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Comparison of Smallest Eigenvalues for Nabla Fractional Boundary Value Problems

Jagan Mohan Jonnalagadda

Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078, Telangana, India.
email: j.jaganmohan@hotmail.com

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Key Words: Nabla fractional difference, boundary value problem, cone, u_0-positive operator, eigenvalue

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1. Introduction

The theories of Krein–Rutman [30] and Krasnosel'skiĭ [29] have been used by many authors to establish the existence of and then compare smallest eigenvalues of boundary value problems for differential equations [5, 10, 11, 12, 13, 32], difference equations [7, 16], dynamic equations on time scales [6, 19], fractional differential equations [8, 9, 18, 28], and delta fractional difference equations [17, 33, 34].

Motivated by these works, in this paper, we obtain the existence of and then compare smallest eigenvalues for the eigenvalue problems

$$\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) + \lambda_1 p(t)u(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{1.1}$$

$$\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) + \lambda_2 q(t)u(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{1.2}$$

satisfying the boundary conditions

$$u(a) = \big(\nabla^{\beta}_{\rho(a)}u\big)(b) = 0. \tag{1.3}$$

Here $1 < \alpha < 2$, $0 \leq \beta \leq 1$, a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, p, $q : \mathbb{N}^{b}_{a+2} \to (0, \infty)$, $\nabla^{\alpha}_{\rho(a)}$ and $\nabla^{\beta}_{\rho(a)}$ are α^{th} and β^{th}-order nabla fractional difference operators, respectively. Observe that the pair of boundary conditions in (1.3) reduces to conjugate [3, 14, 24], right-focal [22] and right-focal type [23] boundary conditions as $\beta \to 0^+$, $\beta \to 1^-$ and $\beta \to (\alpha - 1)$, respectively.

This article is organized as follows: In Section 2, we state the preliminary definitions and results from nabla fractional calculus and the theory of u_0-positive operators with respect to a cone in a Banach space. In Section 3, we define the appropriate Banach space and establish the existence of and then compare smallest eigenvalues of (1.1) - (1.3) and (1.2) - (1.3).

[1]Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad - 500078, Telangana, India. email: j.jaganmohan@hotmail.com

2. Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. For any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$, define

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \text{ and } \mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}.$$

Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1 (See [4]). The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.$$

Definition 2.2 (See [27, 31]). The Euler gamma function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \leq 0$ except for $z \neq 0, -1, -2, \ldots$

Definition 2.3 (See [15]). For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

We use the convention that if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then

$$t^{\overline{r}} := 0.$$

Definition 2.4 (See [4]). Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) := u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) := \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.5 (See [15]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-N} u\big)(t) := \frac{1}{(N-1)!} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{N-1}} u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-N} u\big)(a) = 0$. We define $\big(\nabla_a^{-0} u\big)(t) := u(t)$.

Definition 2.6 (See [15]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\nu-1}} u(s), \quad t \in \mathbb{N}_a,$$

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where by convention $\left(\nabla_a^{-\nu}u\right)(a) = 0$.

Definition 2.7 (See [15]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. Then, we define the ν^{th}-order nabla fractional difference by

$$\left(\nabla_a^{\nu}u\right)(t) := \left(\nabla^N\left(\nabla_a^{-(N-\nu)}u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.8 (See [29]). Let $\mathcal{B}$ be a real Banach space. A set $\mathcal{P} \subset \mathcal{B}$ is called a cone if the following conditions are satisfied:

(1) $\mathcal{P}$ is closed;
(2) if u, $v \in \mathcal{P}$ then $\alpha u + \beta v \in \mathcal{P}$ for all α, $\beta \geq 0$;
(3) if $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ then $u = 0$.

A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^o$, of $\mathcal{P}$, is nonempty. A cone is called reproducing if every element $w \in \mathcal{B}$ can be represented in the form

$$w = u - v, \quad u, v \in \mathcal{P}.$$

Remark 1 (See [29]). Every solid cone is reproducing.

By means of a cone, we define a partial ordering relation in a Banach space as follows.

Definition 2.9 (See [29]). Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. For all u, $v \in \mathcal{B}$, we write $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$.

Further, we also introduce a partial ordering relation on bounded linear operators defined on a Banach space.

Definition 2.10 (See [29]). Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$ and T, $S : \mathcal{B} \to \mathcal{B}$ are bounded linear operators. We write $T \leq S$ with respect to $\mathcal{P}$ if $Tu \leq Su$ for all $u \in \mathcal{P}$.

Definition 2.11 (See [29]). A bounded linear operator $T : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P} \setminus \{0\}$ such that for each $u \in \mathcal{P} \setminus \{0\}$, there exist positive constants $k_1(u)$ and $k_2(u)$ such that $k_1(u_0) \leq Tu \leq k_2(u_0)$ with respect to $\mathcal{P}$.

We use the following three theorems to establish our main results.

Theorem 2.1 (See [29]). *Let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $T : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^o$, then T is u_0-positive.*

Theorem 2.2 (See [29]). *Let $\mathcal{B}$ be a real Banach space, $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone and $T : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive linear operator. Then, T has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.3 (See [25]). *Let $\mathcal{B}$ be a real Banach space, $\mathcal{P} \subset \mathcal{B}$ be a cone and T, $S : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators. Assume at least one of the operators T and S is u_0-positive. If $T \leq S$, $Tu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and $\lambda_1 > 0$, and $Su_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Furthermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. Main Results

The author [21] has derived the Green's function $G(\beta;t,s)$ for

$$\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{3.1}$$

satisfying (1.3) and also obtained a few of its properties.

$$G(\beta;t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}\Big], & t \in \mathbb{N}^{\rho(s)}_{a}, \\ \frac{1}{\Gamma(\alpha)}\Big[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big], & t \in \mathbb{N}^{b}_{s}. \end{cases} \tag{3.2}$$

Theorem 3.1 (See [21]). *The Green's function $G(t,s)$ defined in* (3.2) *satisfies the following properties:*

1. $G(\beta;a,s) = 0$ *for all* $0 \le \beta \le 1$ *and* $s \in \mathbb{N}^{b}_{a+1}$.
2. $G(0;b,s) = 0$ *for all* $s \in \mathbb{N}^{b}_{a+1}$.
3. $G(\beta;t,a+1) = 0$ *for all* $0 \le \beta \le 1$ *and* $t \in \mathbb{N}^{b}_{a}$.
4. $G(0;t,s) > 0$ *for all* $(t,s) \in \mathbb{N}^{b-1}_{a+1} \times \mathbb{N}^{b}_{a+2}$.
5. $G(\beta;t,s) > 0$ *for all* $0 < \beta \le 1$ *and* $(t,s) \in \mathbb{N}^{b}_{a+1} \times \mathbb{N}^{b}_{a+2}$.

Observe that u is a solution of (1.1) - (1.3) if and only if u is a solution of the summation equation

$$u(t) = \lambda_1 \sum_{s=a+2}^{b} G(\beta;t,s)p(s)u(s), \quad t \in \mathbb{N}^{b}_{a}. \tag{3.3}$$

Similarly, u is a solution of (1.2) - (1.3) if and only if u is a solution of the summation equation

$$u(t) = \lambda_2 \sum_{s=a+2}^{b} G(\beta;t,s)q(s)u(s), \quad t \in \mathbb{N}^{b}_{a}. \tag{3.4}$$

Denote by

$$\mathcal{B} = \big\{u : \mathbb{N}^{b}_{a} \to \mathbb{R} \mid u(a) = \big(\nabla^{\beta}_{\rho(a)}u\big)(b) = 0\big\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B}$ is a Banach space equipped with the maximum norm

$$\|u\| := \max_{t \in \mathbb{N}^{b}_{a}} |u(t)|.$$

Define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \ge 0 \text{ for all } t \in \mathbb{N}^{b}_{a}\}.$$

Since

$$\Omega = \{u \in \mathcal{B} \mid u(t) > 0 \text{ for all } t \in \mathbb{N}^{b-1}_{a+1}\} \subset \mathcal{P}^{o},$$

$\mathcal{P}$ is solid and hence it is reproducing. Define the operators

$$(Tu)(t) = \sum_{s=a+2}^{b} G(\beta;t,s)p(s)u(s), \quad t \in \mathbb{N}^{b}_{a}, \tag{3.5}$$

$$(Su)(t) = \sum_{s=a+2}^{b} G(\beta;t,s)q(s)u(s), \quad t \in \mathbb{N}^{b}_{a}. \tag{3.6}$$

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Clearly, T, $S : \mathcal{B} \to \mathcal{B}$ are linear. Note that T and S are summation operators on a discrete finite set. Hence, T and S are compact.

Lemma 3.2. *The operators T and S are u_0-positive with respect to $\mathcal{P}$.*

Proof. We prove this statement for the operator T. For this purpose, we apply Theorem 2.1. Clearly, $\mathcal{P} \subset \mathcal{B}$ is a solid cone and $T : \mathcal{B} \to \mathcal{B}$ is a linear operator. It is enough to show that $T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^o$. To see this, let $u \in \mathcal{P} \setminus \{0\}$. Then, there exists a $t_0 \in \mathbb{N}_{a+2}^{b-1}$ such that $u(t_0) > 0$. Since $G(\beta; t, s) > 0$ for all $0 \leq \beta \leq 1$ and $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^{b}$ and $p(s) > 0$ for all $s \in \mathbb{N}_{a+2}^{b}$, we have

$$\begin{aligned}(Tu)(t) &= \sum_{s=a+2}^{b} G(\beta; t, s)p(s)u(s) \\ &\geq G(\beta; t, t_0)p(t_0)u(t_0) > 0,\end{aligned}$$

for all $t \in \mathbb{N}_{a+1}^{b-1}$. So, $Tu \in \Omega \subset \mathcal{P}^o$. The proof is complete. □

Remark 2. Let λ_1 be a nonzero eigenvalue of (1.1) - (1.3). If u is an eigenvector corresponding to λ_1 of (1.1) - (1.3), then

$$\frac{1}{\lambda_1} u = Tu.$$

So, the eigenvalues of (1.1) - (1.3) are reciprocals of the eigenvalues of (3.5), and conversely.

Theorem 3.3. *T has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue Λ is positive, simple, and larger than the absolute value of any other eigenvalue.*

Proof. We know that $\mathcal{P}$ is a reproducing cone and T is a compact, u_0-positive linear operator. Then, by Theorem 2.2, T has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue Λ is positive, simple, and larger than the absolute value of any other eigenvalue. □

Remark 3. From the proof of Lemma 3.2, we observe that $(Tu)(t) > 0$ for all $t \in \mathbb{N}_{a+1}^{b-1}$ and hence $Tu \in \mathcal{P}^o$. It follows from Theorem 3.3 that $\Lambda u = Tu$. Thus, we obtain

$$u(t) = \frac{1}{\Lambda}(Tu)(t) > 0,$$

for all $t \in \mathbb{N}_{a+1}^{b-1}$. Therefore, $u \in \Omega \subset \mathcal{P}^o$.

Theorem 3.4. *Let $p(s) \leq q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.3 associated with T and S, respectively, with the essentially unique eigenvectors u_1 and u_2 in $\mathcal{P} \setminus \{0\}$. Then, $\Lambda_1 \leq \Lambda_2$. Furthermore, $\Lambda_1 = \Lambda_2$ if and only if $p(s) = q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$.*

Proof. Let $p(s) \leq q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$. Then, for any $u \in \mathcal{P}$ and $t \in \mathbb{N}_{a}^{b}$,

$$(Su - Tu)(t) = \sum_{s=a+2}^{b} G(\beta; t, s)(q(s) - p(s))u(s) \geq 0.$$

So, $(Su - Tu) \in \mathcal{P}$ for all $u \in \mathcal{P}$. That is, $T \leq S$ with respect to $\mathcal{P}$. Then, by Theorem 2.3, we obtain $\Lambda_1 \leq \Lambda_2$.

Now, we prove the second statement of the theorem. If possible, suppose $p(t_0) < q(t_0)$, for some $t_0 \in \mathbb{N}_{a+2}^{b-1}$. Since $u_1 \in \mathcal{P}^o$, we have $u_1(t_0) > 0$. Then, for all $t \in \mathbb{N}_{a+1}^{b-1}$,

$$\begin{aligned}(Su_1 - Tu_1)(t) &= \sum_{s=a+2}^{b} G(\beta; t, s)(q(s) - p(s))u_1(s) \\ &\geq G(\beta; t, t_0)(q(t_0) - p(t_0))u_1(t_0) > 0,\end{aligned}$$

implying that $(Su_1 - Tu_1) \in \Omega \subset \mathcal{P}^o$. So, there exists $\varepsilon > 0$ such that $(S - T)u_1 - \varepsilon u_1 \in \mathcal{P}$. Hence,

$$\Lambda_1 u_1 + \varepsilon u_1 = Tu_1 + \varepsilon u_1 \leq Su_1,$$

which implies

$$(\Lambda_1 + \varepsilon)u_1 \leq Su_1.$$

Since $S \leq S$ and $Su_2 = \Lambda_2 u_2$, by Theorem 2.3, we obtain

$$\Lambda_1 + \varepsilon \leq \Lambda_2 \text{ and } \Lambda_1 < \Lambda_2.$$

Thus, by contrapositive, if $\Lambda_1 = \Lambda_2$, then $p(s) = q(s)$ for all $t \in \mathbb{N}_{a+2}^{b}$. □

By Remark 2, the following theorem is an immediate consequence of Theorems 3.3 and 3.4.

Theorem 3.5. *Assume the hypotheses of Theorem 3.4. Then, there exist smallest positive eigenvalues* λ_1 *and* λ_2 *of* (1.1) - (1.3) *and* (1.2) - (1.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenvectors corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P} \setminus \{0\}$. *Then,* $\lambda_1 \geq \lambda_2$. *Furthermore,* $\lambda_1 = \lambda_2$ *if and only if* $p(s) = q(s)$ *for all* $s \in \mathbb{N}_{a+2}^{b}$.

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