Comparison of Smallest Eigenvalues for Nabla Fractional Boundary Value Problems

Jagan Mohan Jonnalagadda

Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078, Telangana, India.
email: j.jaganmohan@hotmail.com

Abstract: In this article, we establish the existence of and then compare smallest eigenvalues for nabla fractional boundary value problems involving a fractional difference boundary condition, using the theory of \( u_0 \)-positive operators with respect to a cone in a Banach space.

Key Words: Nabla fractional difference, boundary value problem, cone, \( u_0 \)-positive operator, eigenvalue

AMS Classification: 26A33, 34B09, 39A12.
Comparison of Smallest Eigenvalues for Nabla Fractional Boundary Value Problems

Jagan Mohan Jonnalagadda

Abstract: In this article, we establish the existence of and then compare smallest eigenvalues for nabla fractional boundary value problems involving a fractional difference boundary condition, using the theory of $u_0$-positive operators with respect to a cone in a Banach space.

Key Words: Nabla fractional difference, boundary value problem, cone, $u_0$-positive operator, eigenvalue

AMS Classification: 26A33, 34B09, 39A12

1. Introduction

The theories of Krein–Rutman [30] and Krasnosel’skiĭ [29] have been used by many authors to establish the existence of and then compare smallest eigenvalues of boundary value problems for differential equations [5, 10, 11, 12, 13, 32], difference equations [7, 16], dynamic equations on time scales [6, 19], fractional differential equations [8, 9, 18, 28], and delta fractional difference equations [17, 33, 34].

Motivated by these works, in this paper, we obtain the existence of and then compare smallest eigenvalues for the eigenvalue problems

$$
(\nabla_{\rho}^\alpha u)(t) + \lambda_1 p(t) u(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \quad (1.1)
$$

$$
(\nabla_{\rho}^\beta u)(t) + \lambda_2 q(t) u(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \quad (1.2)
$$

satisfying the boundary conditions

$$
u(a) = (\nabla_{\rho}^\beta u)(b) = 0. \quad (1.3)
$$

Here $1 < \alpha < 2$, $0 \leq \beta \leq 1$, $a$, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, $p, q : \mathbb{N}_{a+2}^b \to (0, \infty)$, $\nabla_{\rho}^\alpha$ and $\nabla_{\rho}^\beta$ are $\alpha$th and $\beta$th-order nabla fractional difference operators, respectively. Observe that the pair of boundary conditions in (1.3) reduces to conjugate [3, 14, 24], right-focal [22] and right-focal type [23] boundary conditions as $\beta \to 0^+$, $\beta \to 1^-$ and $\beta \to (\alpha - 1)$, respectively.

This article is organized as follows: In Section 2, we state the preliminary definitions and results from nabla fractional calculus and the theory of $u_0$-positive operators with respect to a cone in a Banach space. In Section 3, we define the appropriate Banach space and establish the existence of and then compare smallest eigenvalues of (1.1) - (1.3) and (1.2) - (1.3).

\footnote{Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad - 500078, Telangana, India. email: j.jaganmohan@hotmail.com}
2. Preliminaries

Denote the set of all real numbers by \( \mathbb{R} \). For any \( a, b \in \mathbb{R} \) such that \( b - a \in \mathbb{N}_1 \), define
\[
N_a := \{a, a + 1, a + 2, \ldots\} \quad \text{and} \quad N^b_a := \{a, a + 1, a + 2, \ldots, b\}.
\]
Assume that empty sums and products are taken to be 0 and 1, respectively.

**Definition 2.1** (See [4]). The backward jump operator \( \rho : N_a \to N_a \) is defined by
\[
\rho(t) = \max\{a, (t - 1)\}, \quad t \in N_a.
\]

**Definition 2.2** (See [27, 31]). The Euler gamma function is defined by
\[
\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.
\]
Using the reduction formula
\[
\Gamma(z + 1) = z\Gamma(z), \quad \Re(z) > 0,
\]
the Euler gamma function can be extended to the half-plane \( \Re(z) \leq 0 \) except for \( z \neq 0, -1, -2, \ldots \).

**Definition 2.3** (See [15]). For \( t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \) and \( r \in \mathbb{R} \) such that \( (t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \), the generalized rising function is defined by
\[
t^r = \frac{\Gamma(t + r)}{\Gamma(t)}.
\]
We use the convention that if \( t \in \{\ldots, -2, -1, 0\} \) and \( r \in \mathbb{R} \) such that \( (t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\} \), then
\[
t^r := 0.
\]

**Definition 2.4** (See [4]). Let \( u : N_a \to \mathbb{R} \) and \( N \in \mathbb{N}_1 \). The first order backward (nabla) difference of \( u \) is defined by
\[
(\nabla u)(t) := u(t) - u(t - 1), \quad t \in N_{a+1},
\]
and the \( N^{th} \)-order nabla difference of \( u \) is defined recursively by
\[
(\nabla^N u)(t) := \left(\nabla\left(\nabla^{N-1} u\right)\right)(t), \quad t \in N_{a+N}.
\]

**Definition 2.5** (See [15]). Let \( u : N_{a+1} \to \mathbb{R} \) and \( N \in \mathbb{N}_1 \). The \( N^{th} \)-order nabla sum of \( u \) based at \( a \) is given by
\[
(\nabla^{-N} u)(t) := \frac{1}{(N - 1)!} \sum_{s=a+1}^{t} (t - \rho(s))^{N-1}u(s), \quad t \in N_a,
\]
where by convention \( (\nabla^{-0} u)(a) = 0 \). We define \( (\nabla^{-0} u)(t) := u(t) \).

**Definition 2.6** (See [15]). Let \( u : N_{a+1} \to \mathbb{R} \) and \( \nu > 0 \). The \( \nu^{th} \)-order nabla sum of \( u \) based at \( a \) is given by
\[
(\nabla^{-\nu} u)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\nu-1}u(s), \quad t \in N_a,
\]
where by convention \((\nabla_{a}^{-\nu}u)(a) = 0\).

**Definition 2.7** (See [15]). Let \(u : \mathbb{N}_{a+1} \to \mathbb{R}, \nu > 0\) and choose \(N \in \mathbb{N}_{1}\) such that \(N - 1 < \nu \leq N\). Then, we define the \(\nu\)-th order nabla fractional difference by
\[
(\nabla_{a}^{\nu}u)(t) := \left(\nabla^{N}(\nabla_{a}^{-(N-\nu)}u)\right)(t), \quad t \in \mathbb{N}_{a+N}.
\]

**Definition 2.8** (See [29]). Let \(\mathcal{B}\) be a real Banach space. A set \(\mathcal{P} \subset \mathcal{B}\) is called a cone if the following conditions are satisfied:

1. \(\mathcal{P}\) is closed;
2. if \(u, v \in \mathcal{P}\) then \(\alpha u + \beta v \in \mathcal{P}\) for all \(\alpha, \beta \geq 0\);
3. if \(u \in \mathcal{P}\) and \(-u \in \mathcal{P}\) then \(u = 0\).

A cone \(\mathcal{P}\) is solid if the interior, \(\mathcal{P}^{o}\), of \(\mathcal{P}\), is nonempty. A cone is called reproducing if every element \(w \in \mathcal{B}\) can be represented in the form

\[
w = u - v, \quad u, v \in \mathcal{P}.
\]

**Remark 1** (See [29]). Every solid cone is reproducing.

By means of a cone, we define a partial ordering relation in a Banach space as follows.

**Definition 2.9** (See [29]). Let \(\mathcal{P}\) be a cone in a real Banach space \(\mathcal{B}\). For all \(u, v \in \mathcal{B}\), we write \(u \leq v\) with respect to \(\mathcal{P}\) if \(v - u \in \mathcal{P}\).

Further, we also introduce a partial ordering relation on bounded linear operators defined on a Banach space.

**Definition 2.10** (See [29]). Let \(\mathcal{P}\) be a cone in a real Banach space \(\mathcal{B}\) and \(T, S : \mathcal{B} \to \mathcal{B}\) are bounded linear operators. We write \(T \leq S\) with respect to \(\mathcal{P}\) if \(Tu \leq Su\) for all \(u \in \mathcal{P}\).

**Definition 2.11** (See [29]). A bounded linear operator \(T : \mathcal{B} \to \mathcal{B}\) is \(u_0\)-positive with respect to \(\mathcal{P}\) if there exists \(u_0 \in \mathcal{P} \setminus \{0\}\) such that for each \(u \in \mathcal{P} \setminus \{0\}\), there exist positive constants \(k_1(u)\) and \(k_2(u)\) such that \(k_1(u_0) \leq Tu \leq k_2(u_0)\) with respect to \(\mathcal{P}\).

We use the following three theorems to establish our main results.

**Theorem 2.1** (See [29]). Let \(\mathcal{P} \subset \mathcal{B}\) be a solid cone. If \(T : \mathcal{B} \to \mathcal{B}\) is a linear operator such that \(T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^{o}\), then \(T\) is \(u_0\)-positive.

**Theorem 2.2** (See [29]). Let \(\mathcal{B}\) be a real Banach space, \(\mathcal{P} \subset \mathcal{B}\) be a reproducing cone and \(T : \mathcal{B} \to \mathcal{B}\) be a compact, \(u_0\)-positive linear operator. Then, \(T\) has an essentially unique eigenvector in \(\mathcal{P}\), and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.

**Theorem 2.3** (See [25]). Let \(\mathcal{B}\) be a real Banach space, \(\mathcal{P} \subset \mathcal{B}\) be a cone and \(T, S : \mathcal{B} \to \mathcal{B}\) be bounded, linear operators. Assume at least one of the operators \(T\) and \(S\) is \(u_0\)-positive. If \(T \leq S, Tu_1 \geq \lambda_1 u_1\) for some \(u_1 \in \mathcal{P}\) and \(\lambda_1 > 0\), and \(Su_2 \leq \lambda_2 u_2\) for some \(u_2 \in \mathcal{P}\) and \(\lambda_2 > 0\), then \(\lambda_1 \leq \lambda_2\). Furthermore, \(\lambda_1 = \lambda_2\) implies \(u_1\) is a scalar multiple of \(u_2\).
3. Main Results

The author [21] has derived the Green’s function \( G(\beta; t, s) \) for

\[
(\nabla^\alpha_{\rho(a)} u)(t) = 0, \quad t \in \mathbb{N}_a^{b+2},
\]

satisfying (1.3) and also obtained a few of its properties.

\[
G(\beta; t, s) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \left[ \frac{(b-s+1)^{\alpha-1} - (t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \right], & t \in \mathbb{N}_a^{b(s)}, \\
\frac{1}{\Gamma(\alpha)} \left[ \frac{(b-a)^{\alpha-1} - (t-a)^{\alpha-1}}{(b-s+1)^{\alpha-1}} \right], & t \in \mathbb{N}_a^{b}. 
\end{cases}
\]

**Theorem 3.1** (See [21]). The Green’s function \( G(t, s) \) defined in (3.2) satisfies the following properties:

1. \( G(\beta; a, s) = 0 \) for all \( 0 \leq \beta \leq 1 \) and \( s \in \mathbb{N}_a^{b+1} \).
2. \( G(0; b, s) = 0 \) for all \( s \in \mathbb{N}_a^{b+1} \).
3. \( G(\beta; t, a + 1) = 0 \) for all \( 0 \leq \beta \leq 1 \) and \( t \in \mathbb{N}_a^{b} \).
4. \( G(0; t, s) > 0 \) for all \( (t, s) \in \mathbb{N}_a^{b+1} \times \mathbb{N}_a^{b+2} \).
5. \( G(\beta; t, s) > 0 \) for all \( 0 < \beta \leq 1 \) and \( (t, s) \in \mathbb{N}_a^{b+1} \times \mathbb{N}_a^{b+2} \).

Observe that \( u \) is a solution of (1.1) - (1.3) if and only if \( u \) is a solution of the summation equation

\[
u(t) = \lambda_1 \sum_{s=a+2}^b G(\beta; t, s)p(s)u(s), \quad t \in \mathbb{N}_a^{b}.
\]

Similarly, \( u \) is a solution of (1.2) - (1.3) if and only if \( u \) is a solution of the summation equation

\[
u(t) = \lambda_2 \sum_{s=a+2}^b G(\beta; t, s)q(s)u(s), \quad t \in \mathbb{N}_a^{b}.
\]

Denote by

\[
\mathcal{B} = \{ u : \mathbb{N}_a^{b} \to \mathbb{R} \mid u(a) = (\nabla^\beta_{\rho(a)} u(b) = 0) \subseteq \mathbb{R}^{b-a+1} \}
\]

Clearly, \( \mathcal{B} \) is a Banach space equipped with the maximum norm

\[
\| u \| := \max_{t \in \mathbb{N}_a^{b}} |u(t)|.
\]

Define the cone

\[
\mathcal{P} = \{ u \in \mathcal{B} \mid u(t) \geq 0 \text{ for all } t \in \mathbb{N}_a^{b} \}.
\]

Since

\[
\Omega = \{ u \in \mathcal{B} \mid u(t) > 0 \text{ for all } t \in \mathbb{N}_a^{b+1} \} \subseteq \mathcal{P}^0,
\]

\( \mathcal{P} \) is solid and hence it is reproducing. Define the operators

\[
(Tu)(t) = \sum_{s=a+2}^b G(\beta; t, s)p(s)u(s), \quad t \in \mathbb{N}_a^{b},
\]

\[
(Su)(t) = \sum_{s=a+2}^b G(\beta; t, s)q(s)u(s), \quad t \in \mathbb{N}_a^{b}.
\]
Clearly, $T, S : \mathcal{B} \to \mathcal{B}$ are linear. Note that $T$ and $S$ are summation operators on a discrete finite set. Hence, $T$ and $S$ are compact.

**Lemma 3.2.** The operators $T$ and $S$ are $u_0$-positive with respect to $\mathcal{P}$.

**Proof.** We prove this statement for the operator $T$. For this purpose, we apply Theorem 2.1. Clearly, $\mathcal{P} \subset \mathcal{B}$ is a solid cone and $T : \mathcal{B} \to \mathcal{B}$ is a linear operator. It is enough to show that $T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^o$. To see this, let $u \in \mathcal{P} \setminus \{0\}$. Then, there exists a $t_0 \in \mathbb{N}^{b-1}_{a+2}$ such that $u(t_0) > 0$. Since $G(\beta; t, s) > 0$ for all $0 \leq \beta \leq 1$ and $(t, s) \in \mathbb{N}^{b-1}_{a+1} \times \mathbb{N}^b_{a+2}$ and $p(s) > 0$ for all $s \in \mathbb{N}^b_{a+2}$, we have

$$(Tu)(t) = \sum_{s=a+2}^{b} G(\beta; t, s)p(s)u(s) \geq G(\beta; t, t_0)p(t_0)u(t_0) > 0,$$

for all $t \in \mathbb{N}^{b-1}_{a+1}$. So, $Tu \in \Omega \subset \mathcal{P}^o$. The proof is complete. \hfill \qed

**Remark 2.** Let $\lambda_1$ be a nonzero eigenvalue of (1.1) - (1.3). If $u$ is an eigenvector corresponding to $\lambda_1$ of (1.1) - (1.3), then

$$\frac{1}{\lambda_1} u = Tu.$$

So, the eigenvalues of (1.1) - (1.3) are reciprocals of the eigenvalues of (3.5), and conversely.

**Theorem 3.3.** $T$ has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue $\Lambda$ is positive, simple, and larger than the absolute value of any other eigenvalue.

**Proof.** We know that $\mathcal{P}$ is a reproducing cone and $T$ is a compact, $u_0$-positive linear operator. Then, by Theorem 2.2, $T$ has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue $\Lambda$ is positive, simple, and larger than the absolute value of any other eigenvalue. \hfill \qed

**Remark 3.** From the proof of Lemma 3.2, we observe that $(Tu)(t) > 0$ for all $t \in \mathbb{N}^{b-1}_{a+1}$ and hence $Tu \in \mathcal{P}^o$. It follows from Theorem 3.3 that $\Lambda u = Tu$. Thus, we obtain

$$u(t) = \frac{1}{\Lambda}(Tu)(t) > 0,$$

for all $t \in \mathbb{N}^{b-1}_{a+1}$. Therefore, $u \in \Omega \subset \mathcal{P}^o$.

**Theorem 3.4.** Let $p(s) \leq q(s)$ for all $s \in \mathbb{N}^b_{a+2}$. Let $\Lambda_1$ and $\Lambda_2$ be the eigenvalues defined in Theorem 3.3 associated with $T$ and $S$, respectively, with the essentially unique eigenvectors $u_1$ and $u_2$ in $\mathcal{P} \setminus \{0\}$. Then, $\Lambda_1 \leq \Lambda_2$. Furthermore, $\Lambda_1 = \Lambda_2$ if and only if $p(s) = q(s)$ for all $s \in \mathbb{N}^b_{a+2}$.

**Proof.** Let $p(s) \leq q(s)$ for all $s \in \mathbb{N}^b_{a+2}$. Then, for any $u \in \mathcal{P}$ and $t \in \mathbb{N}^b_a$,

$$(Su - Tu)(t) = \sum_{s=a+2}^{b} G(\beta; t, s)(q(s) - p(s))u(s) \geq 0.$$
So, \((Su - Tu) \in \mathcal{P}\) for all \(u \in \mathcal{P}\). That is, \(T \leq S\) with respect to \(\mathcal{P}\). Then, by Theorem 2.3, we obtain \(\Lambda_1 \leq \Lambda_2\).

Now, we prove the second statement of the theorem. If possible, suppose \(p(t_0) < q(t_0)\), for some \(t_0 \in \mathbb{N}_{a+2}^{b-1}\). Since \(u_1 \in \mathcal{P}^o\), we have \(u_1(t_0) > 0\). Then, for all \(t \in \mathbb{N}_{a+1}^{b-1}\),

\[
(Su_1 - Tu_1)(t) = \sum_{s=a+2}^{b} G(\beta; t, s)(q(s) - p(s))u_1(s)
\]

\[
\geq G(\beta; t, t_0)(q(t_0) - p(t_0))u_1(t_0) > 0,
\]

implying that \((Su_1 - Tu_1) \in \Omega \subset \mathcal{P}^o\). So, there exists \(\varepsilon > 0\) such that \((S - T)u_1 - \varepsilon u_1 \in \mathcal{P}\). Hence,

\[
\Lambda_1 u_1 + \varepsilon u_1 = Tu_1 + \varepsilon u_1 \leq Su_1,
\]

which implies

\[
(\Lambda_1 + \varepsilon)u_1 \leq Su_1.
\]

Since \(S \leq S\) and \(Su_2 = \Lambda_2 u_2\), by Theorem 2.3, we obtain

\[
\Lambda_1 + \varepsilon \leq \Lambda_2 \text{ and } \Lambda_1 < \Lambda_2.
\]

Thus, by contrapositive, if \(\Lambda_1 = \Lambda_2\), then \(p(s) = q(s)\) for all \(t \in \mathbb{N}_{a+2}^{b}\). \(\square\)

By Remark 2, the following theorem is an immediate consequence of Theorems 3.3 and 3.4.

**Theorem 3.5.** Assume the hypotheses of Theorem 3.4. Then, there exist smallest positive eigenvalues \(\lambda_1\) and \(\lambda_2\) of (1.1) - (1.3) and (1.2) - (1.3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenvectors corresponding to \(\lambda_1\) and \(\lambda_2\) may be chosen to belong to \(\mathcal{P} \setminus \{0\}\). Then, \(\lambda_1 \geq \lambda_2\). Furthermore, \(\lambda_1 = \lambda_2\) if and only if \(p(s) = q(s)\) for all \(s \in \mathbb{N}_{a+2}^{b}\).

**References**


