_0, t_0) , \quad (44)$$

where the finite-time transition probability,  $P(x, t|x_0, t_0)$ , is constructed by iterating the infinitesimal changes described in eq.(13),

$$P(x, t + \Delta t|x_0, t_0) = \int dz P(x, t + \Delta t|z, t) P(z, t|x_0, t_0) . \quad (45)$$

For small times  $\Delta t$  the distribution  $P(x, t + \Delta t|z, t)$ , given in eq. (18), is very sharply peaked at  $x = z$ . In fact, as  $\Delta t \rightarrow 0$  we have  $P(x, t + \Delta t|z, t) \rightarrow \delta(x - z)$ . Such singular behavior cannot be handled directly by Taylor expanding in  $z$  about the point  $x$ . Instead one follows an indirect procedure. Multiply by a smooth test function  $f(x)$  and integrate over  $x$ ,

$$\int dx P(x, t + \Delta t|x_0, t_0) f(x) = \int dz \left[ \int dx P(x, t + \Delta t|z, t) f(x) \right] P(z, t|x_0, t_0) . \quad (46)$$

The test function  $f(x)$  is assumed sufficiently smooth precisely so that it can be expanded about  $z$ . Then as  $\Delta t \rightarrow 0$  the integral in the brackets, dropping all terms of order higher than  $\Delta t$ , is

$$\begin{aligned} [\dots] &= \int dx P(x, t + \Delta t|z, t) \left( f(z) + \frac{\partial f}{\partial z^A} (x^A - z^A) + \dots \right) \\ &= f(z) + v^A(z) \Delta t \frac{\partial f}{\partial z^A} + \dots \end{aligned} \quad (47)$$

where we used eq.(23). Next substitute (47) into the right hand side of (46), divide by  $\Delta t$ , and let  $\Delta t \rightarrow 0$ . Since  $f(x)$  is arbitrary the result is

$$\partial_t P(x, t|x_0, t_0) = -\partial_A [v^A(x) P(x, t|x_0, t_0)] , \quad (48)$$

<sup>11</sup> In the ES type of ED considered in previous papers ( $\gamma = 1$ ) [8]-[10] the probability also satisfies a continuity equation — a Fokker-Planck equation — and the current velocity is the sum of the drift velocity plus an osmotic component

$$u^A = -\hbar m^{AB} \partial_B \log \rho^{1/2}$$

due to diffusion.

288 which is the continuity equation for the finite-time transition probability. Differentiating eq.(44) with  
 289 respect to  $t$ , and substituting (48) completes the derivation of the continuity equation (43).

The continuity equation (43) can be written in another equivalent but very suggestive form involving functional derivatives. For some suitably chosen functional  $\tilde{H}[\rho, \Phi]$  we have

$$\partial_t \rho(x) = -\partial_A \left[ \rho m^{AB} (\partial_B \Phi - \bar{A}_B) \right] = \frac{\delta \tilde{H}}{\delta \Phi(x)}. \quad (49)$$

It is easy to check that the appropriate functional  $\tilde{H}$  is

$$\tilde{H}[\rho, \Phi] = \int dx \frac{1}{2} \rho m^{AB} (\partial_A \Phi - \bar{A}_A) (\partial_B \Phi - \bar{A}_B) + F[\rho], \quad (50)$$

290 where the unspecified functional  $F[\rho]$  is an integration constant.<sup>12</sup>

291 The continuity equation (49) describes a somewhat peculiar OU Brownian motion in which the  
 292 probability density  $\rho(x)$  is driven by the non-dynamical fields  $\Phi$ , and  $\bar{A}$ . This is an interesting ED  
 293 in its own right but it is not QM. Indeed, a *quantum* dynamics consists in the coupled evolution of  
 294 two dynamical fields: the density  $\rho(x)$  and the phase of the wave function. This second field can  
 295 be naturally introduced into ED by allowing the phase field  $\Phi$  in (19) to become dynamical which  
 296 amounts to an ED in which the constraint (5) is continuously updated at each instant in time. Our  
 297 next topic is to propose the appropriate updating criterion. It yields an ED in which the phase field  $\Phi$   
 298 guides the evolution of  $\rho$ , and in return, the evolving  $\rho$  reacts back and induces a change in  $\Phi$ .

## 299 5. The epistemic phase space

In ED we deal with two configuration spaces. One is the *ontic configuration space*  $\mathbf{X}_N = \mathbf{X} \times \mathbf{X} \times \dots$  of all particle positions,  $x = (x_1 \dots x_N) \in \mathbf{X}_N$ . The other is the *epistemic configuration space* or *e-configuration space*  $\mathbf{P}$  of all normalized probabilities,

$$\mathbf{P} = \left\{ \rho \mid \rho(x) \geq 0; \int dx \rho(x) = 1 \right\}. \quad (51)$$

300 To formulate the coupled dynamics of  $\rho$  and  $\Phi$  we need a framework to study paths in the larger space  
 301  $\{\rho, \Phi\}$  that we will call the *epistemic phase space* or *e-phase space*.

302 Given any manifold such as  $\mathbf{P}$  the associated tangent and cotangent bundles, respectively  $T\mathbf{P}$  and  
 303  $T^*\mathbf{P}$ , are geometric objects that are always available to us independently of any physical considerations.  
 304 Both are manifolds in their own right but the cotangent bundle  $T^*\mathbf{P}$  — the space of all probabilities and  
 305 all covectors — is of particular interest because it comes automatically endowed with a rich geometrical  
 306 structure [50]-[56].<sup>13</sup> The point is that cotangent bundles are symplectic manifolds and this singles  
 307 out as “natural” those dynamical laws that happen to preserve some privileged symplectic form. This  
 308 observation will lead us to identify e-phase space  $\{\rho, \Phi\}$  with the cotangent bundle  $T^*\mathbf{P}$  and provides  
 309 the natural criterion for updating constraints, that is, for updating the phase  $\Phi$ .

### 310 5.1. Notation: vectors, covectors, etc.

A point  $X \in T^*\mathbf{P}$  will be represented as

$$X = (\rho(x), \pi(x)) = (\rho^x, \pi_x), \quad (52)$$

<sup>12</sup> Eqs.(49) and (50) show the reason to have introduced the new variable  $\Phi = \hbar\phi$ . With this choice  $\Phi$  will eventually be recognized as the momentum that is canonically conjugate to the generalized coordinate  $\rho$  with Hamiltonian  $\tilde{H}$ .

<sup>13</sup> We deal with  $\infty$ -dimensional spaces. The level of mathematical rigor in what follows is typical of theoretical physics — which is a euphemism for “from very low to none at all.” For a more sophisticated treatment see [53][56].

where  $\rho^x$  represents coordinates on the base manifold  $\mathbf{P}$  and  $\pi_x$  represents some generic coordinates on the space  $T^*\mathbf{P}_\rho$  that is cotangent to  $\mathbf{P}$  at the point  $\rho$ . Curves in  $T^*\mathbf{P}$  allow us to define vectors. Let  $X = X(\lambda)$  be a curve parametrized by  $\lambda$ , then the vector  $\bar{V}$  tangent to the curve at  $X = (\rho, \pi)$  has components  $d\rho^x/d\lambda$  and  $d\pi_x/d\lambda$ , and is written

$$\bar{V} = \frac{d}{d\lambda} = \int dx \left[ \frac{d\rho^x}{d\lambda} \frac{\delta}{\delta\rho^x} + \frac{d\pi_x}{d\lambda} \frac{\delta}{\delta\pi_x} \right], \quad (53)$$

where  $\delta/\delta\rho^x$  and  $\delta/\delta\pi_x$  are the basis vectors. The directional derivative of a functional  $F[X]$  along the curve  $X(\lambda)$  is

$$\frac{dF}{d\lambda} = \tilde{\nabla}F[\bar{V}] = \int dx \left[ \frac{\delta F}{\delta\rho^x} \frac{d\rho^x}{d\lambda} + \frac{\delta F}{\delta\pi_x} \frac{d\pi_x}{d\lambda} \right], \quad (54)$$

where  $\tilde{\nabla}$  is the functional gradient in  $T^*\mathbf{P}$ , that is, the gradient of a generic functional  $F[X] = F[\rho, \pi]$  is

$$\tilde{\nabla}F = \int dx \left[ \frac{\delta F}{\delta\rho^x} \tilde{\nabla}\rho^x + \frac{\delta F}{\delta\pi_x} \tilde{\nabla}\pi_x \right]. \quad (55)$$

311 The tilde ‘ $\tilde{\cdot}$ ’ serves to distinguish the functional gradient  $\tilde{\nabla}$  from the spatial gradient  $\nabla f = \partial_a f \nabla x^a$  on  
312  $\mathbf{R}^3$ .

The fact that the space  $\mathbf{P}$  is constrained to normalized probabilities means that the coordinates  $\rho^x$  are not independent. This technical difficulty is handled by embedding the  $\infty$ -dimensional manifold  $\mathbf{P}$  in a  $(\infty + 1)$ -dimensional manifold  $\mathbf{P}^{+1}$  where the coordinates  $\rho^x$  are unconstrained.<sup>14</sup> Thus, strictly,  $\tilde{\nabla}F$  is a covector on  $T^*\mathbf{P}^{+1}$ , that is,  $\tilde{\nabla}F \in T^*(T^*\mathbf{P}^{+1})_X$  and  $\tilde{\nabla}\rho^x$  and  $\tilde{\nabla}\pi_x$  are the corresponding basis covectors. Nevertheless, the gradient  $\tilde{\nabla}F$  will yield the desired directional derivatives (54) on  $T^*\mathbf{P}$  provided its action is restricted to vectors  $\bar{V}$  that are tangent to the manifold  $\mathbf{P}$ . Such tangent vectors are constrained to obey

$$\frac{d}{d\lambda} \int dx \rho(x) = \int dx \frac{d\rho^x}{d\lambda} = 0. \quad (56)$$

Instead of keeping separate track of the  $\rho^x$  and  $\pi_x$  coordinates it is more convenient to combine them into a single index. A point  $X = (\rho, \pi)$  will then be labelled by its coordinates

$$X^I = (X^{1x}, X^{2x}) = (\rho^x, \pi_x). \quad (57)$$

We will use capital letters from the middle of the Latin alphabet ( $I, J, K \dots$ );  $I = (\alpha, x)$  is a composite index where  $\alpha = 1, 2$  keeps track of whether  $x$  is an upper index ( $\alpha = 1$ ) or a lower index ( $\alpha = 2$ ).<sup>15</sup> Then eqs.(53-55) are written as

$$\bar{V} = V^I \frac{\delta}{\delta X^I}, \quad \text{where} \quad V^I = \frac{dX^I}{d\lambda} = \begin{bmatrix} d\rho^x/d\lambda \\ d\pi_x/d\lambda \end{bmatrix}, \quad (58)$$

$$\frac{dF}{d\lambda} = \tilde{\nabla}F[\bar{V}] = \frac{\delta F}{\delta X^I} V^I \quad \text{and} \quad \tilde{\nabla}F = \frac{\delta F}{\delta X^I} \tilde{\nabla}X^I, \quad (59)$$

313 where the repeated indices indicate a summation over  $\alpha$  and an integration over  $x$ .

## 314 5.2. The symplectic form in ED

315 In classical mechanics with configuration space  $\{q^i\}$  the Lagrangian  $L(q, \dot{q})$  is a function on the  
316 tangent bundle while the Hamiltonian  $H(q, p)$  is a function on the cotangent bundle [78][79]. A

<sup>14</sup> At this point the act of embedding  $\mathbf{P}$  into  $\mathbf{P}^{+1}$  represents no loss of generality because the embedding space  $\mathbf{P}^{+1}$  remains unspecified.

<sup>15</sup> This allows us, among other things, the freedom to switch from  $\rho^x$  to  $\rho_x$  as convenience dictates; from now on  $\rho_x = \rho^x = \rho(x)$ .

317 symplectic form provides a mapping from the tangent to the cotangent bundles. Given a Lagrangian  
 318 the map is defined by  $p_i = \partial L / \partial \dot{q}^i$  and this automatically defines the corresponding symplectic form.  
 319 In ED there is no Lagrangian so in order to define the symplectic map we must look elsewhere. We  
 320 propose that the role played by the Lagrangian in classical mechanics will in ED be played by the  
 321 continuity equation (49).

The fact that the preservation of a symplectic structure must reproduce the continuity equation leads us to identify the phase  $\Phi_x$  as the momentum canonically conjugate to  $\rho^x$ . This identification of the e-phase space  $\{\rho, \Phi\}$  with  $T^*\mathbf{P}$  is highly non-trivial. It amounts to asserting that the phase  $\Phi_x$  transforms as the components of a Poincare 1-form

$$\theta = \int dx \Phi_x d\rho^x, \quad (60)$$

(where  $d$  is the exterior derivative) and the corresponding symplectic 2-form  $\Omega = -d\theta$  is

$$\Omega = \int dx d\rho^x \wedge d\Phi_x = \int dx [\tilde{\nabla}\rho^x \otimes \tilde{\nabla}\Phi_x - \tilde{\nabla}\Phi_x \otimes \tilde{\nabla}\rho^x]. \quad (61)$$

By construction  $\Omega$  is exact ( $\Omega = -d\theta$ ) and closed ( $d\Omega = 0$ ). The action of  $\Omega[\cdot, \cdot]$  on two vectors  $\tilde{V} = d/d\lambda$  and  $\tilde{U} = d/d\mu$  is given by

$$\Omega[\tilde{V}, \tilde{U}] = \int dx [V^{1x}U^{2x} - V^{2x}U^{1x}] = \Omega_{IJ}V^I U^J, \quad (62)$$

so that the components of  $\Omega$  are

$$\Omega_{IJ} = \Omega_{\alpha x, \beta x'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta(x, x'). \quad (63)$$

### 322 5.3. Hamiltonian flows and Poisson brackets

Next we reproduce the  $\infty$ -dimensional  $T^*\mathbf{P}$  analogues of results that are standard in finite-dimensional classical mechanics [78][79]. Given a vector field  $\tilde{V}[X]$  in e-phase space we can integrate  $V^I[X] = dX^I/d\lambda$  to find its integral curves  $X^I = X^I(\lambda)$ . We are particularly interested in those vector fields that generate flows that preserve the symplectic structure,

$$\mathcal{L}_V \Omega = 0, \quad (64)$$

where the Lie derivative is given by

$$(\mathcal{L}_V \Omega)_{IJ} = V^K \tilde{\nabla}_K \Omega_{IJ} + \Omega_{KJ} \tilde{\nabla}_I V^K + \Omega_{IK} \tilde{\nabla}_J V^K. \quad (65)$$

Since by eq.(63) the components  $\Omega_{IJ}$  are constant,  $\tilde{\nabla}_K \Omega_{IJ} = 0$ , we can rewrite  $\mathcal{L}_V \Omega$  as

$$(\mathcal{L}_V \Omega)_{IJ} = \tilde{\nabla}_I (\Omega_{KJ} V^K) - \tilde{\nabla}_J (\Omega_{KI} V^K), \quad (66)$$

which is the exterior derivative (basically, the curl) of the covector  $\Omega_{KI} V^K$ . By Poincare's lemma, requiring  $\mathcal{L}_V \Omega = 0$  (a vanishing curl) implies that  $\Omega_{KI} V^K$  is the gradient of a scalar function, which we will denote  $\tilde{V}[X]$ ,

$$\Omega_{KI} V^K = \tilde{\nabla}_I \tilde{V}. \quad (67)$$

Using (63) this is more explicitly written as

$$\int dx \left[ \frac{d\rho^x}{d\lambda} \tilde{\nabla} \Phi_x - \frac{d\Phi_x}{d\lambda} \tilde{\nabla} \rho^x \right] = \int dx \left[ \frac{\delta \tilde{V}}{\delta \rho^x} \tilde{\nabla} \rho^x + \frac{\delta \tilde{V}}{\delta \Phi_x} \tilde{\nabla} \Phi_x \right], \quad (68)$$

or

$$\frac{d\rho^x}{d\lambda} = \frac{\delta\tilde{V}}{\delta\Phi_x} \quad \text{and} \quad \frac{d\Phi_x}{d\lambda} = -\frac{\delta\tilde{V}}{\delta\rho^x}, \quad (69)$$

323 which we recognize as Hamilton's equations for a Hamiltonian function  $\tilde{V}$ . This justifies calling  $\tilde{V}$  the  
324 Hamiltonian vector field associated to the Hamiltonian function  $\tilde{V}$ .

From (62), the action of the symplectic form  $\Omega$  on two Hamiltonian vector fields  $\tilde{V} = d/d\lambda$  and  $\tilde{U} = d/d\mu$  generated respectively by  $\tilde{V}$  and  $\tilde{U}$  is

$$\Omega[\tilde{V}, \tilde{U}] = \int dx \left[ \frac{d\rho^x}{d\lambda} \frac{d\Phi_x}{d\mu} - \frac{d\Phi_x}{d\lambda} \frac{d\rho^x}{d\mu} \right], \quad (70)$$

which, using (69), gives

$$\Omega[\tilde{V}, \tilde{U}] = \int dx \left[ \frac{\delta\tilde{V}}{\delta\rho^x} \frac{\delta\tilde{U}}{\delta\Phi_x} - \frac{\delta\tilde{V}}{\delta\Phi_x} \frac{\delta\tilde{U}}{\delta\rho^x} \right] \stackrel{\text{def}}{=} \{\tilde{V}, \tilde{U}\}, \quad (71)$$

325 where on the right we introduced the Poisson bracket notation.

To summarize these results: (1) The condition for a flow generated by the vector field  $V^I$  to preserve the symplectic structure,  $\mathcal{L}_V \Omega = 0$ , is that  $V^I$  be the Hamiltonian vector field associated to a Hamiltonian function  $\tilde{V}$ , eq.(69),

$$V^I = \frac{dX^I}{d\lambda} = \{X^I, \tilde{V}\}. \quad (72)$$

(2) The action of  $\Omega$  on two Hamiltonian vector fields (71) is the Poisson bracket of the associated Hamiltonian functions,

$$\Omega[\tilde{V}, \tilde{U}] = \Omega_{IJ} V^I U^J = \{\tilde{V}, \tilde{U}\}. \quad (73)$$

We conclude that the ED that preserves the symplectic structure  $\Omega$  and reproduces the continuity equation (49) is described by the Hamiltonian flow of the scalar functional  $\tilde{H}$  in (50). However, the full dynamics, which will obey the Hamiltonian evolution equations

$$\partial_t \rho^x = \frac{\delta\tilde{H}}{\delta\Phi_x} \quad \text{and} \quad \partial_t \Phi_x = -\frac{\delta\tilde{H}}{\delta\rho^x}, \quad (74)$$

326 is not yet fully determined because the integration constant  $F[\rho]$  in (50) remains to be specified.

#### 327 5.4. The normalization constraint

Since the particular flow that we will associate with time evolution is required to reproduce the continuity equation it will also preserve the normalization constraint,

$$\tilde{N} = 0 \quad \text{where} \quad \tilde{N} = 1 - |\rho| \quad \text{and} \quad |\rho| \stackrel{\text{def}}{=} \int dx \rho(x). \quad (75)$$

Indeed, one can check that

$$\partial_t \tilde{N} = \{\tilde{N}, \tilde{H}\} = 0. \quad (76)$$

The Hamiltonian flow (72) generated by  $\tilde{N}$  and parametrized by  $\alpha$  is given by the vector field

$$\tilde{N} = N^I \frac{\delta}{\delta X^I} \quad \text{with} \quad N^I = \frac{dX^I}{d\alpha} = \{X^I, \tilde{N}\}, \quad (77)$$

or, more explicitly,

$$N^{1x} = \frac{d\rho^x}{d\alpha} = 0 \quad \text{and} \quad N^{2x} = \frac{d\Phi_x}{d\alpha} = 1. \quad (78)$$



The conservation of  $\tilde{N}$ , eq.(76), implies that  $\tilde{N}$  is the generator of a symmetry, namely,

$$\frac{d\tilde{H}}{d\alpha} = \{\tilde{H}, \tilde{N}\} = 0. \quad (79)$$

Integrating (78) one finds the integral curves generated by  $\tilde{N}$ ,

$$\rho^x(\alpha) = \rho^x(0) \quad \text{and} \quad \Phi_x(\alpha) = \Phi_x(0) + \alpha. \quad (80)$$

328 This shows that the symmetry generated by  $\tilde{N}$  is to shift the phase  $\Phi$  by a constant  $\alpha$  without otherwise  
 329 changing the dynamics. This was, of course, already evident in the continuity equation (43) with  
 330 (19) but the implications are very significant. Not only does the constraint  $\tilde{N} = 0$  reduce by one the  
 331 (infinite) number of independent  $\rho^x$  degrees of freedom but the actual number of  $\Phi_x$ s is also reduced  
 332 by one because for any value of  $\alpha$  the phases  $\Phi_x + \alpha$  and  $\Phi_x$  correspond to the same state. (This is  
 333 the ED analogue of the fact that in QM states are represented by rays rather than vectors in a Hilbert  
 334 space.)

The immediate consequence is that two vectors  $\tilde{U}$  and  $\tilde{V}$  at  $X$  that differ by a vector proportional to  $\tilde{N}$ ,

$$\tilde{U} = \tilde{V} + k\tilde{N}, \quad (81)$$

335 are “physically” equivalent. In particular the vector  $\tilde{N}$  is equivalent to zero.

336 The phase space of interest is  $T^*\mathbf{P}$  but to handle the constraint  $|\rho| = 1$  we have been led to using  
 337 coordinates that are more appropriate to the larger embedding space  $T^*\mathbf{P}^{+1}$ . The price we pay for  
 338 introducing one superfluous coordinate is to also introduce a superfluous momentum. We eliminate  
 339 the extra coordinate by imposing the constraint  $\tilde{N} = 0$ . We eliminate the extra momentum by declaring  
 340 it unphysical. All vectors that differ by a vector along the gauge direction  $\tilde{N}$  are declared equivalent;  
 341 they belong to the same equivalence class. The result is a global gauge symmetry.

An equivalence class can be represented by any one of its members and choosing a convenient representative amounts to fixing the gauge. As we shall see below a convenient gauge condition is to impose

$$\int dx \rho^x V^{2x} = 0 \quad \text{or} \quad \langle V^2 \rangle = 0, \quad (82)$$

so that the representative “Tangent Gauge-Fixed” vectors (which we shall refer to as TGF vectors) will satisfy two conditions, eqs.(56) and (82),

$$|V^1| = \int dx V^{1x} = 0 \quad \text{and} \quad \langle V^2 \rangle = \int dx \rho^x V^{2x} = 0. \quad (83)$$

342 The first condition enforces a flow tangent to the  $|\rho| = 1$  surface; the second eliminates a superfluous  
 343 vector component along the gauge direction  $\tilde{N}$ .

344 We end this section with a comment on the symplectic form  $\Omega$  which is non-degenerate on  $T^*\mathbf{P}^{+1}$   
 345 but at first sight appears to be degenerate on  $T^*\mathbf{P}$ . Indeed, we have  $\Omega(\tilde{N}, \tilde{V}) = 0$  for any tangent vector  
 346  $\tilde{V}$ . However, we must recall that  $\tilde{N}$  is equivalent to 0. In fact, since the TGF equivalent of  $\tilde{N}$  is 0,  $\Omega$  is  
 347 not degenerate on  $T^*\mathbf{P}$ .

## 348 6. The information geometry of e-phase space

349 The construction of the ensemble Hamiltonian  $\tilde{H}$  — or *e-Hamiltonian* — is motivated as follows.  
 350 The goal of dynamics is to determine the evolution of the state  $(\rho_t, \Phi_t)$ . From a given initial state  
 351  $(\rho_0, \Phi_0)$  two slightly different Hamiltonians will lead to slightly different final states, say  $(\rho_t, \Phi_t)$  or  
 352  $(\rho_t + \delta\rho_t, \Phi_t + \delta\Phi_t)$ . Will these small changes make any difference? Can we quantify the extent to  
 353 which we can *distinguish* between two neighboring states? This is precisely the kind of question that  
 354 metrics are designed to address. It is then natural that  $\tilde{H}$  be in some way related to some choice of

metric. But although  $\mathbf{P}$  is naturally endowed with a unique information metric the space  $T^*\mathbf{P}$  has none. Thus, our next goal is to construct a metric for  $T^*\mathbf{P}$ .

Once a metric structure is in place we can ask: does the distance between two neighboring states — the extent to which we can *distinguish* them — grow, stay the same, or diminish over time? There are many possibilities here but for pragmatic (and esthetic) reasons we are led to consider the simplest form of dynamics — one that preserves the metric. This leads us to study the Hamilton flows (those that preserve the symplectic structure) that are also Killing flows (those flows that preserve the metric structure).

In ED entropic time is constructed so that time (duration) is defined by a clock provided by the system itself. This leads to require that the generator  $\tilde{H}$  of time translations be defined in terms of the very same clock that provides the measure of time. Thus, the third and final ingredient in the construction of  $\tilde{H}$  is the requirement is that the e-Hamiltonian agree with (50) in order to reproduce the evolution of  $\rho$  given by the continuity equation (49).

In this section our goal is to transform e-phase space  $T^*\mathbf{P}$  from a manifold that is merely symplectic to a manifold that is both symplectic and Riemannian. The implementation of the other two requirements on  $\tilde{H}$  — that it generates a Hamilton-Killing flow and that it agrees with the ED continuity equation — will be tackled in sections 7 and 8.

### 6.1. The metric on the embedding space $T^*\mathbf{P}^{+1}$

The configuration space  $\mathbf{P}$  is a metric space. Our goal here is to extend its metric — given by information geometry — to the full cotangent bundle,  $T^*\mathbf{P}$ . It is convenient to first recall one derivation of the information metric. In the discrete case the statistical manifold is the  $k$ -simplex  $\Sigma = \{p = (p^0 \dots p^k) : \sum_{i=0}^k p^i = 1\}$ . The basic idea is to find the most general metric consistent with a certain symmetry requirement. To suggest what that symmetry might be we change to new coordinates  $\zeta^i = (p^i)^{1/2}$ . In these new coordinates the equation for the  $k$ -simplex  $\Sigma$  — the normalization condition — reads  $\sum_{i=0}^k (\zeta^i)^2 = 1$  which suggests the equation of a sphere.

We take this hint seriously and *declare* that the  $k$ -simplex is a  $k$ -sphere embedded in a generic  $(k+1)$ -dimensional spherically symmetric space  $\Sigma^{+1}$ .<sup>16</sup> In the  $\zeta^i$  coordinates the metric of  $\Sigma^{+1}$  is of the form

$$d\ell^2 = [a(|p|) - b(|p|)] \left( \sum_{i=0}^k \zeta^i d\zeta^i \right)^2 + |p|b(|p|) \sum_{i=0}^k (d\zeta^i)^2, \quad (84)$$

where  $a(|p|)$  and  $b(|p|)$  are two arbitrary smooth and positive functions of  $|p| = \sum_{i=0}^k p^i$ . Expressed in terms of the original  $p^i$  coordinates the metric of  $\Sigma^{+1}$  is

$$d\ell^2 = [a(|p|) - b(|p|)] \left( \sum_{i=0}^k dp^i \right)^2 + |p|b(|p|) \sum_{i=0}^k \frac{1}{p^i} (dp^i)^2. \quad (85)$$

The restriction to normalized states,  $|p| = 1$  with displacements tangent to the simplex,  $\sum_{i=0}^k dp^i = 0$ , gives the information metric induced on the  $k$ -simplex  $\Sigma$ ,

$$d\ell^2 = b(1) \sum_{i=0}^k \frac{1}{p^i} (dp^i)^2. \quad (86)$$

<sup>16</sup> We are effectively determining the metric by imposing a symmetry, namely, rotational invariance. One might be concerned that choosing this symmetry is an ad hoc assumption but the result proves to be very robust. It turns out that exactly the same metric is obtained by several other criteria that may appear more natural in the context of inference and probability. Such criteria include invariance under Markovian embeddings, the geometry of asymptotic inference, and the metrics induced by relative entropy [80][81] (see also [14]).

380 The overall constant  $b(1)$  is not important; it amounts to a choice of the units of distance.

381 To extend the information metric from the  $k$ -simplex  $\Sigma$  to its cotangent bundle  $T^*\Sigma$  we focus on  
382 the embedding spaces  $\Sigma^{+1}$  and  $T^*\Sigma^{+1}$  and require that

- 383 **(a)** the metric on  $T^*\Sigma^{+1}$  be compatible with the metric on  $\Sigma^{+1}$ ; and  
384 **(b)** that the spherical symmetry of the  $(k+1)$ -dimensional space  $\Sigma^{+1}$  be enlarged to full spherical  
385 symmetry for the  $2(k+1)$ -dimensional space  $T^*\Sigma^{+1}$ .

The simplest way to implement **(a)** is to follow as closely as possible the derivation that led to (85). The fact that  $\Phi$  inherits from the drift potential  $\phi$  the topological structure of an angle suggests introducing new coordinates,

$$\xi^i = (p^i)^{1/2} \cos \Phi_i / \hbar \quad \text{and} \quad \eta^i = (p^i)^{1/2} \sin \Phi_i / \hbar. \quad (87)$$

Then the normalization condition reads

$$|p| = \sum_{i=0}^k p^i = \sum_{i=0}^k [(\xi^i)^2 + (\eta^i)^2] = 1 \quad (88)$$

386 which suggests the equation of a  $(2k+1)$ -sphere embedded in  $2(k+1)$  dimensions. To implement **(b)**  
387 we take this spherical symmetry seriously. The most general metric in the embedding space that is  
388 invariant under rotations is

$$\begin{aligned} d\ell^2 = & [a(|p|) - b(|p|)] \left[ \sum_{i=0}^k (\xi^i d\xi^i + \eta^i d\eta^i) \right]^2 \\ & + |p| b(|p|) \sum_{i=0}^k [(d\xi^i)^2 + (d\eta^i)^2], \end{aligned} \quad (89)$$

389 where the two functions  $a(|p|)$  and  $b(|p|)$  are smooth and positive but otherwise arbitrary. Therefore,  
390 changing back to the  $(p^i, \Phi_i)$  coordinates, the most general rotationally invariant metric for the  
391 embedding space  $T^*\Sigma^{+1}$  is

$$\begin{aligned} d\ell^2 = & \frac{1}{4} [a(|p|) - b(|p|)] \left[ \sum_{i=0}^k dp^i \right]^2 \\ & + |p| b(|p|) \frac{1}{2\hbar} \sum_{i=0}^k \left[ \frac{\hbar}{2p^i} (dp^i)^2 + \frac{2p^i}{\hbar} (d\Phi_i)^2 \right]. \end{aligned} \quad (90)$$

Generalizing from the finite-dimensional case to the  $\infty$ -dimensional case yields the metric on the spherically symmetric space  $T^*\mathbf{P}^{+1}$ ,

$$\delta\tilde{\ell}^2 = A \left[ \int dx \delta\rho_x \right]^2 + B \int dx \left[ \frac{\hbar}{2\rho_x} (\delta\rho_x)^2 + \frac{2\rho_x}{\hbar} (\delta\Phi_x)^2 \right]. \quad (91)$$

where we set

$$A(|\rho|) = \frac{1}{4} [a(|\rho|) - b(|\rho|)] \quad \text{and} \quad B(|\rho|) = \frac{1}{2\hbar} |\rho| b(|\rho|). \quad (92)$$

## 392 6.2. The metric induced on $T^*\mathbf{P}$

393 As we saw in section 5.4 the normalization constraint  $|\rho| = 1$  induces a symmetry — points with  
394 phases differing by a constant are identified. Therefore the e-phase space  $T^*\mathbf{P}$  can be obtained from  
395 the spherically symmetric space  $T^*\mathbf{P}^{+1}$  by the restriction  $|\rho| = 1$  and by identifying points  $(\rho_x, \Phi_x)$   
396 and  $(\rho_x, \Phi_x + \alpha)$  that lie on the same gauge orbit, or on the same ray.

Consider two neighboring points  $(\rho_x, \Phi_x)$  and  $(\rho'_x, \Phi'_x)$ . The metric induced on  $T^*\mathbf{P}$  is defined as the shortest  $T^*\mathbf{P}^{+1}$  distance between  $(\rho_x, \Phi_x)$  and points on the ray defined by  $(\rho'_x, \Phi'_x)$ . Setting  $|\delta\rho| = 0$  the  $T^*\mathbf{P}^{+1}$  distance between  $(\rho_x, \Phi_x)$  and  $(\rho_x + \delta\rho_x, \Phi_x + \delta\Phi_x + \delta\alpha)$  is given by

$$\delta\tilde{\ell}^2 = B(1) \int dx \left[ \frac{\hbar}{2\rho_x} (\delta\rho_x)^2 + \frac{2\rho_x}{\hbar} (\delta\Phi_x + \delta\alpha)^2 \right]. \quad (93)$$

Let

$$\delta\tilde{s}^2 = \min_{\delta\alpha} \delta\tilde{\ell}^2. \quad (94)$$

Minimizing over  $\delta\alpha$  gives the metric on  $T^*\mathbf{P}$ ,

$$\delta\tilde{s}^2 = \int dx \left[ \frac{\hbar}{2\rho_x} (\delta\rho_x)^2 + \frac{2\rho_x}{\hbar} (\delta\Phi_x - \langle\delta\Phi\rangle)^2 \right], \quad (95)$$

397 where we set  $B(1) = 1$  which amounts to a choice of units of length. This metric is known as the  
398 Fubini-Study metric.

The scalar product between two vectors  $\bar{V}$  and  $\bar{U}$  is

$$G(\bar{V}, \bar{U}) = \int dx \left[ \frac{\hbar}{2\rho_x} V^{1x} U^{1x} + \frac{2\rho_x}{\hbar} (V^{2x} - \langle V^2 \rangle)(U^{2x} - \langle U^2 \rangle) \right]. \quad (96)$$

It is at this point that we recognize the convenience of imposing the TGF gauge condition (83): the scalar product simplifies to

$$G(\bar{V}, \bar{U}) = \int dx \left[ \frac{\hbar}{2\rho_x} V^{1x} U^{1x} + \frac{2\rho_x}{\hbar} V^{2x} U^{2x} \right]. \quad (97)$$

An analogous expression can be written for the length  $\delta\tilde{s}$  of a displacement  $(\delta\rho_x, \delta\Phi_x)$ ,

$$\delta\tilde{s}^2 = \int dx \left[ \frac{\hbar}{2\rho_x} (\delta\rho_x)^2 + \frac{2\rho_x}{\hbar} (\delta\Phi_x)^2 \right], \quad (98)$$

where it is understood that  $(\delta\rho_x, \delta\Phi_x)$  satisfies the TGF condition

$$|\delta\rho| = 0 \quad \text{and} \quad \langle\delta\Phi\rangle = 0. \quad (99)$$

In index notation the metric (98) of  $T^*\mathbf{P}$  is written as

$$\delta\tilde{s}^2 = G_{IJ} \delta X^I \delta X^J = \int dx dx' G_{\alpha x, \beta x'} \delta X^{\alpha} \delta X^{\beta} \quad (100)$$

where the metric tensor  $G_{IJ}$  is<sup>17</sup>

$$G_{IJ} = G_{\alpha x, \beta x'} = \begin{bmatrix} \frac{\hbar}{2\rho_x} \delta_{xx'} & 0 \\ 0 & \frac{2}{\hbar} \rho_x \delta_{xx'} \end{bmatrix}. \quad (101)$$

<sup>17</sup> The tensor  $G_{IJ}$  in eq.(101) can act on arbitrary vectors whether they satisfy the TGF condition or not. It is only when  $G_{IJ}$  acts on TGF vectors that it is interpreted as a metric tensor on  $T^*\mathbf{P}$ .

## 399 6.3. A complex structure

Consider contracting the symplectic form  $\Omega_{IJ}$ , eq.(63), with the inverse of the metric tensor,

$$G^{IJ} = G^{\alpha x, \beta x'} = \begin{bmatrix} \frac{2}{\hbar} \rho_x \delta_{xx'} & 0 \\ 0 & \frac{\hbar}{2\rho_x} \delta_{xx'} \end{bmatrix}. \quad (102)$$

The result is a mixed tensor  $J$  with components

$$J^I{}_J = -G^{IK} \Omega_{KJ} = \begin{bmatrix} 0 & -\frac{2}{\hbar} \rho_x \delta_{xx'} \\ \frac{\hbar}{2\rho_x} \delta_{xx'} & 0 \end{bmatrix}. \quad (103)$$

(The reason for the negative sign will become clear below.) The tensor  $J^I{}_J$  maps vectors to vectors — as any mixed (1,1) tensor should. What makes the tensor  $J$  special is that — as one can easily check — its action on a TGF vector  $\bar{V}$  yields another vector  $J\bar{V}$  that is also TGF and, furthermore, its square is

$$J^I{}_K J^K{}_J = -\delta_{xx'} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\delta^I{}_J. \quad (104)$$

400 In words, when acting on vectors tangent to  $T^*\mathbf{P}$  the action of  $J^2$  (or  $\Omega^2$ ) is equivalent to multiplying  
401 by  $-1$ . This means that  $J$  plays the role of a complex structure.

402 We conclude that the cotangent bundle  $T^*\mathbf{P}$  has a symplectic structure  $\Omega$ , as all cotangent bundles  
403 do; that it can be given a Riemannian structure  $G_{IJ}$ ; and that the mixed tensor  $J$  provides it with a  
404 complex structure.

## 405 6.4. Complex coordinates

The fact that  $T^*\mathbf{P}$  is endowed with a complex structure suggests introducing complex coordinates,

$$\Psi_x = \rho_x^{1/2} \exp i\Phi_x / \hbar. \quad (105)$$

A point  $\Psi \in T^*\mathbf{P}^{+1}$  has coordinates

$$\Psi^{\mu x} = \begin{pmatrix} \Psi^{1x} \\ \Psi^{2x} \end{pmatrix} = \begin{pmatrix} \Psi_x \\ i\hbar \Psi_x^* \end{pmatrix}, \quad (106)$$

406 where the index  $\mu$  takes two values,  $\mu = 1, 2$ .

We can check that the transformation from real coordinates  $(\rho, \Phi)$  to complex coordinates  $(\Psi, i\hbar\Psi^*)$  is canonical. Indeed, the action of  $\Omega$  on two infinitesimal vectors  $\delta X^I$  and  $\delta' X^J$  is

$$\Omega_{IJ} \delta X^I \delta' X^J = \int dx (\delta \rho_x \delta' \Phi_x - \delta \Phi_x \delta' \rho_x),$$

which, when expressed in  $\Psi$  coordinates, becomes

$$\Omega_{IJ} \delta X^I \delta' X^J = \int dx (\delta \Psi \delta' i\hbar \Psi^* - \delta i\hbar \Psi^* \delta' \Psi) = \Omega_{\mu x, \nu x'} \delta \Psi^{\mu x} \delta \Psi^{\nu x'} \quad (107)$$

where

$$\Omega_{\mu x, \nu x'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{xx'}, \quad (108)$$

407 retains the same form as (63).

Expressed in  $\Psi$  coordinates the Hamiltonian flow generated by the normalization constraint (75),

$$\tilde{N} = 0 \quad \text{with} \quad \tilde{N} = 1 - \int dx \Psi_x^* \Psi_x, \tag{109}$$

and parametrized by  $\alpha$  is given by the vector field

$$\tilde{N} = - \left( \frac{\Psi_x / i\hbar}{i\hbar (\Psi_x / i\hbar)^*} \right). \tag{110}$$

Its integral curves are

$$\Psi_x(\alpha) = \Psi_x(0) e^{i\alpha/\hbar}. \tag{111}$$

The constraint  $\tilde{N} = 0$  induces a gauge symmetry which leads us to restrict our attention to vectors  $\tilde{V} = d/d\lambda$  satisfying two real TGF conditions (83). In  $\Psi$  coordinates this is replaced by the single complex TGF condition,

$$\int dx \Psi_x^* \frac{d\Psi_x}{d\lambda} = 0. \tag{112}$$

In  $\Psi$  coordinates the metric on  $T^*\mathbf{P}$ , eq.(98), becomes

$$\delta\tilde{s}^2 = -2i \int dx \delta\Psi_x \delta i\hbar \Psi_x^* = \int dx dx' G_{\mu x, \nu x'} \delta\Psi^{\mu x} \delta\Psi^{\nu x'}, \tag{113}$$

where the metric tensor and its inverse are

$$G_{\mu x, \nu x'} = -i\delta_{xx'} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad G^{\mu x, \nu x'} = i\delta_{xx'} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{114}$$

Finally, using  $G^{\mu x, \nu x'}$  to raise the first index of  $\Omega_{\nu x', \gamma x''}$  gives the  $\Psi$  components of the tensor  $J$

$$J^{\mu x}_{\gamma x''} \stackrel{\text{def}}{=} -G^{\mu x, \nu x'} \Omega_{\nu x', \gamma x''} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \delta_{xx'}. \tag{115}$$

## 408 7. Hamilton-Killing flows

Our next goal will be to find those Hamiltonian flows  $Q^I$  that also happen to preserve the metric tensor, that is, we want  $Q^I$  to be a Killing vector. The condition for  $Q^I$  is

$$(\mathcal{L}_Q G)_{IJ} = Q^K \tilde{\nabla}_K G_{IJ} + G_{KJ} \tilde{\nabla}_I Q^K + G_{IK} \tilde{\nabla}_J Q^K = 0. \tag{116}$$

In complex coordinates eq.(114) gives  $\tilde{\nabla}_K G_{IJ} = 0$ , and the Killing equation simplifies to

$$(\mathcal{L}_Q G)_{IJ} = G_{KJ} \tilde{\nabla}_I Q^K + G_{IK} \tilde{\nabla}_J Q^K = 0, \tag{117}$$

or

$$(\mathcal{L}_Q G)_{\mu x, \nu x'} = -i \begin{bmatrix} \frac{\delta Q^{2x'}}{\delta \Psi_x} + \frac{\delta Q^{2x}}{\delta \Psi_{x'}}; & \frac{\delta Q^{1x'}}{\delta \Psi_x} + \frac{\delta Q^{2x}}{\delta i\hbar \Psi_{x'}^*} \\ \frac{\delta Q^{2x'}}{\delta i\hbar \Psi_x^*} + \frac{\delta Q^{1x}}{\delta \Psi_{x'}}; & \frac{\delta Q^{1x'}}{\delta i\hbar \Psi_x^*} + \frac{\delta Q^{1x}}{\delta i\hbar \Psi_{x'}^*} \end{bmatrix} = 0. \tag{118}$$

If we further require that  $Q^I$  be a Hamiltonian flow,  $\mathcal{L}_Q \Omega = 0$ , then we substitute

$$Q^{1x} = \frac{\delta \tilde{Q}}{\delta i\hbar \Psi_x^*} \quad \text{and} \quad Q^{2x} = -\frac{\delta \tilde{Q}}{\delta \Psi_x} \tag{119}$$

into (118) to get

$$\frac{\delta^2 \tilde{Q}}{\delta \Psi_x \delta \Psi_{x'}} = 0 \quad \text{and} \quad \frac{\delta^2 \tilde{Q}}{\delta \Psi_x^* \delta \Psi_{x'}^*} = 0. \quad (120)$$

Therefore in order to generate a flow that preserves both  $G$  and  $\Omega$  the functional  $\tilde{Q}[\Psi, \Psi^*]$  must be linear in both  $\Psi$  and  $\Psi^*$ ,

$$\tilde{Q}[\Psi, \Psi^*] = \int dx dx' \Psi_x^* \hat{Q}_{xx'} \Psi_{x'}, \quad (121)$$

409 where  $\hat{Q}_{xx'}$  is a possibly non-local kernel. The actual Hamilton-Killing flow is

$$\frac{d\Psi_x}{d\lambda} = Q^{1x} = \frac{\delta \tilde{Q}}{\delta i\hbar \Psi_x^*} = \frac{1}{i\hbar} \int dx' \hat{Q}_{xx'} \Psi_{x'}, \quad (122)$$

$$\frac{di\hbar \Psi_x^*}{d\lambda} = Q^{2x} = -\frac{\delta \tilde{Q}}{\delta \Psi_x} = -\int dx' \Psi_{x'}^* \hat{Q}_{xx'}. \quad (123)$$

Taking the complex conjugate of (122) and comparing with (123), shows that the kernel  $\hat{Q}_{xx'}$  is Hermitian,

$$\hat{Q}_{xx'}^* = \hat{Q}_{x'x}, \quad (124)$$

and we can check that the corresponding Hamiltonian functionals  $\tilde{Q}$  are real,

$$\tilde{Q}[\Psi, \Psi^*]^* = \tilde{Q}[\Psi, \Psi^*].$$

The Hamiltonian flows that might potentially be of interest are those that generate symmetry transformations. For example, the generator of translations is total momentum. Under a spatial displacement by  $\varepsilon^a$ ,  $g(x) \rightarrow g_\varepsilon(x) = g(x - \varepsilon)$ , the change in  $f[\rho, \Phi]$  is

$$\delta_\varepsilon f[\rho, \Phi] = \int dx \left( \frac{\delta f}{\delta \rho_x} \delta_\varepsilon \rho_x + \frac{\delta f}{\delta \Phi_x} \delta_\varepsilon \Phi_x \right) = \{f, \tilde{P}_a \varepsilon^a\} \quad (125)$$

where

$$\tilde{P}_a = \int dx \rho \sum_n \frac{\partial \Phi}{\partial x_n^a} = \int dx \rho \frac{\partial \Phi}{\partial X^a} \quad (126)$$

is interpreted as the expectation of the total momentum, and  $X^a$  are the coordinates of the center of mass,

$$X^a = \frac{1}{M} \sum_n m_n x_n^a. \quad (127)$$

In complex coordinates,

$$\tilde{P}_a = \int dx \Psi^* \left( \sum_n \frac{\hbar}{i} \frac{\partial}{\partial x_n^a} \right) \Psi = \int dx \Psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial X^a} \right) \Psi, \quad (128)$$

and the corresponding kernel  $\hat{P}_{axx'}$  is

$$\hat{P}_{axx'} = \delta_{xx'} \sum_n \frac{\hbar}{i} \frac{\partial}{\partial x_n^a} = \delta_{xx'} \frac{\hbar}{i} \frac{\partial}{\partial X^a}. \quad (129)$$

## 410 8. The e-Hamiltonian

In previous sections we supplied the symplectic e-phase space  $T^*\mathbf{P}$  with a Riemannian metric and, as a welcome by-product, also with a complex structure. Then we showed that the condition for

the simplest form of dynamics — one that preserves all the metric, symplectic, and complex structures — is a Hamilton-Killing flow generated by a Hamiltonian  $\tilde{H}$  that is linear in both  $\Psi$  and  $\Psi^*$ ,

$$\tilde{H}[\Psi, \Psi^*] = \int dx dx' \Psi_x^* \hat{H}_{xx'} \Psi_{x'} . \quad (130)$$

411 The last ingredient in the construction of  $\tilde{H}$  is that the e-Hamiltonian has to agree with (50) in order to  
412 reproduce the entropic evolution of  $\rho$  given by the continuity eq.(49).

To proceed we use the identity

$$\frac{1}{2} \rho m^{AB} (\partial_A \Phi - \bar{A}_A) (\partial_B \Phi - \bar{A}_B) = \frac{\hbar^2}{2} m^{AB} (D_A \Psi)^* D_B \Psi - \frac{\hbar^2}{8\rho^2} m^{AB} \partial_A \rho \partial_B \rho \quad (131)$$

where

$$D_A = \partial_A - \frac{i}{\hbar} \bar{A}_A \quad \text{and} \quad \bar{A}_A(x) = \hbar \beta_n A_n(x_n) . \quad (132)$$

Rewriting  $\tilde{H}[\rho, \Phi]$  in (50) in terms of  $\Psi$  and  $\Psi^*$  we get

$$\tilde{H}[\Psi, \Psi^*] = \int dx \left( -\frac{\hbar^2}{2} m^{AB} \Psi^* D_A D_B \Psi \right) + F'[\rho] . \quad (133)$$

where

$$F'[\rho] = F[\rho] - \frac{\hbar^2}{8\rho^2} m^{AB} \partial_A \rho \partial_B \rho . \quad (134)$$

According to (121) in order for  $\tilde{H}[\Psi, \Psi^*]$  to generate an HK flow we must impose that  $F'[\rho]$  be linear in both  $\Psi$  and  $\Psi^*$ ,

$$F'[\rho] = \int dx dx' \Psi_x^* \hat{V}_{xx'} \Psi_{x'} \quad (135)$$

for some Hermitian kernel  $\hat{V}_{xx'}$ , but  $F'[\rho]$  must remain independent of  $\Phi$ ,

$$\frac{\delta F'[\rho]}{\delta \Phi_x} = 0 . \quad (136)$$

Substituting  $\Psi = \rho^{1/2} e^{i\Phi/\hbar}$  into (135) and using  $\hat{V}_{x'x}^* = \hat{V}_{xx'}$  leads to

$$\frac{\delta F'}{\delta \Phi_x} = \frac{2}{\hbar} \rho_x^{1/2} \int dx' \rho_{x'}^{1/2} \text{Im} \left( \hat{V}_{xx'} e^{-i(\Phi_x - \Phi_{x'})/\hbar} \right) = 0 \quad (137)$$

This equation must be satisfied for all choices of  $\rho_{x'}$ , which implies

$$\text{Im} \left( \hat{V}_{xx'} e^{-i(\Phi_x - \Phi_{x'})/\hbar} \right) = 0 , \quad (138)$$

and also for all choices of  $\Phi_x$  and  $\Phi_{x'}$ . Therefore, the kernel  $\hat{V}_{xx'}$  must be local in  $x$ ,

$$\hat{V}_{xx'} = \delta_{xx'} V_x \quad (139)$$

413 where  $V_x = V(x)$  is some real function.

We conclude that the Hamiltonian that generates a Hamilton-Killing flow and agrees with the ED continuity equation must be of the form

$$\tilde{H}[\Psi, \Psi^*] = \int dx \Psi^* \left( -\frac{\hbar^2}{2} m^{AB} D_A D_B + V(x) \right) \Psi . \quad (140)$$



The evolution of  $\Psi$  is given by the Hamilton equation,

$$\partial_t \Psi_x = \{\Psi_x, \tilde{H}\} = \frac{\delta \tilde{H}}{\delta(i\hbar \Psi^*(x))}, \quad (141)$$

which is the Schrödinger equation,

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2} m^{AB} D_A D_B \Psi + V \Psi. \quad (142)$$

In more standard notation it reads

$$i\hbar \partial_t \Psi = \sum_n \frac{-\hbar^2}{2m_n} \delta^{ab} \left( \frac{\partial}{\partial x_n^a} - i\beta_n A_a(x_n) \right) \left( \frac{\partial}{\partial x_n^b} - i\beta_n A_b(x_n) \right) \Psi + V \Psi. \quad (143)$$

At this point we can finally provide the physical interpretation of the various constants introduced along the way. Since the Schrödinger equation (143) is the tool we use to analyze experimental data we can identify  $\hbar$  with Planck's constant,  $m_n$  will be interpreted as the particles' masses, and the  $\beta_n$  are related to the particles' electric charges  $q_n$  by

$$\beta_n = \frac{q_n}{\hbar c}. \quad (144)$$

414 For completeness we write the Hamiltonian in the  $(\rho, \Phi)$  variables,

$$\begin{aligned} \tilde{H}[\rho, \Phi] = & \int d^{3N} x \rho \left[ \sum_n \frac{\delta^{ab}}{2m_n} \left( \frac{\partial \Phi}{\partial x_n^a} - \frac{q_n}{c} A_a(x_n) \right) \left( \frac{\partial \Phi}{\partial x_n^b} - \frac{q_n}{c} A_b(x_n) \right) \right. \\ & \left. + \sum_n \frac{\hbar^2}{8m_n} \frac{\delta^{ab}}{\rho^2} \frac{\partial \rho}{\partial x_n^a} \frac{\partial \rho}{\partial x_n^b} + V(x_1 \dots x_n) \right]. \end{aligned} \quad (145)$$

The Hamilton equations for  $\rho$  and  $\Phi$  are the continuity equation (49),

$$\partial_t \rho = \frac{\delta \tilde{H}}{\delta \Phi} = - \sum_n \frac{\partial}{\partial x_n^a} \left[ \rho \frac{\delta^{ab}}{m_n} \left( \frac{\partial \Phi}{\partial x_n^b} - \frac{q_n}{c} A_b(x_n) \right) \right], \quad (146)$$

415 and the quantum analogue of the Hamilton-Jacobi equation,

$$\begin{aligned} \partial_t \Phi = & - \frac{\delta \tilde{H}}{\delta \rho} = \sum_n \frac{-\delta^{ab}}{2m_n} \left( \frac{\partial \Phi}{\partial x_n^a} - \frac{q_n}{c} A_a(x_n) \right) \left( \frac{\partial \Phi}{\partial x_n^b} - \frac{q_n}{c} A_b(x_n) \right) \\ & + \sum_n \frac{\hbar^2}{2m_n} \frac{\delta^{ab}}{\rho^{1/2}} \frac{\partial^2 \rho^{1/2}}{\partial x_n^a \partial x_n^b} - V(x_1 \dots x_n). \end{aligned} \quad (147)$$

416 We have just shown that an ED that preserves both the symplectic and metric structures of the e-phase  
417 space  $T^*\mathbf{P}$  leads to a linear Schrödinger equation. In particular, such an ED reproduces the quantum  
418 potential in (147) with the correct coefficients  $\hbar^2/2m_n$ .

## 419 9. Entropic time, physical time, and time reversal

420 Now that the dynamics has been fully developed we revisit the question of time. The derivation  
421 of laws of physics as examples of inference led us to introduce the notion of entropic time which  
422 includes assumptions about the concept of instant, of simultaneity, of ordering, and of duration. It is  
423 clear that entropic time is useful but is this the actual, real, "physical" time? The answer is yes. By  
424 deriving the Schrödinger equation (from which we can obtain the classical limit) we have shown that  
425 the  $t$  that appears in the laws of physics is entropic time. Since these are the equations that we routinely

4.26 use to design and calibrate our clocks we conclude that *what clocks measure is entropic time*. No notion  
 4.27 of time that is in any way deeper or more “physical” is needed. Most interestingly, the entropic model  
 4.28 automatically includes an arrow of time.

The statement that the laws of physics are invariant under time reversal has nothing to do with particles travelling backwards in time. It is instead the assertion that the laws of physics exhibit a certain symmetry. For a classical system described by coordinates  $q$  and momenta  $p$  the symmetry is the statement that if  $\{q_t, p_t\}$  happens to be one solution of Hamilton’s equations then we can construct another solution  $\{q_t^T, p_t^T\}$  where

$$q_t^T = q_{-t} \quad \text{and} \quad p_t^T = -p_{-t}, \quad (148)$$

4.29 but both solutions  $\{q_t, p_t\}$  and  $\{q_t^T, p_t^T\}$  describe evolution forward in time. An alternative statement of  
 4.30 time reversibility is the following: if there is one trajectory of the system that takes it from state  $\{q_0, p_0\}$   
 4.31 at time  $t_0$  to state  $\{q_1, p_1\}$  at the later time  $t_1$ , then there is another trajectory that takes the system from  
 4.32 state  $\{q_1, -p_1\}$  at time  $t_0$  to state  $\{q_0, -p_0\}$  at the later time  $t_1$ . The merit of this re-statement is that it  
 4.33 makes clear that nothing needs to travel back in time. Indeed, rather than time reversal the symmetry  
 4.34 might be more appropriately described as momentum or motion reversal.

Since ED is a Hamiltonian dynamics one can expect that similar considerations will apply to QM and indeed they do. It is straightforward to check that given one solution  $\{\rho_t(x), \Phi_t(x)\}$  that evolves forward in time, we can construct another solution  $\{\rho_t^T(x), \Phi_t^T(x)\}$  that is also evolving forward in time. The reversed solution is

$$\rho_t^T(x) = \rho_{-t}(x) \quad \text{and} \quad \Phi_t^T(x) = -\Phi_{-t}(x). \quad (149)$$

These transformations constitute a symmetry — *i.e.*, the transformed  $\Psi_t^T(x)$  is a solution of the Schrödinger equation — provided the motion of the sources of the external potentials is also reversed, that is, the potentials  $A_a(\vec{x}, t)$  and  $V(x, t)$  are transformed according to

$$A_a^T(\vec{x}, t) = -A_a(\vec{x}, -t) \quad \text{and} \quad V^T(x, t) = V(x, -t). \quad (150)$$

Expressed in terms of wave functions the time reversal transformation is

$$\Psi_t^T(x) = \Psi_{-t}^*(x). \quad (151)$$

4.35 The proof that this is a symmetry is straightforward; just take the complex conjugate of (143), and let  
 4.36  $t \rightarrow -t$ .

## 4.37 10. Linearity and the superposition principle

The Schrödinger equation is linear, that is, a linear combination of solutions is a solution too. However, this *mathematical* linearity does not guarantee the *physical* linearity that is usually referred to as the superposition principle. The latter is the physical assumption that if there is one experimental setup that prepares a system in the (epistemic) state  $\Psi_1$  and there is another setup that prepares the system in the state  $\Psi_2$  then, at least in principle, it is possible to construct yet a third setup that can prepare the system in the superposition

$$\Psi_3 = \alpha_1 \Psi_1 + \alpha_2 \Psi_2, \quad (152)$$

4.38 where  $\alpha_1$  and  $\alpha_2$  are arbitrary complex numbers. Mathematical linearity refers to the fact that solutions  
 4.39 can be expressed as sums of solutions. There is no implication that any of these solutions will  
 4.40 necessarily describe physical situations. Physical linearity on the other hand — the Superposition  
 4.41 Principle — refers to the fact that the superposition of physical solutions is also a physical solution.

442 The point to be emphasized is the Superposition Principle is not a principle; it is a physical hypothesis  
443 that need not be universally true.

#### 444 10.1. The single-valuedness of $\Psi$

445 The question “Why should wave functions be single-valued?” has been around for a long time.  
446 In this section we build on and extend recent work [82] to argue that the single- or multi-valuedness of  
447 the wave functions is closely related to the question of linearity and the superposition principle. Our  
448 discussion parallels that by Schrödinger [83].<sup>18</sup> (See also [84]–[89].)<sup>19</sup>

To show that the mathematical linearity of (143) is not sufficient to imply the superposition principle we argue that even when  $|\Psi_1|^2 = \rho_1$  and  $|\Psi_2|^2 = \rho_2$  are probabilities it is not generally true that  $|\Psi_3|^2$ , eq.(152), will also be a probability. Consider moving around a closed loop  $\Gamma$  in configuration space. Since phases  $\Phi(x)$  can be multi-valued the corresponding wave functions can be multi-valued too. Let a generic  $\Psi$  change by a phase factor,

$$\Psi \rightarrow \Psi' = e^{i\delta}\Psi, \quad (153)$$

then the superposition  $\Psi_3$  of two wave functions  $\Psi_1$  and  $\Psi_2$  changes into

$$\Psi_3 \rightarrow \Psi'_3 = \alpha_1 e^{i\delta_1} \Psi_1 + \alpha_2 e^{i\delta_2} \Psi_2. \quad (154)$$

The problem is that even if  $|\Psi_1|^2 = \rho_1$  and  $|\Psi_2|^2 = \rho_2$  are single-valued (because they are probability densities), the quantity  $|\Psi_3|^2$  need not in general be single-valued. Indeed,

$$|\Psi_3|^2 = |\alpha_1|^2 \rho_1 + |\alpha_2|^2 \rho_2 + 2\text{Re}[\alpha_1 \alpha_2^* \Psi_1 \Psi_2^*], \quad (155)$$

changes into

$$|\Psi'_3|^2 = |\alpha_1|^2 \rho_1 + |\alpha_2|^2 \rho_2 + 2\text{Re}[\alpha_1 \alpha_2^* e^{i(\delta_1 - \delta_2)} \Psi_1 \Psi_2^*], \quad (156)$$

so that in general

$$|\Psi'_3|^2 \neq |\Psi_3|^2, \quad (157)$$

449 which precludes the interpretation of  $|\Psi_3|^2$  as a probability. That is, even when the epistemic states  $\Psi_1$   
450 and  $\Psi_2$  describe actual physical situations, their superposition need not.

The problem does not arise when

$$e^{i(\delta_1 - \delta_2)} = 1. \quad (158)$$

451 If we were to group the wave functions into classes each characterized by its own  $\delta$  then we could  
452 have a limited version of the superposition principle that applies within each class. We conclude  
453 that beyond the linearity of the Schrödinger equation we have a superselection rule that restricts the  
454 validity of the superposition principle to wave functions belong to the same  $\delta$ -class.

To find the allowed values of  $\delta$  we argue as follows. It is natural to assume that if  $\{\rho, \Phi\}$  (at some given time  $t_0$ ) is a physical state then the state with reversed momentum  $\{\rho, -\Phi\}$  (at the same time  $t_0$ )

<sup>18</sup> Schrödinger invoked time reversal invariance which was a very legitimate move back in 1938 but today it is preferable to develop an argument which does not invoke symmetries that are already known to be violated.

<sup>19</sup> The answer proposed by Pauli is also worthy of note [84][85]. He proposed that admissible wave functions must form a basis for representations of the transformation group that happens to be pertinent to the problem at hand. In particular, Pauli's argument serves to discard double-valued wave functions for describing the orbital angular momentum of scalar particles. The question of single-valuedness was revived by Takabayashi [86] in the context of the hydrodynamical interpretation of QM, and later rephrased by Wallstrom [88][89] as an objection to Nelson's stochastic mechanics: are these theories equivalent to QM or do they merely reproduce a subset of its solutions? Wallstrom's objection is that Nelson's stochastic mechanics leads to phases and wave functions that are either both multi-valued or both single-valued. Both alternatives are unsatisfactory because on one hand QM requires single-valued wave functions, while on the other hand single-valued phases exclude states that are physically relevant (e.g., states with non-zero angular momentum).

is an equally reasonable physical state. Basically, the idea is that if particles can be prepared to move in one direction, then they can also be prepared to move in the opposite direction. In terms of wave functions the statement is that if  $\Psi_{t_0}$  is a physically allowed initial state, then so is  $\Psi_{t_0}^*$ .<sup>20</sup> Next we consider a generic superposition

$$\Psi_3 = \alpha_1 \Psi + \alpha_2 \Psi^* . \quad (159)$$

455 Is it physically possible to construct superpositions such as (159)? The answer is that while constructing  
 456  $\Psi_3$  for an arbitrary  $\Psi$  might not be feasible in practice there is strong empirical evidence that there  
 457 exist no superselection rules to prevent us from doing so in principle. Indeed, it is easy to construct  
 458 superpositions of wavepackets with momentum  $\vec{p}$  and  $-\vec{p}$ , or superpositions of states with opposite  
 459 angular momenta,  $Y_{\ell m}$  and  $Y_{\ell, -m}$ . We shall assume that in principle the superpositions (159) are physically  
 460 possible.

According to eq.(153) as one moves in a closed loop  $\Gamma$  the wave function  $\Psi_3$  will transform into

$$\Psi'_3 = \alpha_1 e^{i\delta} \Psi + \alpha_2 e^{-i\delta} \Psi^* , \quad (160)$$

and the condition (158) for  $|\Psi_3|^2$  to be single-valued is

$$e^{2i\delta} = 1 \quad \text{or} \quad e^{i\delta} = \pm 1 . \quad (161)$$

461 Thus, we are restricted to two discrete possibilities  $\pm 1$ . Since the wave functions are assumed  
 462 sufficiently well behaved (continuous, differentiable, etc.) we conclude that they must be either  
 463 single-valued,  $e^{i\delta} = 1$ , or double-valued,  $e^{i\delta} = -1$ .

464 We conclude that the Superposition Principle appears to be valid in a sufficiently large number of  
 465 cases to be a useful rule of thumb but it is restricted to single-valued (or double-valued) wave functions.  
 466 The argument above does not exclude the possibility that a multi-valued wave function might describe  
 467 an actual physical situation. What the argument implies is that the Superposition Principle would not  
 468 extend to such states.

## 469 10.2. Charge quantization

From now on we confine our attention to systems that are described by single-valued wave functions ( $e^{i\delta} = +1$ )<sup>21</sup> and obey the superposition principle. The condition for the wave function to be single-valued is

$$\Delta \frac{\Phi}{\hbar} = \oint_{\Gamma} d\ell^A \partial_A \frac{\Phi}{\hbar} = 2\pi k_{\Gamma} , \quad (162)$$

where  $k_{\Gamma}$  is an integer that depends on the loop  $\Gamma$ . Under a local gauge transformation

$$A_a(\vec{x}) \rightarrow A_a(\vec{x}) + \partial_a \chi(\vec{x}) \quad (163)$$

the phase  $\Phi$  transforms according to (38),

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \sum_n \frac{q_n}{c} \chi(\vec{x}_n) . \quad (164)$$

<sup>20</sup> We make no symmetry assumptions such as parity or time reversibility. It need not be the case that there is any symmetry that relates the time evolution of  $\Psi_{t_0}^*$  to that of  $\Psi_{t_0}$ .

<sup>21</sup> Double-valued wave functions with  $e^{i\delta} = -1$  will, of course, find use in the description of spin-1/2 particles. [71]

The requirement that the gauge symmetry be compatible with the superposition principle amounts to requiring that the gauge transformed states also be single-valued,

$$\Delta \frac{\Phi'}{\hbar} = \oint_{\Gamma} d\ell^A \partial_A \frac{\Phi'}{\hbar} = 2\pi k'_{\Gamma}, \quad (165)$$

Thus, the allowed gauge transformations are restricted to functions  $\chi(\vec{x})$  such that

$$\sum_n \frac{q_n}{\hbar c} \oint_{\Gamma} d\ell_n^a \partial_{na} \chi(\vec{x}_n) = 2\pi \Delta k_{\Gamma} \quad (166)$$

where  $\Delta k_{\Gamma} = k'_{\Gamma} - k_{\Gamma}$  is an integer. Consider now a loop  $\gamma$  in which we follow the coordinates of the  $n$ th particle around some closed path in 3-dimensional space while all the other particles are kept fixed. Then

$$\frac{q_n}{\hbar c} \oint_{\gamma} d\ell_n^a \partial_{an} \chi(\vec{x}_n) = 2\pi \Delta k_{n\gamma}, \quad (167)$$

470 where  $\Delta k_{n\gamma}$  is an integer. Since the gauge function  $\chi(\vec{x})$  is just a function in 3-dimensional space it is  
 471 the same for all particles and the integral on the left is independent of  $n$ . This implies that the charge  $q_n$   
 472 divided by an integer  $\Delta k_{n\gamma}$  must be independent of  $n$  which means that  $q_n$  must be an integer multiple  
 473 of some basic charge  $q_0$ . We conclude that the charges  $q_n$  are quantized.

474 The issue of charge quantization is ultimately the issue of deciding which is the gauge group  
 475 that generates electromagnetic interactions. We could for example decide to restrict the gauge  
 476 transformations to single-valued gauge functions  $\chi(\vec{x})$  so that (167) is trivially satisfied irrespective of  
 477 the charges being quantized or not. Under such a restricted symmetry group the single-valued (or  
 478 double-valued) nature of the wave function is unaffected by gauge transformations. If, on the other  
 479 hand, the gauge functions  $\chi(\vec{x})$  are allowed to be multi-valued, then the compatibility of the gauge  
 480 transformation (163-164) with the superposition principle demands that charges be quantized.

481 The argument above cannot fix the value of the basic charge  $q_0$  because it depends on the units  
 482 chosen for the vector potential  $A_a$ . Indeed since the dynamical equations show  $q_n$  and  $A_a$  appearing  
 483 only in the combination  $q_n A_a$  we can change units by rescaling charges and potentials according to  
 484  $Cq_n = q'_n$  and  $A_a/C = A'_a$  so that  $q_n A_a = q'_n A'_a$ . For conventional units such that the basic charge is  
 485  $q_0 = e/3$  with  $\alpha = e^2/\hbar c = 1/137$  the scaling factor is  $C = (\alpha\hbar c)^{1/2}/3q_0$ . A more natural set of units  
 486 might be to set  $q_0 = \hbar c$  so that all  $\beta_n$ s are integers and the gauge functions  $\chi(\vec{x})$  are angles.

A similar conclusion — that charge quantization is a reflection of the compactness of the gauge group — can be reached following an argument due to C. N. Yang [90]. Yang's argument assumes that a Hilbert space has been established and one has access to the unitary representations of symmetry groups. Yang considers a gauge transformation

$$\Psi(x) \rightarrow \Psi(x) \exp i \sum_n \frac{q_n}{c} \chi(\vec{x}_n), \quad (168)$$

487 with  $\chi(\vec{x})$  independent of  $\vec{x}$ . If the  $q_n$ s are not commensurate there is no value of  $\chi$  (except 0) that  
 488 makes (168) be the identity transformation. The gauge group — translations on the real line — would  
 489 not be compact. If, on the other hand, the charges are integer multiples of a basic charge  $q_0$ , then  
 490 two values of  $\chi$  that differ by an integer multiple of  $2\pi c/q_0$  give identical transformations and the  
 491 gauge group is compact. In the present ED derivation, however, we deal with the space  $T^*\mathbf{P}$  which  
 492 is a complex projective space. We cannot adopt Yang's argument because a gauge transformation  $\chi$   
 493 independent of  $\vec{x}$  is already an identity transformation — it leads to an equivalent state in the same ray  
 494 — and cannot therefore lead to any constraints on the allowed charges.

## 495 11. The classical limits and the Bohmian limit

### 496 11.1. Classical limits

497 There are two classical limits that one might wish to consider. One is the mathematical limit  $\hbar \rightarrow 0$ .  
 498 Taking  $\hbar \rightarrow 0$  leaves unchanged both the velocities  $v_n^a$  of the particles, eq.(19), and the probability flow,  
 499 eq.(146). The main effect is to suppress the quantum potential so that eq.(147) becomes the classical  
 500 Hamilton-Jacobi equation. The symplectic form, eq.(63), survives unscathed but the metric and the  
 501 complex structures, eqs.(101) and (103), do not. But this is not quite classical mechanics. Since the  
 502 velocity fluctuations, eq.(25), remain unaffected the resulting dynamics is a non-dissipative version of  
 503 the classical Ornstein-Uhlenbeck Brownian motion. To recover a deterministic classical mechanics  
 504 one must also take the limit  $\eta \rightarrow 0$ .

The other classical limit arises in the more physically relevant situation where one deals with a system with a large number  $N$  of particles — for example, a speck of dust — and one wishes to study the motion of an effective macrovariable such as the center of mass (CM), eq.(127). The large  $N$  limit of ED with particles undergoing an ES Brownian motion was studied in [66]. The same argument goes through essentially unchanged for the OU Brownian motion discussed here. Skipping all details we find that as a consequence of the central limit theorem the continuity equation for  $\rho_{\text{cm}}(X^a)$  and the velocity fluctuations are given by the analogues of (43) and (25) for a single particle of mass  $M = \sum_{n=1}^N m_n$ ,

$$\partial_t \rho_{\text{cm}} = \frac{\partial}{\partial X^a} (\rho_{\text{cm}} V^a) \quad \text{with} \quad V^a = \frac{\langle \Delta X^a \rangle}{\Delta t} = \frac{1}{M} \frac{\partial \Phi_{\text{cm}}}{\partial X^a}, \quad (169)$$

$$\left\langle \left( \frac{\Delta X^a}{\Delta t} - V^a \right) \left( \frac{\Delta X^b}{\Delta t} - V^b \right) \right\rangle = \frac{\eta \Delta t}{M}. \quad (170)$$

We also find that under rather general conditions the CM motion decouples from the motion of the component particles and obeys the single particle HJ equation

$$-\partial_t \Phi_{\text{cm}} = \frac{1}{2M} \left( \frac{\partial \Phi_{\text{cm}}}{\partial X^a} \right)^2 - \frac{\hbar^2}{2M} \frac{\nabla^2 \rho_{\text{cm}}^{1/2}}{\rho_{\text{cm}}^{1/2}} + V_{\text{ext}}(X). \quad (171)$$

505 In the large  $N$  limit  $M \sim O(N)$  and we obtain a finite velocity  $V^a$  in (169) provided  $\Phi_{\text{cm}} \sim O(N)$ . In  
 506 eq.(171) we see that for a sufficiently large system the quantum potential for the CM motion vanishes.  
 507 Therefore, for  $N \rightarrow \infty$ , the CM follows smooth trajectories described by a classical Hamilton-Jacobi  
 508 equation. Furthermore, eq.(170) shows that as  $N \rightarrow \infty$  the velocity fluctuations vanish irrespective of  
 509 the value of  $\eta$ . This is a truly deterministic classical mechanics.

510 An important feature of this derivation is that  $\hbar$  and  $\eta$  remain finite which means that a mesoscopic  
 511 or macroscopic object will behave classically while all its component particles remain fully quantum  
 512 mechanical.

### 513 11.2. The Bohmian limit

514 ED models with different values of  $\eta$  lead to the same Schrödinger equation. In other words,  
 515 different sub-quantum models lead to the same emergent quantum behavior. The limit of vanishing  $\eta$   
 516 deserves particular attention because the velocity fluctuations, eq.(25), are suppressed and the motion  
 517 becomes deterministic. This means that ED includes the Bohmian form of quantum mechanics [45]-[47]  
 518 as a special limiting case — but with the important caveat that the difference in physical interpretation  
 519 remains enormous. It is only with respect to the mathematical formalism that ED includes Bohmian  
 520 mechanics as a special case.

521 Bohmian mechanics attempts to provide an actual description of reality. In the Bohmian view the  
 522 universe consists of real particles that have definite positions and their trajectories are guided by a real  
 523 field, the wave function  $\Psi$ . Not only does this pilot wave live in  $3N$ -dimensional configuration space

524 but it manages to act on the particles without the particles reacting back upon it. These are peculiarities  
 525 that have stood in the way of a wider acceptance of the Bohmian interpretation. In contrast, ED's  
 526 pragmatic goal is much less ambitious: to make the best possible predictions on the basis of very  
 527 incomplete information. As in Bohmian mechanics, in ED the particles also have definite positions and  
 528 its formalism includes a function  $\Phi$  that plays the role of a pilot wave. But  $\Phi$  is an epistemic tool for  
 529 reasoning; it is not meant to represent anything real. There is no implication that the particles move  
 530 the way they do because they are pushed around by a pilot wave or by some stochastic force. In fact  
 531 ED is silent on the issue of what if anything is pushing the particles. What the probability  $\rho$  and the  
 532 phase  $\Phi$  are designed to do is not to guide the particles but to guide our inferences. They guide our  
 533 expectations of where and when to find the particles but they do not exert any causal influence on the  
 534 particles themselves.

## 535 12. Hilbert space

536 The formulation of the ED of spinless particles is now complete. We note, in particular, that the  
 537 notion of Hilbert spaces turned out to be unnecessary to the formulation of quantum mechanics. As we  
 538 shall see next, while strictly unnecessary in principle, the introduction of Hilbert spaces is nevertheless  
 539 very convenient for calculational purposes.

540 **A vector space** — As we saw above the infinite-dimensional e-phase space — the cotangent  
 541 bundle  $T^*\mathbf{P}$  — is difficult to handle. The problem is that the natural coordinates are probabilities  
 542  $\rho_x$  which, due to the normalization constraint, are not independent. In a discrete space one could  
 543 single out one of the coordinates and its conjugate momentum and then proceed to remove them.  
 544 Unfortunately, with a continuum of coordinates and momenta the removal is not feasible. The solution  
 545 is to embed  $T^*\mathbf{P}$  in a larger space  $T^*\mathbf{P}^{+1}$ . This move allows us to keep the natural coordinates  $\rho_x$  but  
 546 there is a price: we are forced to deal with a constrained system and its attendant gauge symmetry.

We also saw that the geometry of the embedding space was not fully determined: any spherically  
 symmetric space would serve our purposes. This is a freedom we can further exploit. For calculational  
 purposes the linearity of the Schrödinger equation (143) is very convenient but its usefulness is severely  
 limited by the normalization constraint. If  $\Psi_1$  and  $\Psi_2$  are flows in  $T^*\mathbf{P}$  then the superposition  $\Psi_3$  in  
 (152) will also be a flow in  $T^*\mathbf{P}$  but only when constrained by

$$|\alpha_1|^2 + |\alpha_2|^2 = 1. \quad (172)$$

547 This restriction can be removed by choosing the extended embedding space  $T^*\mathbf{P}^{+1}$  to be flat — just set  
 548  $A = 0$  and  $B = 1$  in eq.(91). (The fact that this space is flat is evident in the metric (89) for the discrete  
 549 case.) We emphasize that this choice is not at all obligatory; it is optional.

550 The fact that in the flat space  $T^*\mathbf{P}^{+1}$  superpositions are allowed for arbitrary constants  $\alpha_1$  and  
 551  $\alpha_2$  means that  $T^*\mathbf{P}^{+1}$  is not just a manifold; it is also a vector space. Each point  $\Psi$  in  $T^*\mathbf{P}^{+1}$  is itself a  
 552 vector. Furthermore, since the vector tangent to a curve is just a difference of two vectors  $\Psi$  we see that  
 553 that points on the manifold and vectors tangent to the manifold are objects of the same kind. In other  
 554 words, the tangent spaces  $T[T^*\mathbf{P}^{+1}]_\Psi$  are identical to the space  $T^*\mathbf{P}^{+1}$  itself.

The symplectic form  $\Omega$  and the metric tensor  $G$  on the extended space  $T^*\mathbf{P}^{+1}$  are given by eq.(108)  
 and (114). These expressions now hold for all points  $\Psi \in T^*\mathbf{P}^{+1}$  and not just for those that happen to  
 be normalized and gauge fixed according to (83). These tensors are meant to act on vectors but now  
 they can also act on points. For example, the action of the mixed tensor  $J$ , eq.(115), on a wave function  
 $\Psi$  is

$$J^{\mu x}_{\nu x'} \Psi^{\nu x'} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{pmatrix} \Psi_x \\ i\hbar\Psi_x^* \end{pmatrix} = \begin{pmatrix} i\Psi_x \\ i\hbar(i\Psi_x)^* \end{pmatrix}, \quad (173)$$

555 which indicates that  $J$  plays the role of multiplication by  $i$ , that is, when acting on a point  $\Psi$  the action  
 556 of  $J$  is  $\Psi \xrightarrow{J} i\Psi$ .

**Dirac notation** — We can at this point introduce the Dirac notation to represent the wave functions  $\Psi_x$  as vectors  $|\Psi\rangle$  in a Hilbert space. The scalar product  $\langle\Psi_1|\Psi_2\rangle$  is defined using the metric  $G$  and the symplectic form  $\Omega$ ,

$$\langle\Psi_1|\Psi_2\rangle \stackrel{\text{def}}{=} \frac{1}{2\hbar} \int dx dx' (\Psi_{1x}, i\hbar\Psi_{1x}^*) (G + i\Omega) \begin{pmatrix} \Psi_{2x'} \\ i\hbar\Psi_{2x'}^* \end{pmatrix}. \quad (174)$$

A straightforward calculation gives

$$\langle\Psi_1|\Psi_2\rangle = \int dx \Psi_1^* \Psi_2. \quad (175)$$

The map  $\Psi_x \leftrightarrow |\Psi\rangle$  is defined by

$$|\Psi\rangle = \int dx |x\rangle \Psi_x \quad \text{where} \quad \Psi_x = \langle x|\Psi\rangle, \quad (176)$$

where, in this “position” representation, the vectors  $\{|x\rangle\}$ , form a basis that is orthogonal and complete,

$$\langle x|x'\rangle = \delta_{xx'} \quad \text{and} \quad \int dx |x\rangle \langle x| = \hat{1}. \quad (177)$$

**Hermitian and unitary operators** — The bilinear Hamilton functionals  $\tilde{Q}[\Psi, \Psi^*]$  with kernel  $\hat{Q}(x, x')$  in eq.(121) can now be written in terms of a Hermitian operator  $\hat{Q}$  and its matrix elements,

$$\tilde{Q}[\Psi, \Psi^*] = \langle\Psi|\hat{Q}|\Psi\rangle \quad \text{and} \quad \hat{Q}(x, x') = \langle x|\hat{Q}|x'\rangle. \quad (178)$$

The corresponding Hamilton-Killing flows are given by

$$i\hbar \frac{d}{d\lambda} \langle x|\Psi\rangle = \langle x|\hat{Q}|\Psi\rangle \quad \text{or} \quad i\hbar \frac{d}{d\lambda} |\Psi\rangle = \hat{Q}|\Psi\rangle. \quad (179)$$

These flows are described by unitary transformations

$$|\Psi(\lambda)\rangle = \hat{U}_Q(\lambda) |\Psi(0)\rangle \quad \text{where} \quad \hat{U}_Q(\lambda) = \exp\left(-\frac{i}{\hbar} \hat{Q} \lambda\right). \quad (180)$$

**Commutators** — The Poisson bracket of two Hamiltonian functionals  $\tilde{U}[\Psi, \Psi^*]$  and  $\tilde{V}[\Psi, \Psi^*]$ ,

$$\{\tilde{U}, \tilde{V}\} = \int dx \left( \frac{\delta\tilde{U}}{\delta\Psi_x} \frac{\delta\tilde{V}}{\delta i\hbar\Psi_x^*} - \frac{\delta\tilde{U}}{\delta i\hbar\Psi_x^*} \frac{\delta\tilde{V}}{\delta\Psi_x} \right),$$

can be written in terms of the commutator of the associated operators, then

$$\{\tilde{U}, \tilde{V}\} = \frac{1}{i\hbar} \langle\Psi|[\hat{U}, \hat{V}]|\Psi\rangle. \quad (181)$$

Thus the Poisson bracket is the expectation of the commutator. This *identity* is much sharper than Dirac’s pioneering discovery that the quantum commutator of two  $q$ -variables is *analogous* to the Poisson bracket of the corresponding classical variables. Further parallels between the geometric and the Hilbert space formulation of QM can be found in [50]-[58].

### 13. Remarks on ED and Quantum Bayesianism

Having discussed the ED approach in some detail it is now appropriate to comment on how ED differs from the interpretations known as Quantum Bayesianism [16]-[18] and its closely related descendant QBism [19][20]; for simplicity, I shall refer to both as QB. Both ED and QB adopt an epistemic degree-of-belief concept of probability but there are important differences:



567 (a) QB adopts a personalistic de Finetti type of Bayesian interpretation while ED adopts an  
 568 impersonal entropic Bayesian interpretation somewhat closer but not identical to Jaynes' [12]-[15].  
 569 In ED the probabilities do not reflect the subjective beliefs of any particular person. They are tools  
 570 designed to assist us in those all too common situations in which we are confused and due to insufficient  
 571 information we do not know what to believe. The probabilities will then provide guidance as to what  
 572 agents ought to believe if only they were ideally rational. More explicitly, probabilities in ED describe  
 573 the objective degrees of belief of ideally rational agents who have been supplied with the maximal  
 574 allowed information about a particular quantum system.

575 (b) ED derives or reconstructs the mathematical framework of QM — it explains where the  
 576 symplectic, metric, and complex structures, including Hilbert spaces and time evolution come from.  
 577 In contrast, at its current stage of development QB consists of appending a Bayesian interpretation  
 578 to an already existing mathematical framework. Indeed, assumptions and concepts from quantum  
 579 information are central to QB and are implicitly adopted from the start. For example, a major QB  
 580 concern is the justification of the Born rule starting from the Hilbert space framework while ED starts  
 581 from probabilities and its goal is to justify the construction of wave functions; the Born rule follows as  
 582 a trivial consequence.

583 (c) ED is an application of entropic/Bayesian inference. Of course, the choices of variables  
 584 and of the constraints that happen to be physically relevant are specific to our subject matter —  
 585 quantum mechanics — but the inference method itself is of universal applicability. It applies to  
 586 electrons just as well as to the stock market or to medical trials. In contrast, in QB the personalistic  
 587 Bayesian framework is not of universal validity. For those special systems that we call 'quantum'  
 588 the inference framework is itself modified into a new "Quantum-Bayesian coherence" in which the  
 589 standard Bayesian inference must be supplemented with concepts from quantum information theory.  
 590 The additional technical ingredient is a hypothetical structure called a "symmetric informationally  
 591 complete positive-operator-valued measure". In short, Born's Rule is not derived but constitutes an  
 592 addition beyond the raw probability theory.

593 (d) QB is an anti-realist neo-Copenhagen interpretation; it accepts complementarity. (Here  
 594 complementarity is taken to be the common thread that runs through all Copenhagen interpretations.)  
 595 Probabilities in QB refer to the outcomes of experiments and not to ontic pre-existing values. In  
 596 contrast, in ED probabilities refer to ontic positions — including the ontic positions of pointer variables  
 597 — and thus contributes to solve the problem of quantum measurement (see [21][22]).

#### 598 14. Some final remarks

599 We conclude with a summary of the main assumptions:

- 600 ● Particles have definite but unknown positions and follow continuous trajectories.
- 601 ● The probability of a short step is given by the method of maximum entropy subject to a drift  
 602 potential constraint that introduces directionality and correlations, plus gauge constraints  
 603 that account for external electromagnetic fields.
- 604 ● The accumulation of short steps requires a notion of time as a book-keeping device. This  
 605 involves the introduction of the concept of an instant and a convenient definition of the  
 606 duration between successive instants.
- 607 ● The e-phase space  $\{\rho, \Phi\}$  has a natural symplectic geometry that results from treating the  
 608 pair  $(\rho_x, \Phi_x)$  as canonically conjugate variables.
- 609 ● The information geometry of the space of probabilities is extended to the full e-phase space  
 610 by imposing the latter be spherically symmetric.
- 611 ● The drift potential constraint is updated instant by instant in such a way as to preserve both  
 612 the symplectic and metric geometries of the e-phase space.

613 The resulting entropic dynamics is described by the Schrödinger equation. Different sub-quantum  
 614 Brownian motions all lead to the same emergent quantum mechanics. In previous work we dealt with

615 an Einstein-Smoluchowski process; here we have explored an Ornstein-Uhlenbeck process. Other  
616 “fractional” Brownian motions might be possible but have not yet been studied.

617 A natural question is whether these different sub-quantum Brownian motions might have  
618 observable consequences. At this point our answer can only be tentative. To the extent that we  
619 have succeeded in deriving QM and not some other theory one should not expect deviations in the  
620 predictions for the standard experiments that are the subject of the standard quantum theory — at least  
621 not in the nonrelativistic regime. As the ED program is extended to other regimes involving higher  
622 energies and/or gravity it is quite possible that those different sub-quantum motions might not be  
623 empirically equivalent.

624 ED achieves ontological clarity by sharply separating the ontic elements — the positions of  
625 particles — from the epistemic elements — the probabilities  $\rho$  and the phases  $\Phi$ . ED provides a  
626 dynamics of probabilities and not a dynamics of particles. Of course, if probabilities at one instant are  
627 large in one place and at a later time they are large in some other place one infers that the particles  
628 must have moved — but nothing in ED describes what it is that has pushed the particles around. ED is  
629 a mechanics without a mechanism.

630 We can elaborate on this point from a different direction. The empirical success of ED suggests  
631 that its epistemic probabilities are in agreement with ontic features of the physical world. It is highly  
632 desirable to clarify the precise nature of this agreement. Consider, for example, a fair die. Its property  
633 of being a perfect cube is an ontic property of the die which is reflected at the epistemic level in the  
634 equal assignment of probabilities to each face of the die. In this example we see that the epistemic  
635 probabilities achieve objectivity, and therefore usefulness, by corresponding to something ontic. The  
636 situation in ED is similar except for one crucial aspect. The ED probabilities are objective and they  
637 are empirically successful. They must therefore reflect something real. However, it is not yet known  
638 what those underlying ontic properties might possibly be. Fortunately, for the purposes of making  
639 predictions knowing those epistemic probabilities is all we need.

640 The trick of embedding the e-phase space  $T^*\mathbf{P}$  in a *flat* vector space  $T^*\mathbf{P}^{+1}$  is clever but optional.  
641 It allows one to make use of the calculational advantages of linearity. This recognition that Hilbert  
642 spaces are not fundamental is one of the significant contributions of the entropic approach to our  
643 understanding of QM. The distinction — whether Hilbert spaces are necessary in principle as opposed  
644 to merely convenient in practice — is not of purely academic interest. It can be important in the search  
645 for a quantum theory that includes gravity: Shall we follow the usual approaches to quantization  
646 that proceed by replacing classical dynamical variables by an algebra of linear operators acting on  
647 some abstract space? Or, in the spirit of an entropic dynamics, shall we search for an appropriately  
648 constrained dynamics of probabilities and information geometries? First steps towards formulating a  
649 first-principles theory along this lines are given in [70][91].

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