On a class of Humbert-Hermite polynomials

M. A. Pathan[1] and Waseem A. Khan[2]

[1]Centre for Mathematical and Statistical Sciences (CMSS), Peechi P.O., Thrissur, Kerala-680653, India

[2]Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India

E-mail: mapathan@gmail.com, waseem08_khan@rediffmail.com

Abstract. A unified presentation of a class of Humbert's polynomials in two variables which generalizes the well known class of Gegenbauer, Humbert, Legendre, Chebycheff, Pincherle, Horadam, Kinnsy, Horadam-Pethe, Djordjević , Gould, Milovanović and Djordjević, Pathan and Khan polynomials and many not so called 'named' polynomials has inspired the present paper and the authors define here generalized Humbert-Hermite polynomials of two variables. Several expansions of Humbert-Hermite polynomials, Hermite-Gegenbaurer (or ultraspherical) polynomials and Hermite-Chebyshev polynomials are proved.

Keywords: Hermite polynomials, Humbert polynomials, Gegenbauer polynomials, Chebyshev polynomials, Pathan-Khan polynomials, hypergeometric function.
2010 Mathematics Subject Classification: 33C45, 33C55, 33C99.

1. Introduction

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [3] and [7] defined as

$$H_n(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}, \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y=-1$ and x is replaced by $2x$.

Next, we recall the definition of N-variable generalized Hermite polynomials $H_n(\{x\}_1^N)$ defined by Dattoli et al. [6, p.602]:

$$\exp\sum_{s=1}^{N}x_s t^s=\sum_{n=0}^{\infty}H_n(\{x\}_1^N)\frac{t^n}{n!}, \tag{1.3}$$

where $\{x\}_1^N=x_1,x_2,...,x_N$.

Generalized Hermite polynomials $H_n(\{x\}_1^N)$ for $N = 3$ also belong to those of Bell type as shown in [8, p.403(26)]. The Gould-Hooper polynomials $g_n^m(x,y)$ (see [5] and [10]) is a special case of (1.3). The notation $H_n^m(x,y)$ or $g_n^m(x,y)$ was given by Dattoli et al. [5]. These are specified by

$$e^{xt+yt^m}=\sum_{n=0}^{\infty}H_n^m(x,y)\frac{t^n}{n!} \tag{1.4}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2019 doi:10.20944/preprints201909.0068.v1

Another generalization of Hermite polynomials which we wish to consider in this paper is given by $H_{n,m,\nu}(x,y)$ in the form of the generating function (see [16])

$$e^{\nu(x+y)t-(xy+1)t^m} = \sum_{n=0}^{\infty} H_{n,m,\nu}(x,y)\frac{t^n}{n!}, \tag{1.5}$$

which reduces to the ordinary Hermite polynomials $H_n(x)$ when $\nu = 2, x = 0$ or $\nu = 2, y = 0$.

We draw attention to familiar generating relations given by

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.6}$$

where $P_n(x)$ is Legendre's polynomial of first kind.

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.7}$$

where $U_n(x)$ is Chebychev polynomial of second kind.

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n, \tag{1.8}$$

where $C_n^{\nu}(x)$ is Gegenbauer's polynomial.

$$(1-mxt+t^m)^{-\nu} = \sum_{n=0}^{\infty} h_{n,m}^{\nu}(x)t^n, \tag{1.9}$$

$$h_{n,m}^{\nu}(x) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k(\nu)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!},$$

where $h_{n,m}^{\nu}(x)$ is Humbert polynomial and m is a positive integer. The Pochammer symbol $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \left[\begin{array}{ll} 1 & \text{if } n=0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n=1,2,3\cdots \end{array} \right.$$

In 1965, Gould [11] gave the following generating relation

$$(c-mxt+yt^m)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,c)t^n, \tag{1.10}$$

where m is a positive integer and other parameters are unrestricted in general. $P_n(m,x,y,p,c)$ is defined explicitly by [11, p.699]:

$$P_n(m,x,y,p,c) = \sum_{k=0}^{[\frac{n}{m}]} \left(\begin{array}{c} p \\ k \end{array}\right)\left(\begin{array}{c} p-k \\ n-mk \end{array}\right) c^{p-n+(m-1)k}y^k(-mx)^{n-mk}. \tag{1.11}$$

In 1989, Sinha [19] gave the following generating relation

$$\left[1-2xt+t^2(2x-1)\right]^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x)t^n, \tag{1.12}$$

where

$$S_n^{\nu}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}(2x-1)^k}{k!(n-2k)!}, \tag{1.13}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2019 doi:10.20944/preprints201909.0068.v1

$S_n^{\nu}(x)$ is the generalization of Shrestha polynomial $S_n(x)$ (see [16]).

In 1991, Milovanović and Djordjević [14](see also [15])gave the following generating relation

$$(1-2xt+t^m)^{-\lambda}=\sum_{n=0}^{\infty}p_{n,m}^{\lambda}(x)t^n, \tag{1.14}$$

where $m\in\mathbb{N}$ and $\lambda>-\frac{1}{2}$ and

$$p_{n,m}^{\lambda}(x)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(\lambda)_{n-(m-1)_k}(2x)^{n-mk}}{k!(n-mk)!}. \tag{1.15}$$

It is to be noted that the polynomials represented by $p_{n,1}^{\lambda}(x)$, $p_{n,2}^{\lambda}(x)$ and $p_{n,3}^{\lambda}(x)$ are known as Horadam polynomials [12], Gegenbauer polynomials and Horadam-Pethe polynomials [13], respectively.

Many interesting generalizations to these polynomials appeared in the literature. In particular in 1997, Pathan and Khan [16,p.54] generalized these polynomials and gave the following generating relation

$$[c-axt+bt^m(2x-1)^d]^{-\nu}=\sum_{n=0}^{\infty}p_{n,m,a,b,c,d}^{\nu}(x)t^n$$

$$=\sum_{n=0}^{\infty}\Theta_n(x)t^n, \tag{1.16}$$

where

$$\Theta_n(x)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^kc^{-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k}{k!(n-mk)!}. \tag{1.17}$$

Djordjević [9] provided a generalization of various polynomials of two variables in the form

$$[1-2(x+y)t+t^m(2xy+1)]^{-\alpha}=\sum_{n=0}^{\infty}G_n^{\alpha,m}(x,y)t^n, \tag{1.18}$$

where

$$G_n^{\alpha,m}(x,y)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(\alpha)_{n-(m-1)k}(2x+2y)^{n-mk}(2xy+1)^k}{k!\ (n-mk)!}. \tag{1.19}$$

Note that $G_n^{1,m}(x,y)=C_n^m(x,y)$ and $G_n^{1/2,m}(x,y)=P_n^m(x,y)$ where $C_n^m(x,y)$ and $U_n^m(x,y)$ are Chebyshev and Legendre polynomials of two variables, respectively.

For $m=2$, $G_n^{\alpha,m}(x,y)$ reduces to a polynomial studied by Dave [4]. For $m=2$ and $y=0$, $G_n^{\alpha,m}(x,y)$ reduces to Gegenbauer polynomial and for $m=3$ and $y=0$, $G_n^{\alpha,m}(x,y)$ are Horadam-Pethe polynomials [13]. Further, for $y=0$, $G_n^{\alpha,m}(x,y)$ reduces to a polynomial $p_{n,m}^{\alpha}(x)$ studied by Milovanović and Djordjević ([14]and [15]).

A generalization and unification of various polynomials mentioned above is provided by the definition of generalized Humbert polynomials in two variables given recently by Pathan and Khan [17] which has the generating function

$$[a-(bx+cy)t+dt^m(exy-1)^g]^{-h}=\sum_{n=0}^{\infty}Q_{n,m,g,h}^{a,b,c,d,e}(x,y)t^n=\sum_{n=0}^{\infty}Q_n(x,y)t^n, \quad (1.20)$$

where $m\in\mathbb{N}$, $h>0$ and the other parameters are unrestricted in general.
In (1.20), if we put $a=1$, $b=c=2$, $d=-1$, $e=-2$ and $g=1$, then we get a generating relation (1.18) studied by Djordjević [9]. For $y=1$, $e=2$ and $c=0$, we get a generating relation (1.16) studied by Pathan and Khan [16]. For $a=1$, $b=2$, $c=0$, $d=1$ and $g=0$, we get a generating relation (1.14) studied by Milovanović - Djordjević [15]. For $a=1$, $b=2$, $m=2$, $y=1$, $e=2$ and $g=1$, we get a polynomial defined by Sinha [19] and for $c=0$, $g=0$, $d=y$ and $h=-p$, we get a generating relation (1.4) given by Gould [11]. Some more interesting special cases which are recorded by G.B. Djordjević and G.V. Milovanović in [10] can be established similarly.

2. On a class of Humbert-Hermite polynomials

A generalization and unification of various polynomials mentioned above is provided by the definition of generalized Humbert-Hermite polynomials ${}_HG_n^{\nu,\alpha,m}(x,y)$ in two variables which has the generating function

$$[1-2(x+y)t+t^m(2xy+1)]^{-\nu}e^{\alpha(x+y)t-(xy+1)t^m}=\sum_{n=0}^{\infty}{}_HG_n^{\nu,\alpha,m}(x,y)t^n, \quad (2.1)$$

where $m\in\mathbb{N}$, $\alpha,\nu>0$ and the other parameters are unrestricted in general. This is interesting since, as will be shown, the polynomials ${}_HG_n^{\nu,\alpha,m}(x,y)$ contain a number of known polynomials (see [4], [9], [10], [11], [12], [13], [14], [16], [17] and [18]).

Using the definitions of $H_{n,m,\nu}(x,y)$ and $G_n^{\alpha,m}(x,y)$ given by (1.5) and (1.18) in (2.1), we find the representation

$${}_HG_n^{\nu,\alpha,m}(x,y)=\sum_{k=0}^{n}\frac{n!H_{k,m,\alpha}(x,y)G_{n-k}^{\nu,m}(x,y)}{k!}. \quad (2.2)$$

Some special cases of (2.2) are

$${}_HG_n^{\nu,1,m}(x,y)={}_HC_n^{\nu,m}(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)C_{n-k}^{\nu,m}(x,y)}{k!}.$$

Here ${}_HC_n^{\nu,m}(x,y)$ are Hermite-Gegenbaurer polynomials of two variables.

$${}_HC_n^{1,m}(x,y)={}_HU_n^m(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)U_{n-k}^m(x,y)}{k!},$$

where ${}_HU_n^m(x,y)$ are Hermite-Chebychev polynomials of two variables.

$${}_HC_n^{1/2,m}(x,y)={}_HP_n^m(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)P_{n-k}^m(x,y)}{k!},$$

where ${}_HP_n^m(x,y)$ are Hermite-Legendre polynomials of two variables.

As a special case, let $y=0$ and $\alpha=2$ be chosen in (2.1) so that generalized Humbert-Hermite polynomial ${}_HG_n^{\nu,\alpha,m}(x,y)$ of two variables reduces to Humbert-Hermite polynomial ${}_HG_n^{\nu,2,m}(x,0)={}_HG_n^{\nu,m}(x)$ of one variable. Then (2.1) yields the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 6 September 2019 doi:10.20944/preprints201909.0068.v1

generating function

$$[1-2xt+t^m]^{-\nu}e^{2xt-t^m}=\sum_{n=0}^{\infty}{}_HG_n^{\nu,m}(x)t^n. \tag{2.3}$$

Furthermore, the Hermite-Gegenbaurer (or ultraspherical) polynomials ${}_HC_n^{\nu,2}(x)={}_HC_n^{\nu}(x)$ of one variable, for nonnegative integer ν are given by

$$e^{2xt-t^2}(1-2xt+t^2)^{-\nu}=\sum_{n=0}^{\infty}{}_HC_n^{\nu}(x)\frac{t^n}{n!}. \tag{2.4}$$

Letting $\nu=1/2$ and $\nu=1$ in (2.4) gives

$$e^{2xt-t^2}(1-2xt+t^2)^{-1/2}=\sum_{n=0}^{\infty}{}_HP_n(x)\frac{t^n}{n!}, \tag{2.5}$$

where ${}_HP_n(x)$ are Hermite-Legendre polynomials and

$$e^{2xt-t^2}(1-2xt+t^2)^{-1}=\sum_{n=0}^{\infty}{}_HU_n(x)\frac{t^n}{n!}, \tag{2.6}$$

where ${}_HU_n(x)$ are Hermite-Chebyshev polynomials.

3. On expansions of Hermite-Chebyshev and Hermite-Gegenbaurer polynomials

In this section, we prove several theorems on the expansions of Hermite-Gegenbaurer and Hermite-Chebyshev polynomials of two variables. We will start with (2.1), (2.3) and a special case of (2.1) for $\nu=1$,

$$[1-2(x+y)t+t^m(2xy+1)]^{-1}e^{\alpha(x+y)t-(xy+1)t^m}=\sum_{n=0}^{\infty}{}_HU_n^{\alpha,m}(x,y)\frac{t^n}{n!}, \tag{3.1}$$

which will be used in obtaining the corollaries of the following theorem.

Theorem 3.1. For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$

$$\sum_{r=0}^{n}\frac{H_r^m(\alpha k(x+y),-k(xy+1))G_{n-r}^{\nu k,m}(x,y)}{r!}$$
$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots{}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.2}$$

Proof. The definition of ${}_HG_n^{\nu,\alpha,m}(x,y)$ given in (2.1) can be written as

$$\left[[1-2(x+y)t+t^m(2xy+1)]^{-\nu}e^{\alpha(x+y)t-(xy+1)t^m}\right]^k$$
$$=[1-2(x+y)t+t^m(2xy+1)]^{-\nu k}e^{\alpha k(x+y)t-k(xy+1)t^m}=\left[\sum_{n=0}^{\infty}{}_HG_n^{\nu,\alpha,m}(x,y)\frac{t^n}{n!}\right]^k.$$

Using (1.4), we can write

$$e^{\alpha k(x+y)t-k(xy+1)t^m}=\sum_{r=0}^{\infty}H_r^m(\alpha k(x+y),-k(xy+1))\frac{t^r}{r!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty} G_n^{\nu k,m}(x,y)t^n \sum_{r=0}^{\infty} H_r^m(\alpha k(x+y), -k(xy+1))\frac{t^r}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots {}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}t^n.$$

An application of manipulation of series yields

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{H_r^m(\alpha k(x+y), -k(xy+1))G_{n-r}^{\nu k,m}(x,y)}{r!}t^n$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots {}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.1. Setting $\nu = 1$ in Theorem 3.1, the result reduces to

Corollary 3.1. For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$

$$\sum_{r=0}^{n} \frac{H_r^m(\alpha k(x+y), -k(xy+1))C_{n-r}^{k,m}(x,y)}{r!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HU_{n_1}^{\alpha,m}(x,y)\,{}_HU_{n_2}^{\alpha,m}(x,y)\cdots {}_HU_{n_k}^{\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.3}$$

Remark 3.2. Setting $\nu = 0$ in Theorem 3.1, the result reduces to

Corollary 3.2. For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$

$$\frac{H_n^m(\alpha k(x+y), -k(xy+1))}{n!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^{\alpha,m}(x,y)H_{n_2}^{\alpha,m}(x,y)\cdots H_{n_k}^{\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.4}$$

Remark 3.3. Setting $\alpha = m = 2$, $\nu, y = 0$ in Theorem 3.1, the result reduces to known result of Batahan and Shehata [2,p.50.,Eq.(2.1)].

Corollary 3.3. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$

$$\sum_{r=0}^{[\frac{n}{2}]} \frac{(-k)^r(2kx)^{n-2r}}{(n-2r)r!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)}{n_1!n_2!\cdots n_k!}. \tag{3.5}$$

Theorem 3.2. For $k \in \mathbb{N}$ and $X, Y \in \mathbb{C}$

$$\sum_{r=0}^{n} \frac{H_r^m(\alpha k(X+Y), -k(XY+1))G_{n-r}^{\nu k,m}(X,Y)}{r!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y)\cdots {}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}, \tag{3.6}$$

where $X = \sum_{i=0}^{k} x_i$ and $Y = \sum_{j=0}^{k} y_j$.

Proof. The definition of ${}_HG_n^{\nu,\alpha,m}(x,y)$ can be written as

$$\left[[1-2(X+Y)t+t^m(2XY+1)]^{-\nu}\, e^{\alpha(X+Y)t-(XY+1)t^m}\right]^k$$

$$=[1-2(X+Y)t+t^m(2XY+1)]^{-\nu k}\, e^{\alpha k(X+Y)t-k(XY+1)t^m}$$

$$=\left[\sum_{n=0}^{\infty} {}_HG_n^{\nu,\alpha,m}(x_1+x_2+\cdots+x_k, y_1+y_2+\cdots+y_k)\frac{t^n}{n!}\right]^k.$$

Using (1.4), we can write

$$e^{\alpha k(X+Y)t-k(XY+1)t^m}=\sum_{r=0}^{\infty} H_n^m(\alpha k(X+Y),-k(XY+1))\frac{t^r}{r!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty} G_n^{\nu k,m}(X,Y)t^n \sum_{r=0}^{\infty} H_n^m(\alpha k(X+Y),-k(XY+1))\frac{t^r}{r!}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y)\ldots {}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}t^n.$$

An application of manipulation of series yields

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{H_r^m(\alpha k(X+Y),-k(XY+1))G_{n-r}^{\nu k,m}(X,Y)}{r!}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y\cdots {}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}t^n.$$

Now equating coefficients of t on both sides of the resulting equation will give the required result. ☐

Remark 3.4. Setting $\nu=1$ in Theorem 3.2, the result reduces to

Corollary 3.4. For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$, we have

$$\sum_{r=0}^{n} \frac{H_r^m(\alpha k(X+Y),-k(XY+1))C_{n-r}^{k,m}(X,Y)}{r!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HU_{n_1}^{\alpha,m}(X,Y)\,{}_HU_{n_2}^{\alpha,m}(X,Y)\cdots {}_HU_{n_k}^{\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}. \tag{3.7}$$

Remark 3.5. Setting $\nu=0$ in Theorem 3.2, the result reduces to

Corollary 3.5. For $k\in\mathbb{N}$ and $X,Y\in\mathbb{C}$, we have

$$\frac{H_n^m(\alpha k(X+Y),-k(XY+1))}{n!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^{\alpha,m}(X,Y)H_{n_2}^{\alpha,m}(X,Y)\cdots H_{n_k}^{\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}. \tag{3.8}$$

Remark 3.6. Setting $\alpha=m=2$, $\nu=0$, $x_2=\cdots x_k=0$, $y_1=\cdots y_k=0$ and replacing x_1 by x in Theorem 3.2, the result reduces to known result of Batahan and Shehata [2,p.51.,Eq.(2.4)].

Corollary 3.6. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$, we have

$$\sum_{r=0}^{[\frac{n}{2}]} \frac{(-k)^r (2kx)^{n-2r}}{(n-2r)r!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)}{n_1!n_2!\cdots n_k!}. \tag{3.9}$$

Theorem 3.3. For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$, we have

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (\nu k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!}$$
$$= \sum_{n_1+n_2+\cdots+n_k=n} G_{n_1}^{\nu,m}(x,y) G_{n_2}^{\nu,m}(x,y) \cdots G_{n_k}^{\nu,m}(x,y). \tag{3.10}$$

Proof. Using the power series of $[1-2(x+y)t+t^m(2xy+1)]^{-k}$ and making the necessary series arrangements gives

$$[1-2(x+y)t+t^m(2xy+1)]^{-\nu k} = \sum_{n=0}^{\infty} \sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (\nu k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!} t^n.$$

In addition to this, we can write

$$[1-2(x+y)t+t^m(2xy+1)]^{-k} = \left[[1-2(x+y)t+t^m(2xy+1)]^{-\nu}\right]^k = \left[\sum_{n=0}^{\infty} G_n^{\nu,m}(x,y) t^n\right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} G_{n_1}^{\nu,m}(x,y) G_{n_2}^{\nu,m}(x,y) \cdots G_{n_k}^{\nu,m}(x,y) t^n.$$

Now equating coefficients of t on both sides of the resulting equation will give the required result. □

Remark 3.7. For $\nu = 1$ in Theorem 3.3, the result reduces to

Corollary 3.7. For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$, we have

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!}$$
$$= \sum_{n_1+n_2+\cdots+n_k=n} U_{n_1}^{m}(x,y) U_{n_2}^{m}(x,y) \cdots U_{n_k}^{m}(x,y). \tag{3.11}$$

Acknowledgement: The first author M. A. Pathan would like to thank the Department of Science and Technology, Government of India, for the financial assistance for this work under project number SR/S4/MS:794/12.

References

[1] Andrews, L. C, Special functions for engineers and mathematicians, Macmillan Co. New York, 1985.

[2] Batahan, R. S and Shehata, A, Hermite-Chebyshev polynomials with their generalization form, J. Math. Sci. Adv. Appl., 29 (2014), 47-59.

[3] Bell, E. T, Exponential polynomials, Ann. of Math., 35(1934), 258-277.

[4] Dave, C. K, Another generalization of Gegenbauer polynomials, J. Indian Acad. Math., 1, 2 (1978), 42-45.

[5] Dattoli, G, Chiccoli, C, Lorenzutta, S, Maimo, G and Torre, A, Generalized Bessel functions and generalized Hermite polynomials, J. Math. Anal. Appl., 178(1993), 509-516.
[6] Dattoli, G, Maimo, G, Torre, A and Cesarano, C, Generalized Hermite polynomials and super-Gaussian forms, J. Math. Anal. Appl., 233(1996), 597-609.
[7] Dattoli, G, Lorenzutta, S and Cesarano, C, Finite sums and generalized forms of Bernoulli polynomials, Rendiconti di Mathematica, 19(1999), 385-391.
[8] Dattoli, G, Torre, A and Lorenzutta, S, Operational identities and properties of ordinary and generalized special functions, J. Math. Anal. Appl., 236(1999), 399-414.
[9] Djordjević , G. B, A generalization of Gegenbauer polynomial with two variables, (To appear in Indian J. Pure Appl. Math).
[10] Djordjević , G. B and Milovanović ,G.V, Special Classes of Polynomials, University of Nis, Faculty of Technology, Leskovac, 2014.
[11] Gould, H. W, Inverse series relation and other expansions involving Humbert polynomials, Duke Math. J., 32(1965), 697-711.
[12] Horadam, A, Gegenbauer polynomials revisited, The Fibonacci Quart., 23 (1985), 295-299, 307.
[13] Horadam, A and Pethe, S, Polynomials associated with Gegenbauer polynomials, The Fibonacci Quart., 19(1981), 393-398.
[14] Milovanović , G. V and Djordjević , G. B, On some properties of Humberts polynomials-I, Fibonacci Quart., 25 (1987), 356-360
[15] Milovanović , G. V and Djordjević , G. B, On some properties of Humberts polynomials-II, Facta Universitatis (Nis) Math. Inform., 6 (1991), 23-30.
[16] Pathan, M. A and Khan, M. A, On polynomials associated with Humberts polynomials, Publ. Inst. Math., (Beograd), 62(76)(1997), 53-62.
[17] Pathan, M. A and Khan, N. U, On unified presentation of a class of generalized Humberts polynomials of two variables, ROMAI J., 11(2)(2015), 185-199.
[18] Shrestha, N. B, Polynomial associated with Legendre polynomials, Nepali Math. Sci. Rep. Triv. Univ., 2:1 (1977).
[19] Sinha, S. K, On a polynomial associated with Gegenbauer polynomials, Proc. Nat. Acad. Sci., India, 59 (A):III (1989), 439-455.