A Generalised CIR Process with Externally-exciting and Self-exciting Jumps and its Applications in Insurance and Finance

Angelos Dassios*
London School of Economics

Jiwook Jang†
Macquarie University

Hongbiao Zhao‡
Shanghai University of Finance and Economics

26th August 2019

Abstract

In this paper, we study a generalised CIR process with externally-exciting and self-exciting jumps, and focus on the distributional properties and applications of this process and its aggregated process. The first and second moments of this jump-diffusion process are used to calculate the insurance premium based on mean-variance principle. The Laplace transform of aggregated process is derived, and this leads to an application for pricing default-free bonds which could capture the impacts of both exogenous and endogenous shocks. Illustrative numerical examples and comparisons with other models are also provided.

Keywords: Contagion risk; Insurance premium; Aggregate claims; Default-free bond pricing; Self-exciting process; Hawkes process; CIR process

JEL Classification: G22 · G13 · C02

*Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Email: a.dassios@lse.ac.uk

†Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney NSW 2109, Australia, Email: jiwook.jang@mq.edu.au

‡School of Statistics and Management, Shanghai University of Finance and Economics, No. 777 Guoding Road, Shanghai 200433, China; Shanghai Institute of International Finance and Economics, No. 777 Guoding Road, Shanghai 200433, China. Email: h.zhao1@lse.ac.uk

1 Introduction

Over recent years, self-exciting processes, especially the Hawkes point processes, have been brought to bear in the modelling and analysis of phenomena as diverse as earth quakes, credit defaults and arrivals of orders in the limit-order books in financial markets. Numerous papers have looked at modelling finance and insurance risk based on them. The theoretical foundation can be traced back to a series of papers written by Hawkes (1971a,b), Hawkes and Oakes (1974), Brémaud and Massoulié (1996, 2001, 2002), and more recently, Dassios and Zhao (2011), Zhu (2013a, 2015), Jaisson and Rosenbaum (2015) and Boumezoued (2016). Early applications concentrated on the fields such as seismology, see Vere-Jones (1975, 1978), Adamopoulos (1976), Ozaki (1979), Vere-Jones and Ozaki (1982) and Ogata (1988).

Recently, rapidly growing applications have emerged in market microstructure and finance, see Chavez-Demoulin et al. (2005), Bowsher (2007), Bauwens and Hautsch (2009), Embrechts et al. (2011), Bacry et al. (2013) and Aït-Sahalia et al. (2015). Moreover, reduced-form models for credit default risk based on these processes can also be found in Errais et al. (2010), Dassios and Zhao (2011) and Aït-Sahalia et al. (2014). On the other hand, Stabile and Torrisi (2010) first applied Hawkes process in the context of insurance risk to study the asymptotic behavior of the infinite- and finite-horizon ruin probabilities. Dassios and Zhao (2012) adopted a generalised version as a claim-arrival process, and estimated the ruin probabilities via importance sampling. Jang and Dassios (2013) followed this work by studying a bivariate version and applied to the insurance premium calculations. In addition, Zhu (2013b) investigated the asymptotics of ruin probabilities based on the large deviation principle.

In particular, these papers related to the aforementioned insurance assumed that interest rates equal zero, except for Jang and Dassios (2013) where the interest rate is assumed to be constant. Previous works dealing with the effect of constant interest rates in terms of premium setting can be found in Léveillé and Garrido (2001), Jang (2004) and Jang and Krvavych (2004). Considering the claim inflation experienced cancels out interest earned, we can ignore the effect of the rate of interest. However, the interest rate might be more variable than the claims themselves. Hence, in this paper, we consider a stochastic interest rate to the aggregate claim amounts. Besides, to further accommodate the clustering effects of claims due to increases in the frequency of natural or man-made disasters, improved models are required to predict claims arising from catastrophic events. For these, we now study a generalised CIR process with externally-exciting and self-exciting jumps, which can be considered as a model extension of Zhu (2013a) or Dassios and Zhao (2017a,b). It is also a generalisation of Jang (2007), where he studied a stochastic interest rate for the aggregate claim amounts using a jump-diffusion process without the self-exciting component.

Since the global financial crisis of 2007, interest rates have been lowered to avoid a great recession, and developed countries have delayed the rises of interest rates due to their fragile economies. However, this low interest rate regime will not continue forever. Also, recent Greece's 'No' vote on the bailout conditions proposed by the relevant international institutions (EU, IMF and ECB) brought about the increases of the yields on the country's government bonds as well as the yields on Italian and Spanish government bonds. Even though Greece and the rest of eurozone reached an agreement that could lead to a third bailout and keep the country in the eurozone, undoubtedly there would be sudden jumps in the yields on above government bonds due to the clustering arrivals of shocks, such as news of failing to reign in their budget deficits and debts. There have been also sudden interest rate rises in the market in the past, for instance, when the UK crashed out of the ERM in 1992, the East Asian financial crisis of 1997, and the European sovereign debt crisis since 2009. We attempt to model the evolution of interest rates in a continuous-time setting by using a flexible stochastic process that includes a *mean-reverting diffusion*, *externally-exciting* and *self-exciting* jumps all together within a single framework. The arrivals of externally-exciting jumps are assumed to be distributed according to a simple Poisson process. We then calculate the prices of default-free zero-coupon bonds at time 0 paying \$1 at time t.

This article is structured as follows. We define and characterise the generalised CIR process with externally-exciting and self-exciting jumps in Section 2. It is then followed by Section 3 analysing its theoretical distributional properties based on martingale methodology. Examining variations of this process in modelling the aggregate claim amounts with/without interest rate and also with/without a cluster of claims, we provide insurance premium calculations based on these moments in Section 4.1; The comparisons between the moments of aggregate claims with/without self-exciting jumps and with/without a diffusion coefficient are also made. In Section 4.2, we apply the results in Section 2 to modelling interest rates and pricing government zero-coupon bonds. The comparisons between the bond prices with/without self-exciting component are also made. The sensitivities are also shown with respect to the underlying parameters in this section. Section 5 contains some concluding remarks.

2 Definition

In this section, let us first provide a mathematical definition as below for this generalised CIR process.

Definition 2.1 (Generalised CIR Process with Externally-exciting and Self-exciting Jumps)**.** *Generalised CIR process with externally-exciting and self-exciting jumps is a jump-diffusion process*

$$
\begin{aligned}
S_t \quad = \quad & a + (S_0 - a)\, e^{-\delta t} + \sigma \int_0^t e^{-\delta(t-s)} \sqrt{S_s} \mathrm{d}W_s \\
& + \sum_{0 \le T_i^{(X)} < t} X_i e^{-\delta\left(t - T_i^{(X)}\right)} + \sum_{0 \le T_j^{(Y)} < t} Y_j e^{-\delta\left(t - T_j^{(Y)}\right)}, \qquad t \ge 0, \qquad (2.1)
\end{aligned}
$$

where

- $S_0 > 0$ *is the initial value at time* $t = 0$*;*
- $a \ge 0$ *is the constant mean-reversion level;*
- $\delta > 0$ *is the constant mean-reversion rate;*
- $\sigma > 0$ *is the constant that governs the volatility;*
- $\{W_t\}_{t \ge 0}$ *is a standard Brownian motion;*
- $\{X_i\}_{i=1,2,\ldots}$ *are the sizes of externally-exciting jumps, a sequence of i.i.d. positive r.v.s with distribution function* $H(y), y > 0$*, occurring at the corresponding random times* $\left\{T_i^{(X)}\right\}_{i=1,2,\ldots}$ *following a Poisson process* $N_t^{(X)}$ *of constant rate* $\varrho > 0$*;*
- $\{Y_j\}_{j=1,2,\ldots}$ *are the sizes of self-exciting jumps, a sequence of i.i.d. positive r.v.s with distribution function* $G(y), y > 0$*, occurring at the corresponding random times* $N \equiv \left\{T_j^{(Y)}\right\}_{j=1,2,\ldots}$*, and this point process* N_t *has a stochastic intensity linearly dependent on* S_t*, i.e.*

$$
\lambda_t = b + cS_t, \qquad b, c \ge 0; \qquad (2.2)
$$

- *the sequences* $\{X_i\}_{i=1,2,\ldots}$*,* $\{Y_j\}_{j=1,2,\ldots}$*,* $\left\{T_i^{(X)}\right\}_{i=1,2,\ldots}$ *and* $\{W_t\}_{t \ge 0}$ *are assumed to be independent of each other.*

Equivalently, (2.1) can be expressed by the stochastic differential equation (SDE)

$$
\mathrm{d}S_t = \delta\,(a - S_t)\,\mathrm{d}t + \sigma \sqrt{S_t}\mathrm{d}W_t + \mathrm{d}J_t^{(X)} + \mathrm{d}J_t^{(Y)}, \qquad (2.3)
$$

where $J_t^{(X)} := \sum_{i=1}^{N_t^{(X)}} X_i$ and $J_t^{(Y)} := \sum_{j=1}^{N_t} Y_j$. Basically, this stochastic process S_t has four terms:

- The first two terms correspond to the classical *square-root process* (Feller, 1951) or *CIR process* (Cox et al., 1985);
- The third term corresponds to the impact of *exogenous* shocks;
- The last term corresponds to the impact of past *endogenous* shocks acting on the future intensity, and this term corresponds to the self-exciting component in a generalised Hawkes framework.

The resulting process can be considered either as a natural generalisation of a CIR process or a Markovian Hawkes process[1]. Hence, it can be considered as the extensions of some recent models proposed by Zhu (2013a) and Dassios and Zhao (2017a,b). This process presents some unique features which might be suitable for mimicking the dynamics of some financial quantities, such as the aggregate losses for insurance companies and interest rates in the fixed-income markets. In particular, a crucial relationship between the process level and the jump arrivals is specified by (2.2)[2], and it essentially controls the degree of "contagion" effects: when the level of process is high, more jump arrivals are expected to follow afterwards, hence, contagion spreads accordingly. Some simulated sample paths within different time horizons are presented in Figure 1.

For notational simplification, we denote the moments and Laplace transforms by

$$\mu_{1_H} := \int_0^\infty x \mathrm{d}H\left(x\right), \qquad \mu_{2_H} := \int_0^\infty x^2 \mathrm{d}H\left(x\right), \qquad \hat{h}\left(u\right) := \int_0^\infty e^{-ux} \mathrm{d}H(x),$$

$$\mu_{1_G} := \int_0^\infty y \mathrm{d}G(y), \qquad \mu_{2_G} := \int_0^\infty y^2 \mathrm{d}G(y), \qquad \hat{g}\left(u\right) := \int_0^\infty e^{-uy} \mathrm{d}G(y),$$

and the *aggregated process* by $Z_t := \int_0^t S_u \mathrm{d}u$. For the well-posedness issue of the process, $\delta > \mu_{1_G}$ is the *stationary condition* for the original Hawkes process, and we also need it in some parts of this paper. However, the conventional *Feller's condition* $2\delta a \geq \sigma^2$ for the original CIR process is not required throughout this paper as we allow the process to reach the zero level flexibly.

[1]A Markovian Hawkes process is the one with exponential fertility rate.

[2]A similar setup as (2.2) for constructing dependency between the interest rate and default rate was presented in Lando (2004, p.123).

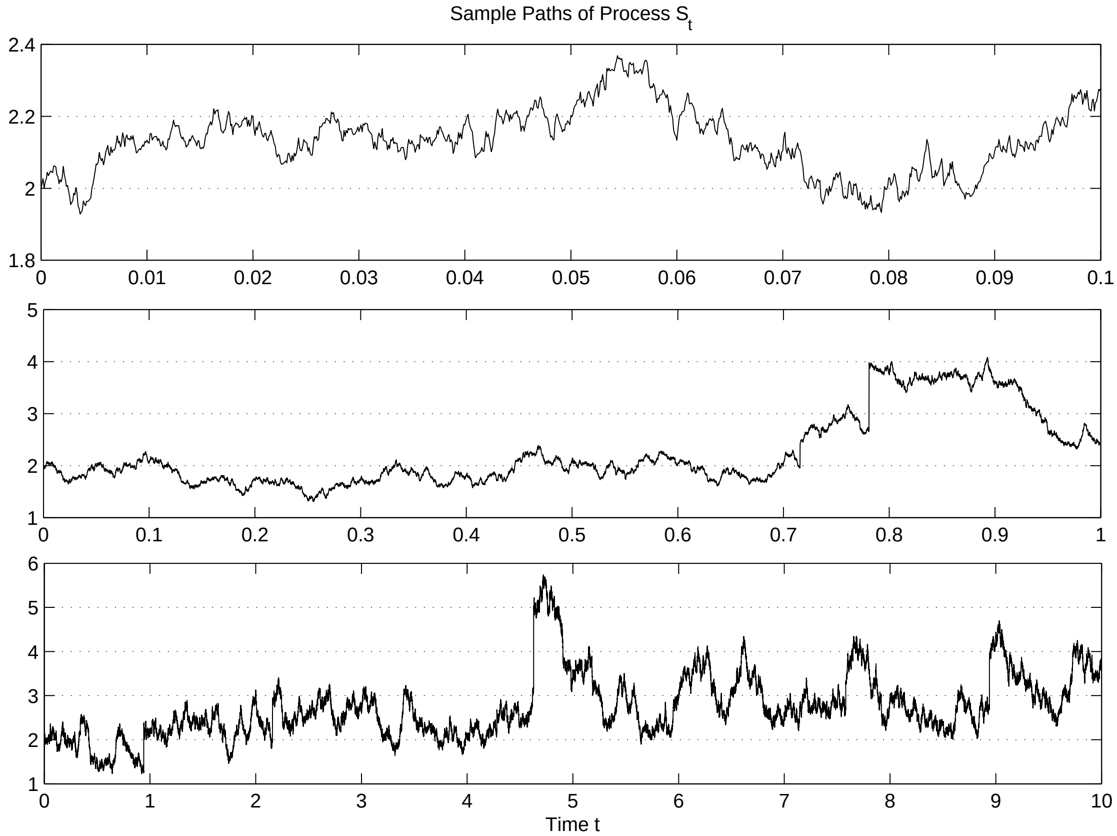

Figure 1: Sample paths of simulated process S_t for three different time horizons

3 Distributional Properties

Note that, this model is still within the classical *affine framework* (Duffie et al., 2000, 2003). Without losing generality, in this paper, we only consider the canonical case when $b = 0$ and $c = 1$ for the intensity process (2.2), as indeed it is mathematically trivial to derive all associated results below for a general setup based on $b, c \geq 0$, see also in Zhu (2014). Let us first provide the joint Laplace transform of the distribution of (S_t, Z_t):

Proposition 3.1. *For constants $\nu, \xi \geq 0$, we have the conditional joint Laplace transform*

$$\mathbb{E}\left[e^{-\nu S_T} e^{-\xi Z_T} \mid S_t, Z_t\right] = e^{-\left(D(T)-D(t)\right)} e^{-C(t)S_t} e^{-\xi Z_t}, \qquad t \in [0, T], \tag{3.1}$$

where $C(t)$ is determined by the non-linear ordinary differential equation (ODE)

$$-C'(t) + \delta C(t) + \left[\hat{g}(C(t)) - 1\right] + \frac{1}{2}\sigma^2 C^2(t) - \xi = 0, \tag{3.2}$$

with the boundary condition $C(T) = \nu$; and $D(T) - D(t)$ is determined by

$$D(T) - D(t) = a\delta \int_t^T C(s)\,\mathrm{d}s + \varrho \int_t^T \left[1 - \hat{h}\left(C(s)\right)\right] \mathrm{d}s. \tag{3.3}$$

The proof is provided in Appendix A, which is based on the martingale approach, see also Dassios and Embrechts (1989) and Dassios and Jang (2003).

Theorem 3.1. *Under the condition $\delta > \mu_{1_G}$, for any $\nu \in [0, a^+)$ and $\xi > 0$, the joint Laplace transform of (S_T, Z_T) conditional on S_0 is given by*

$$\mathbb{E}\left[e^{-\nu S_T} e^{-\xi Z_T} \mid S_0\right] = e^{-\mathcal{G}_{\nu,\xi}^{-1}(T) S_0} \times \exp\left(- \int_{\nu}^{\mathcal{G}_{\nu,\xi}^{-1}(T)} \frac{a\delta u + \varrho\left[1 - \hat{h}(u)\right]}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u \right), \tag{3.4}$$

where

$$\mathcal{G}_{\nu,\xi}(A) := \int_{\nu}^{A} \frac{1}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u, \qquad A \in [v, a^+);$$

and a^+ is the unique positive solution to the equation $1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2 = 0$.

Proof. By setting $t = 0$ in (3.1), we have

$$\mathbb{E}\left[e^{-\nu S_T} e^{-\xi Z_T} \mid \mathcal{F}_0\right] = e^{-C(0) S_0} e^{-\left(D(T) - D(0)\right)}, \qquad t \in [0, T], \tag{3.5}$$

where $C(0)$ is uniquely determined by the non-linear ODE

$$-C'(t) + \delta C(t) + \left[\hat{g}\left(C(t)\right) - 1\right] + \frac{1}{2}\sigma^2 C^2(t) - \xi = 0, \tag{3.6}$$

with the boundary condition $C(T) = \nu$. Under the condition $\delta > \mu_{1_G}$, it can be solved by the following steps:

1. Set $C(t) = A(T - t)$ and $\tau = T - t$. Then, (3.6) becomes

$$\frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} = 1 - \delta A(\tau) - \hat{g}\left(A(\tau)\right) - \frac{1}{2}\sigma^2 A^2(\tau) + \xi, \tag{3.7}$$

with the initial condition $A(0) = \nu \geq 0$; we define the right-hand side of (3.7) as the function $f_1(A)$, i.e.

$$f_1(A) := 1 + \xi - \delta A - \hat{g}(A) - \frac{1}{2}\sigma^2 A^2.$$

2. Under the condition of $\delta > \mu_{1_G}$, we have

$$\frac{\partial f_1(A)}{\partial A} = \int_0^\infty y e^{-Ay} \mathrm{d}G(y) - \delta - \sigma^2 A \le \int_0^\infty y \mathrm{d}G(y) - \delta = \mu_{1_G} - \delta < 0, \qquad A \ge 0,$$

then, $f_1(A)$ is a strictly decreasing function of $A \ge 0$. So, we have $f_1(A) < \xi$ for $A > 0$, since $f_1(0) = \xi > 0$; there is one unique positive solution a^+ to $f_1(A) = 0$ for $A \ge 0$, and $f_1(A) > 0$ for $A \in [0, a^+)$.

3. As ν should be approachable to zero, we assume $A(0) = \nu \in [0, a^+)$, we have $A(\tau) \in [v, a^+)$ and $f_1(A(\tau)) > 0$, then, (3.7) can be written as

$$\frac{\mathrm{d}A(\tau)}{1 + \xi - \delta A(\tau) - \hat{g}\left(A(\tau)\right) - \frac{1}{2}\sigma^2 A^2(\tau)} = \mathrm{d}\tau.$$

Integrate both sides from time 0 to τ with the initial condition $A(0) = \nu \ge 0$, then we have

$$\int_\nu^A \frac{1}{1 + \xi - \delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u = \tau, \qquad A \in [v, a^+).$$

Define the function on the left-hand side as

$$\mathcal{G}_{\nu,\xi}(A) := \int_\nu^A \frac{1}{1 + \xi - \delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u, \qquad A \in [v, a^+),$$

then, we have $\mathcal{G}_{\nu,\xi}(A) = \tau$, it is obvious that $A \to \nu$ when $\tau \to 0$.

4. By convergence test, we have

$$\begin{aligned} \lim_{u \uparrow a^+} \frac{\frac{1}{a^+ - u}}{\frac{1}{1+\xi-\delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2}} &= \lim_{u \uparrow a^+} \frac{1 + \xi - \delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2}{a^+ - u} \\ &= \lim_{v \downarrow 0} \frac{1 + \xi - \delta(a^+ - v) - \hat{g}\left(a^+ - v\right) - \frac{1}{2}\sigma^2 (a^+ - v)^2}{v} \\ &= \delta - \int_0^\infty y e^{-a^+ y} \mathrm{d}G(y) + \sigma^2 a^+ > \delta - \mu_{1_G} + \sigma^2 a^+ > 0. \end{aligned}$$

Obviously, $\displaystyle\int_v^{a^+} \frac{1}{a^+ - u} \mathrm{d}u = \infty$, then,

$$\int_v^{a^+} \frac{1}{1 + \xi - \delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u = \infty,$$

so $A \to a^+$ when $\tau \to \infty$. Therefore $\mathcal{G}_{\nu,\xi}(A) = \tau : [\nu, a^+) \to [0,\infty)$ is a well defined (strictly increasing) function and its inverse function $\mathcal{G}^{-1}_{\nu,\xi}(\tau) = A : [0,\infty) \to [\nu, a^+)$ exists.

5. The unique solution is found by $A(\tau) = \mathcal{G}^{-1}_{\nu,\xi}(\tau) = \mathcal{G}^{-1}_{\nu,\xi}(T-t)$. Hence, $C(0) = A(T) = \mathcal{G}^{-1}_{\nu,\xi}(T)$.

6. Now, $D(T) - D(0)$ is determined by

$$D(T) - D(0) = \varrho \int_0^T \left[1 - \hat{h}\left(\mathcal{G}^{-1}_{\nu,\xi}(\tau)\right)\right] \mathrm{d}\tau + a\delta \int_0^T \mathcal{G}^{-1}_{\nu,\xi}(\tau)\mathrm{d}\tau.$$

By the change of variable $\mathcal{G}^{-1}_{\nu,\xi}(\tau) = u$, we have $\tau = \mathcal{G}_{\nu,\xi}(u)$, and

$$\begin{aligned}
\int_0^T \left[1 - \hat{h}\left(\mathcal{G}^{-1}_{\nu,\xi}(\tau)\right)\right] \mathrm{d}\tau &= \int_{\mathcal{G}^{-1}_{\nu,\xi}(0)}^{\mathcal{G}^{-1}_{\nu,\xi}(T)} \left[1 - \hat{h}(u)\right] \frac{\partial \tau}{\partial u}\mathrm{d}u = \int_{\nu}^{\mathcal{G}^{-1}_{\nu,\xi}(T)} \frac{1 - \hat{h}(u)}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u, \\
\int_0^T \mathcal{G}^{-1}_{\nu,\xi}(\tau)\mathrm{d}\tau &= \int_{\mathcal{G}^{-1}_{\nu,\xi}(0)}^{\mathcal{G}^{-1}_{\nu,\xi}(T)} u \frac{\partial \tau}{\partial u}\mathrm{d}u = \int_{\nu}^{\mathcal{G}^{-1}_{\nu,\xi}(T)} \frac{u}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u.
\end{aligned}$$

7. Finally, substitute $C(0)$ and $D(T) - D(0)$ into (3.5) and the result follows.

□

Corollary 3.1. *The Laplace transform of aggregated process Z_T conditional on S_0 is given by*

$$\mathbb{E}\left[e^{-\xi Z_T} \mid S_0\right] = e^{-\mathcal{G}^{-1}_{0,\xi}(T) S_0} \times \exp\left(- \int_0^{\mathcal{G}^{-1}_{0,\xi}(T)} \frac{a\delta u + \varrho\left[1 - \hat{h}(u)\right]}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u \right), \tag{3.8}$$

where

$$\mathcal{G}_{0,\xi}(A) := \int_0^A \frac{1}{1 + \xi - \delta u - \hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u, \qquad A \in [0, a^+).$$

Proof. Setting $\nu = 0$ in (3.4), the result follows immediately. □

If we set $T \to \infty$, then $\mathbb{E}\left[e^{-\xi Z_T} \mid S_0\right] \to 0$, which means that $Z_T \to \infty$ almost surely when $T \to \infty$.

Note that, to derive the Laplace transform of S_T, we cannot trivially set $\xi = 0$ in (3.4), since a^+ does not exist when $\xi = 0$. Dassios and Zhao (2017b) derived the Laplace transform of S_T and its moments, for which we state the means and variances directly from their results as follows:

Proposition 3.2. *The expectation of S_t conditional on S_0 is given by*

$$\begin{aligned}\mathbb{E}\left[S_t \mid S_0\right] &= S_0 e^{-\iota t} + \frac{\mu_{1_H}\varrho + a\delta}{\iota}\left(1 - e^{-\iota t}\right), \qquad \text{for } \iota \neq 0,\\ \mathbb{E}\left[S_t \mid S_0\right] &= S_0 + \left(\mu_{1_H}\varrho + a\delta\right) t, \qquad \text{for } \iota = 0,\end{aligned}$$

where $\iota := \delta - \mu_{1_G}$.

Proposition 3.3. *The variance of S_t conditional on S_0 is given by*

$$\begin{aligned}\mathrm{Var}\left[S_t \mid S_0\right] &= \frac{1}{2\iota}\left[\frac{\left(\mu_{2_G} + \sigma^2\right)\left(\mu_{1_H}\varrho + a\delta\right)}{\iota} - \mu_{2_H}\varrho - 2\left(\mu_{2_G} + \sigma^2\right) S_0\right] e^{-2\iota t}\\ &\quad + \frac{\left(\mu_{2_G} + \sigma^2\right)}{\iota}\left(S_0 - \frac{\mu_{1_H}\varrho + a\delta}{\iota}\right) e^{-\iota t}\\ &\quad + \frac{1}{2\iota}\left[\mu_{2_H}\varrho + \frac{\left(\mu_{2_G} + \sigma^2\right)\left(\mu_{1_H}\varrho + a\delta\right)}{\iota}\right], \qquad \text{for } \iota \neq 0,\\ \mathrm{Var}\left[S_t \mid S_0\right] &= \frac{1}{2}\left(\mu_{2_G} + \sigma^2\right)\left(\mu_{1_H}\varrho + a\delta\right) t^2 + \left[\left(\mu_{2_G} + \sigma^2\right) S_0 + \mu_{2_H}\varrho\right] t, \qquad \text{for } \iota = 0.\end{aligned}$$

Similar results for some special cases could also be found in Zhu (2014).

4 Applications

In this section, we first provide an application to insurance for calculating insurance premium by using the moments of S_t from Proposition 3.2 and 3.3. We then provide an application to finance for pricing default-free zero-coupon bonds based on Corollary 3.1.

4.1 An Application in Insurance: Insurance Premium Calculation

By setting $a = S_0 = 0$ in (2.1) or (2.3), the process follows

$$\mathrm{d}S_t = -\delta S_t \mathrm{d}t + \sigma\sqrt{S_t}\mathrm{d}W_t + \mathrm{d}J_t^{(X)} + \mathrm{d}J_t^{(Y)}. \tag{4.1}$$

We further consider two special cases as below:

1. If there are no self-exciting jumps and no diffusion in (4.1), it becomes a simple Poisson shot-noise process, denoted by L_t, i.e.

$$\mathrm{d}L_t = -\delta L_t \mathrm{d}t + \mathrm{d}J_t^{(X)}. \tag{4.2}$$

   This process has been used for actuarial applications as a *discounted aggregate loss process*, see Jang (2004, 2007) and Jang and Krvavych (2004). If we assume (often implicitly) that interest rate is zero, i.e. $\delta = 0$, it becomes a simple compound Poisson process $L_t = \sum_{i=1}^{N_t^{(X)}} X_i$.

2. If we replace $-\delta$ by δ and set $\sigma = 0$ in (4.1) and $S_0 = 0$, then we have a process of

$$M_t := \sum_{0 \le T_i^{(X)} < t} X_i e^{\delta\left(t - T_i^{(X)}\right)} + \sum_{0 \le T_j^{(Y)} < t} Y_j e^{\delta\left(t - T_j^{(Y)}\right)}, \tag{4.3}$$

with the SDE

$$\mathrm{d}M_t = \delta M_t \mathrm{d}t + \mathrm{d}J_t^{(X)} + \mathrm{d}J_t^{(Y)}. \tag{4.4}$$

Remark 4.1. This shot-noise self-exciting jump process (4.4) may be interpreted in the context of non-life insurance. A single event (e.g. natural catastrophe) may induce losses for a line of business. Each loss may produce a cluster of losses according to a *branching structure* (Dassios and Zhao, 2011). Both losses are accumulated on a constant risk-free force of interest rate δ.

If there are no self-exciting jumps, from (4.3) we have

$$\Gamma_t := \sum_{i=1}^{N_t^{(X)}} X_i e^{\delta(t - T_i)},$$

with the SDE given by

$$\mathrm{d}\Gamma_t = \delta \Gamma_t \mathrm{d}t + \mathrm{d}J_t^{(X)}, \tag{4.5}$$

which recovers the aggregate loss process used in Jang (2004, 2007).

3. In contrast, now let us consider a stochastic interest rate to the aggregate loss amounts up to time t, denoted by L_t, as it is not deterministic in practice. So, if we replace $-\delta$ by $\eta > 0$ in (4.1), then we can extend our study from (4.4) and (4.5) to

$$\mathrm{d}L_t = \eta L_t \mathrm{d}t + \sigma \sqrt{L_t} \mathrm{d}W_t + \mathrm{d}J_t^{(X)} + \mathrm{d}J_t^{(Y)}. \tag{4.6}$$

Remark 4.2. This shot-noise self-exciting jump-diffusion process (4.6) may be also interpreted in the context of non-life insurance. Similarly, a single event (e.g. a natural catastrophe) may induce losses for a line of business. Compared with (4.5), both losses are accumulated on a *stochastic* force of interest rate. The proposed model captures the effect of sudden intensity increases due to external events, together with the accumulation of losses on a *stochastic* interest rate. Hence, it does have a potential interest in the insurance field.

4.1.1 Expectation of Loss Process L_t

From Proposition 3.2, by setting $a = S_0 = 0$ and replacing δ by $-\eta$, the conditional expectation of L_t of (4.6) is given by

$$\mathbb{E}\left[L_t \mid L_0\right] = L_0 e^{\zeta t} + \frac{\mu_{1_H}\varrho}{\eta + \mu_{1_G}}\left(e^{\zeta t} - 1\right), \tag{4.7}$$

where $\zeta := \eta + \mu_{1_G}$. We consider two special cases as below:

1. If there are no self-exciting jumps, from (4.7), we have

$$\mathbb{E}\left[L_t \mid L_0\right] = L_0 e^{\eta t} + \frac{\mu_{1_H}\varrho}{\eta}\left(e^{\eta t} - 1\right). \tag{4.8}$$

2. If we only consider self-exciting jumps, i.e. set $\mu_{1_H} = 0$ in (4.7), we have

$$\mathbb{E}\left[L_t \mid L_0\right] = L_0 e^{\zeta t}. \tag{4.9}$$

4.1.2 Variance of Loss Process L_t

Similarly, from Proposition 3.3, by setting $a = S_0 = 0$ and replacing δ by $-\eta$, the conditional variance of L_t of (4.6) is given by

$$\begin{aligned}\mathrm{Var}\left[L_t \mid L_0\right] &= \frac{1}{2\zeta}\left[\frac{\left(\mu_{2_G} + \sigma^2\right)\mu_{1_H}\varrho}{\zeta} + \mu_{2_H}\varrho + 2\left(\mu_{2_G} + \sigma^2\right)L_0\right]e^{2\zeta t} \\ &\quad -\frac{\left(\mu_{2_G} + \sigma^2\right)}{\zeta}\left(L_0 + \frac{\mu_{1_H}\varrho}{\zeta}\right)e^{\zeta t} - \frac{\varrho}{2\zeta}\left[\mu_{2_H} - \frac{\left(\mu_{2_G} + \sigma^2\right)\mu_{1_H}}{\zeta}\right],\end{aligned} \tag{4.10}$$

and we consider three special cases as below:

1. If there are no self-exciting jumps, from (4.10), we have

$$\begin{aligned}\mathrm{Var}\left[L_t \mid L_0\right] &= \frac{\sigma^2}{2\eta}\left[\frac{\mu_{1_H}\varrho}{\eta} + \mu_{2_H}\varrho + 2L_0\right]e^{2\eta t} \\ &\quad -\frac{\sigma^2}{\eta}\left(L_0 + \frac{\mu_{1_H}\varrho}{\eta}\right)e^{\eta t} - \frac{\varrho}{2\eta}\left(\mu_{2_H} - \frac{\sigma^2\mu_{1_H}}{\eta}\right).\end{aligned} \tag{4.11}$$

2. If we only consider self-exciting jumps, i.e. set $\mu_{1_H} = \mu_{2_H} = 0$ in (4.10), we have

$$\mathrm{Var}\left[L_t \mid L_0\right] = \frac{\left(\mu_{2_G} + \sigma^2\right)L_0}{\zeta}\left(e^{2\zeta t} - e^{\zeta t}\right). \tag{4.12}$$

3. If we set $\sigma = 0$ in (4.10) and denote the special case of L_t by V_t, then it is given by

$$\begin{aligned}\mathrm{Var}\left[V_t \mid V_0\right] &= \frac{1}{2\zeta}\left(\frac{\mu_{2G}\mu_{1H}\varrho}{\zeta} + \mu_{2H}\varrho + 2\mu_{2_G}V_0\right)e^{2\zeta t} \\ &\quad -\frac{\mu_{2G}}{\zeta}\left(V_0 + \frac{\mu_{1H}\varrho}{\zeta}\right)e^{\zeta t} - \frac{\varrho}{2\zeta}\left(\mu_{2H} - \frac{\mu_{2G}\mu_{1H}}{\zeta}\right).\end{aligned} \tag{4.13}$$

4.1.3 Numerical Examples

Let us illustrate the calculations for the moments of aggregate losses up to time t using the expressions above. For the purpose of illustration, we choose exponential distributions for $H(x)$ and $G(y)$, i.e.

$$h(x) = \alpha e^{-\alpha x}, \qquad g(y) = \beta e^{-\beta y}, \qquad x, y, \alpha, \beta > 0.$$

Then we have their moments and Laplace transforms

$$\mu_{1H} = \frac{1}{\alpha}, \quad \mu_{2H} = \frac{2}{\alpha^2}, \quad \hat{h}(u) = \frac{\alpha}{\alpha + u}, \quad \mu_{1G} = \frac{1}{\beta}, \quad \mu_{2G} = \frac{2}{\beta^2}, \quad \hat{g}(u) = \frac{\beta}{\beta + u}.$$

We assume that an insurance company's standard loss frequency rate is 5 per unit time period (say, per year) with the average of losses 1. The mean of after-losses (which are unknown at the arrival times of standard losses from a catastrophic event) is assumed to be 2. We assume that the risk-free force of interest rate is 0.05, the volatility is 1, and the initial loss amount that has been carried over is 1. Hence, the parameter values to calculate the moments of aggregate claim amounts are

$$L_0 = 1, \quad \eta = \delta = 0.05, \quad \varrho = 5, \quad \sigma = 1, \quad \alpha = 1, \quad \beta = 0.5, \quad t = 1.$$

The calculations of expectations of aggregate loss amounts with stochastic interest rate based on (4.7), (4.8) and (4.9) are shown in Table 1. The calculations of variances of aggregate loss amounts with stochastic interest rate based on (4.10), (4.11) and (4.12) are shown in Table 2.

Table 1: The expectation of loss: $\mathbb{E}[L_t \mid L_0]$

$\mathbb{E}[L_t \mid L_0]$	(4.7)	24.28
$\mathbb{E}[L_t \mid L_0]$, if there are no self-exciting jumps	(4.8)	6.18
$\mathbb{E}[L_t \mid L_0]$, if we only consider self-exciting jumps	(4.9)	7.77

Table 2: The variance of loss: $\mathrm{Var}[L_t \mid L_0]$

$\mathrm{Var}[L_t \mid L_0]$	(4.10)	620.77
$\mathrm{Var}[L_t \mid L_0]$, if there are no self-exciting jumps	(4.11)	14.22
$\mathrm{Var}[L_t \mid L_0]$, if we only consider self-exciting jumps	(4.12)	230.81

Remark 4.3. Table 1 and 2 show that, the expectation and variance of accumulated premium values calculated based on (4.7) and (4.10) are much higher than their counterparts calculated based on (4.8) and (4.11). This is mainly because they grow exponentially. It becomes much clear if we only consider self-exciting jumps, i.e. given time t, μ_{1_G} (or equivalently $e^{\mu_{1_G} t}$) which is the mean of after-losses, is the main driver to raise the variance of accumulated premium value extremely higher than its counterpart. Hence, the significance of loss-clustering impacts (i.e. after-losses' impacts) from a catastrophic event depends on after-loss size measure $\mathrm{d}G(y)$. It would be of interest to examine them using other heavy-tailed distributions for after-loss size measures.

The calculations of variances of aggregate loss amounts with stochastic/deterministic interest rate and their differences by changing the values of diffusion coefficient σ are shown in Table 3. It is used by (4.10). The calculations of the moments of aggregate claim amounts by changing the values of β for the magnitude of self-exciting jump sizes are shown in Table 4.

Table 3: The variance of loss: $\text{Var}\,[L_t \mid L_0]$

σ	$\text{Var}\,[L_t \mid L_0]$ (4.10)	$\text{Var}\,[V_t \mid V_0]$ (4.13)	$\text{Var}\,[L_t \mid L_0] - \text{Var}\,[V_t \mid V_0]$
0.0	567.88	567.88	0.00
0.5	581.10	567.88	13.22
0.6	586.92	567.88	19.04
0.7	593.80	567.88	25.92
0.8	601.73	567.88	33.85
0.9	610.72	567.88	42.84
1.0	620.77	567.88	52.89

Table 4: Sensitivity analysis of means and variances for the parameter β

β	$\mathbb{E}\,[L_t \mid L_0]$ (4.8)	$\text{Var}\,[L_t \mid L_0]$ (4.11)	$\mathbb{E}\,[L_t \mid L_0]$ (4.9)	$\text{Var}\,[L_t \mid L_0]$ (4.12)	$\mathbb{E}\,[L_t \mid L_0]$ (4.10)	$\text{Var}\,[V_t \mid V_0]$ (4.13)
10.00	6.18	14.22	1.16	1.28	15.91	11.75
5.00	6.18	14.22	1.28	1.58	18.03	13.35
1.00	6.18	14.22	2.86	15.17	72.77	59.89
0.50	6.18	14.22	7.77	230.81	620.77	567.88
0.25	6.18	14.22	57.40	26,376.00	46,440.00	45,156.00

Remark 4.4. If $\sigma = 0$, the insurance companies have the same variance of aggregate claim amounts even if they consider a stochastic interest rate to the aggregate claim amounts. However, the higher the value of diffusion coefficient is, the higher the variance of aggregate claim amounts is (see Table 3). If insurance companies adopt the *mean-variance principle* (Bühlmann, 1970; Gerber, 1979; Goovaerts et al., 1984) for their premium calculations, they become higher than those calculated using a deterministic interest rate. Therefore, it is necessary for insurance companies to charge higher premiums when the interest rate is expected to be more volatile than usual. Also as shown in Table 4, the insurance premium charged by mean-variance principle could be very large when after-losses' impacts (represented by β) are significant.

4.2 An Application in Finance: Default-free Bond Pricing

CIR process with externally-exciting and self-exciting jumps presented in the general form of (2.3) offers a versatile model, interesting both from a theoretical and a practical point view. If we ignore both *externally-exciting* and *self-exciting* jumps, it becomes the celebrated CIR model (Cox et al., 1985) for interest rates, denoted by r_t i.e.

$$\mathrm{d}r_t = \delta\left(a - r_t\right)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t. \tag{4.14}$$

In this paper, we assume the evolution of interest rate r_t follows this generalised process, i.e. $r_t \equiv S_t$ for any time t. Similar assumptions have been also proposed in Zhu (2014). By setting $\xi = 1$ in (3.8), we can calculate the prices of default-free zero-coupon bonds paying \$100 at time T by

$$\begin{aligned} B(0,T) &= \$100 \times \mathbb{E}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right) \mid r_0\right] & (4.15)\\ &= \$100 \times e^{-\mathcal{G}_{0,1}^{-1}(T)r_0} \times \exp\left(-\int_0^{\mathcal{G}_{0,1}^{-1}(T)} \frac{a\delta u + \varrho\left[1-\hat{h}\left(u\right)\right]}{2-\delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right), & (4.16)\end{aligned}$$

where

$$\mathcal{G}_{0,1}(A) := \int_0^A \frac{1}{2-\delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u, \qquad A \in [0, a^+).$$

We consider two special cases as below:

1. If there are no self-exciting jumps, from (4.16) we have

$$\mathbb{E}\left[e^{-Z_T} \mid r_0\right] = e^{-\mathcal{G}_{0,1}^{-1}(T)r_0} \times \exp\left(-\int_0^{\mathcal{G}_{0,1}^{-1}(T)} \frac{a\delta u + \varrho\left[1-\hat{h}\left(u\right)\right]}{2-\delta u - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right), \tag{4.17}$$

   where

$$\mathcal{G}_{0,1}(A) := \int_0^A \frac{1}{2-\delta u - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u.$$

2. If we only consider the self-exciting jumps, i.e. $\varrho = 0$ in (4.16), we have

$$\mathbb{E}\left[e^{-Z_T} \mid r_0\right] = e^{-\mathcal{G}_{0,1}^{-1}(T)r_0} \times \exp\left(-\int_0^{\mathcal{G}_{0,1}^{-1}(T)} \frac{a\delta u}{2-\delta u - \hat{g}\left(u\right) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right), \tag{4.18}$$

where

$$\mathcal{G}_{0,1}(A) := \int_0^A \frac{1}{2 - \delta u - \hat{g}\,(u) - \frac{1}{2}\sigma^2 u^2} \mathrm{d}u.$$

4.2.1 Numerical Examples

We assume that the frequency rate of externally-exciting events (e.g. news on Greek debt crisis) is 3 per unit time period (say, per year) with their average magnitude of 0.01. The mean of self-exciting event (e.g. news of failing to reign in their budget deficits and debts) magnitudes, which are unknown at the arrival times of externally-exciting events, is assumed to be 0.02. The risk-free force of interest rate is 0.05 and that the initial rate of interest is 0.05. Hence, the parameter values to calculate the price of a default-free zero-coupon bond are

$$r_0 = a = 0.05, \quad \delta = 0.05, \quad \varrho = 3, \quad \sigma = 0.8, \quad \alpha = 100, \quad \beta = 50, \quad T = 1.$$

The calculations of bond prices based on (4.16), (4.17) and (4.18) are shown in Table 5.

Table 5: Bond price $B(0,1)$

$B(0,1)$	(4.16)	94.19
$B(0,1)$, if there are no self-exciting jumps	(4.17)	94.23
$B(0,1)$, if we only consider self-exciting jumps	(4.18)	95.52

Using (4.16), the calculations of bond prices by changing the values of coefficient σ are shown in Table 6 and Figure 2. Note that, for any time T, when $\sigma \to \infty$, the unique positive root $a^+ \to 0$ and $\mathcal{G}_{0,1}^{-1}(T) \to 0$, hence, $\mathbb{E}\left[e^{-Z_T} \mid r_0\right] \to 1$.

Remark 4.5. Having considered both the upward externally-exciting and self-exciting jumps in S_t, we are expecting higher interest rates as time goes by. So, Table 5 shows that the bond price calculated based on (4.16) is the lowest. The price calculated based on (4.18) where self-exciting jumps are only considered, is higher than its counterpart calculated based on (4.17) as the self-exciting jump frequency rate is lower than the externally-exciting jump frequency rate, $\varrho = 3$ (see Figure 3). Also, the more volatile the interest rate is (meaning more uncertainty for future consumption),

Table 6: Bond price $B(0,1)$

σ	$B(0,1)$
0.01	93.68
0.1	93.69
0.5	93.89
0.8	94.19
10	98.89
∞	100

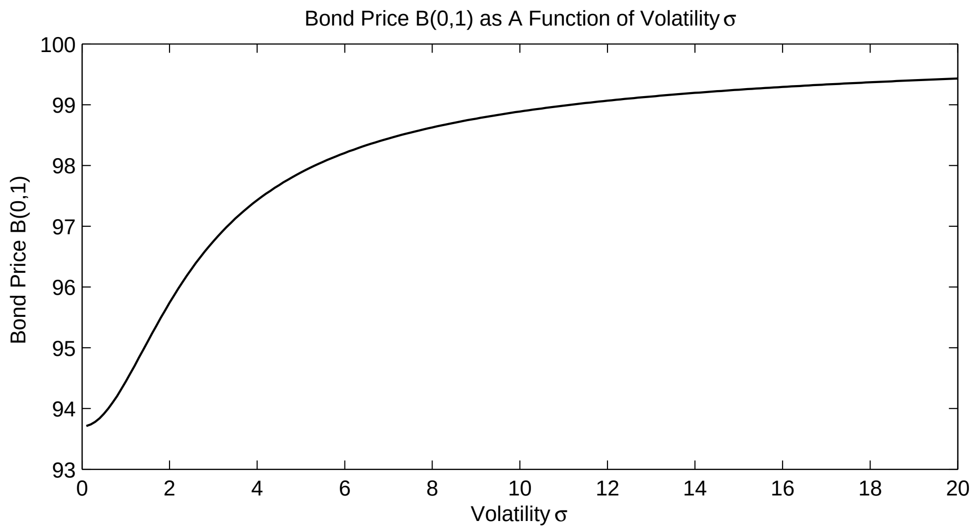

Figure 2: Bond price $B(0,1)$ as a function of volatility σ

the more attractive purchasing a bond that pays guaranteed $\$100$ is. Hence, the higher σ is, the higher the bond price is (see Table 6 and Figure 2), which is the same result as the CIR case.

The calculations of bond prices by changing the values of α and its frequency rate ϱ are shown in Tables 7 and 8, respectively. The bigger the magnitude of positive jumps is, the less attractive purchasing a bond is. Hence, the smaller α is, the lower the bond price is (see Table 7). Also the higher ϱ is, the lower the bond price is (see Table 8).

Table 7: Sensitivity analysis of bond price $B(0,1)$ for the parameter α

α	$B(0,1)$ (4.16)	$B(0,1)$ (4.17)
∞	95.5201	95.5585
100	94.1880	94.2340
90	94.0422	94.0889
70	93.6278	93.6768
50	92.8904	92.9434
30	91.2734	91.2116
5	74.2420	74.3715
1	39.1674	39.3072

Table 8: Sensitivity analysis of bond price $B(0,1)$ for the parameter ϱ

ϱ	$B(0,1)$ (4.16)	$B(0,1)$ (4.17)
100	59.8136	60.0077
50	75.5870	75.7248
30	83.0054	83.1095
20	86.9833	87.0677
10	91.1518	91.2143
5	93.3104	93.3612
3	94.1880	94.2340
2	94.6300	94.6734
0	95.5201	95.5585

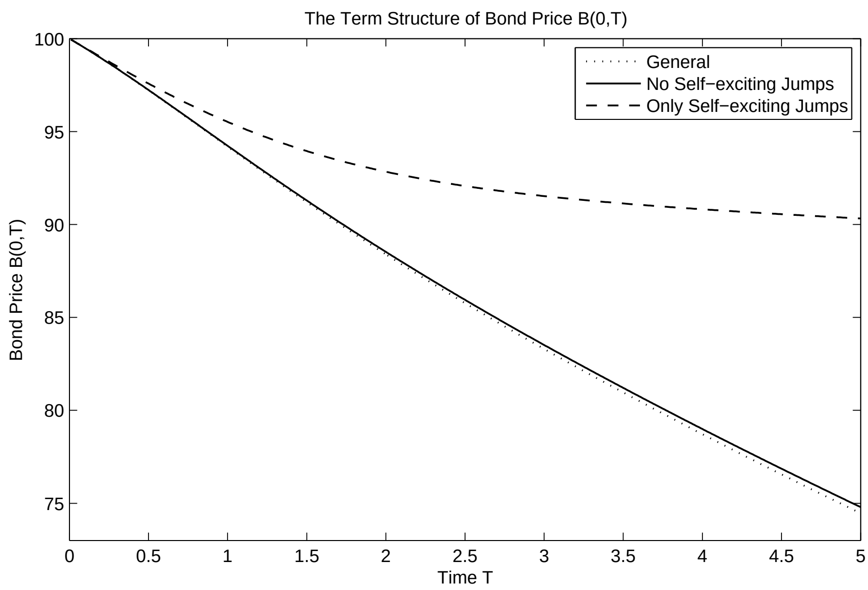

Figure 3: Term structure of bond price $B(0, T)$

We are particularly interested in the special cases where the frequencies and jump sizes are the same. To compare the effect of externally-exciting and self-exciting jumps, we have to make sure that they arrive with an equal frequency on the long run and have the same distribution for their sizes, i.e.

$$\underbrace{\frac{\mu_{1_F}\varrho + a\delta}{\delta - \mu_{1_G}}}_{\text{Frequency rate of self-exciting jumps}} = \underbrace{\varrho}_{\text{Frequency rate of externally-exciting jumps}}.$$

Setting parameters as

$$r_0 = 0.05, \quad a = 0.6, \quad \delta = 0.05, \quad \varrho = 3, \quad \sigma = 0.8, \quad \alpha = \beta = 50, \quad T = 1,$$

bond prices calculated based on (4.16), (4.17) and (4.18) are provided in Table 9.

Table 9: Bond price $B(0, 1)$

$B(0, 1)$	(4.16)	91.6950
$B(0, 1)$, if there are no-self exciting jumps	(4.17)	91.7546
$B(0, 1)$, if we only consider self-exciting jumps	(4.18)	94.2909

Remark 4.6. Table 9 shows that if externally-exciting and self-exciting jumps arrive with an equal frequency on the long run and with the same distribution for their sizes, externally-exciting events matter more. We observe that the bond prices calculated based on (4.16) and (4.17) are almost the same. Considering only self-exciting events, the bond price calculated based on (4.18) is higher than its counterpart (4.17). It indicates that their impact on interest rates is not as large as the impact of externally-exciting events. Of course, in practice, it is unlikely that the two kinds of events will arrive with the same frequency and the same distributions of jump sizes; this exercise was to study the impact of the *nature of jumps*.

5 Conclusion

We studied a generalised CIR process with externally-exciting and self-exciting jumps, and examined the distributional properties. The joint Laplace transform of the process and its integrated process was derived by applying the standard martingale theory. Using the first and second moments of the process, we provided insurance premium calculations and their comparisons with/without self-exciting jumps, and with/without a diffusion coefficient. As a financial application, we presented how this Laplace transform can be used for pricing default-free zero-coupon bonds. Numerical calculations for bond prices were illustrated with/without self-exciting jumps, and with/without a diffusion coefficient. Changing the relevant parameters of the process, their sensitivities were also presented for both applications.

Appendices

A Proof for Proposition 3.1

Proof. Note that, the infinitesimal generator of the joint process (Z_t, S_t, N_t, t) acting on a function $f(z,s,n,t)$ within its domain $\Omega(\mathcal{A})$ is specified by

$$\begin{aligned}\mathcal{A}f(z,s,n,t) &= \frac{\partial f}{\partial t} + s\frac{\partial f}{\partial z} - \delta(s-a)\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s\frac{\partial^2 f}{\partial s^2} \\ &+\varrho\left(\int_0^\infty f(z,s+x,n,t)\,\mathrm{d}H(x) - f(z,s,n,t)\right) \\ &+s\left(\int_0^\infty f(z,s+y,n+1,t)\,\mathrm{d}G(y) - f(z,s,n,t)\right), \qquad \text{(A.1)}\end{aligned}$$

where $\Omega(\mathcal{A})$ is the domain of generator $\mathcal{A}$ such that $f(z,s,n,t)$ is differentiable with respect to z, s and t, and

$$\begin{aligned}\left|\int_0^\infty f(z,s+x,n,t)\,\mathrm{d}H(x) - f(z,s,n,t)\right| &< \infty, \\ \left|\int_0^\infty f(z,s+y,n+1,t)\,\mathrm{d}G(y) - f(z,s,n,t)\right| &< \infty.\end{aligned}$$

Consider a function $f(s,z,t)$ of an exponential affine form $f(s,z,t) = e^{-C(t)S_t}e^{-\xi Z_t}e^{D(t)}$, substitute into $\mathcal{A}f = 0$ in (A.1), we have

$$
\begin{aligned}
&-C'(t)\,s + D'(t) - s\xi - a\delta C(t) + \delta s C(t) \\
&+\varrho\left[\hat{h}(C(t)) - 1\right] + s\left[\hat{g}(C(t)) - 1\right] + \frac{1}{2}\sigma^2 s C^2(t) = 0.
\end{aligned}
$$

Since this equation holds for any s, it is equivalent to solving two separated equations

$$
\begin{aligned}
-C'(t) + \delta C(t) + \left[\hat{g}(C(t)) - 1\right] + \frac{1}{2}\sigma^2 C^2(t) - \xi &= 0, \\
D'(t) + \varrho[\hat{h}(C(t)) - 1] - a\delta C(t) &= 0.
\end{aligned}
\tag{A.2}
$$

With the boundary condition $C(T) = \nu$, we have the ODE as (3.2). By (A.2), the integration as (3.3) follows. Since $e^{-C(t)S_t}e^{-\xi Z_t}e^{D(t)}$ is a martingale by the property of infinitesimal generator, we have

$$
\mathbb{E}\left[e^{-C(T)S_T}e^{-\xi Z_T}e^{D(T)} \mid S_t, Z_t\right] = e^{-C(t)S_t}e^{-\xi Z_t}e^{D(t)}.
$$

Then, based on the boundary condition $C(T) = \nu$, (3.1) follows. $\square$

References

Adamopoulos, L. (1976). Cluster models for earthquakes: Regional comparisons. *Mathematical Geology*, 8(4):463–475.

Aït-Sahalia, Y., Cacho-Diaz, J., and Laeven, R. J. (2015). Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117(3):585–606.

Aït-Sahalia, Y., Laeven, R. J., and Pelizzon, L. (2014). Mutual excitation in Eurozone sovereign CDS. *Journal of Econometrics*, 183(2):151–167.

Bacry, E., Delattre, S., Hoffmann, M., and Muzy, J.-F. (2013). Modelling microstructure noise with mutually exciting point processes. *Quantitative Finance*, 13(1):65–77.

Bauwens, L. and Hautsch, N. (2009). Modelling financial high frequency data using point processes. In Andersen, T. G., Davis, R. A., Kreiß, J.-P., and Mikosch, T. V., editors, *Handbook of Financial Time Series*, pages 953–979. Springer, Berlin.

Boumezoued, A. (2016). Population viewpoint on Hawkes processes. *Advances in Applied Probability*, 48(2):463–480.

Bowsher, C. G. (2007). Modelling security market events in continuous time: Intensity based, multivariate point process models. *Journal of Econometrics*, 141(2):876–912.

Brémaud, P. and Massoulié, L. (1996). Stability of nonlinear Hawkes processes. *Annals of Probability*, 24(3):1563–1588.

Brémaud, P. and Massoulié, L. (2001). Hawkes branching point processes without ancestors. *Journal of Applied Probability*, 38(1):122–135.

Brémaud, P. and Massoulié, L. (2002). Power spectra of general shot noises and Hawkes point processes with a random excitation. *Advances in Applied Probability*, 34(1):205–222.

Bühlmann, H. (1970). *Mathematical Methods in Risk Theory*. Springer-Verlag, Berlin.

Chavez-Demoulin, V., Davison, A. C., and McNeil, A. J. (2005). Estimating value-at-risk: a point process approach. *Quantitative Finance*, 5(2):227–234.

Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Dassios, A. and Embrechts, P. (1989). Martingales and insurance risk. *Stochastic Models*, 5(2):181–217.

Dassios, A. and Jang, J. (2003). Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity. *Finance and Stochastics*, 7(1):73–95.

Dassios, A. and Zhao, H. (2011). A dynamic contagion process. *Advances in Applied Probability*, 43(3):814–846.

Dassios, A. and Zhao, H. (2012). Ruin by dynamic contagion claims. *Insurance: Mathematics and Economics*, 51(1):93–106.

Dassios, A. and Zhao, H. (2017a). Efficient simulation of clustering jumps with CIR intensity. *Operations Research*, 65(6):1494–1515.

Dassios, A. and Zhao, H. (2017b). A generalised contagion process with an application to credit risk. *International Journal of Theoretical and Applied Finance*, 20(1):1–33.

Duffie, D., Filipovic, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Probability*, 13(3):984–1053.

Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376.

Embrechts, P., Liniger, T., and Lin, L. (2011). Multivariate Hawkes processes: an application to financial data. *Journal of Applied Probability*, 48A:367–378.

Errais, E., Giesecke, K., and Goldberg, L. R. (2010). Affine point processes and portfolio credit risk. *SIAM Journal on Financial Mathematics*, 1(1):642–665.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*, 54(1):173–182.

Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation for Insurance Education, Philadelphia.

Goovaerts, M., De Vylder, F., and Haezendonck, J. (1984). *Insurance Premiums*. North-Holland, Amsterdam.

Hawkes, A. G. (1971a). Point spectra of some mutually exciting point processes. *Journal of the Royal Statistical Society. Series B (Methodological)*, 33(3):438–443.

Hawkes, A. G. (1971b). Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1):83–90.

Hawkes, A. G. and Oakes, D. (1974). A cluster process representation of a self-exciting process. *Journal of Applied Probability*, 11:493–503.

Jaisson, T. and Rosenbaum, M. (2015). Limit theorems for nearly unstable Hawkes processes. *Annals of Applied Probability*, 25(2):600–631.

Jang, J. (2004). Martingale approach for moments of discounted aggregate claims. *Journal of Risk and Insurance*, 71(2):201–211.

Jang, J. (2007). Jump diffusion processes and their applications in insurance and finance. *Insurance: Mathematics and Economics*, 41(1):62–70.

Jang, J. and Dassios, A. (2013). A bivariate shot noise self-exciting process for insurance. *Insurance: Mathematics and Economics*, 53(3):524–532.

Jang, J. and Krvavych, Y. (2004). Arbitrage-free premium calculation for extreme losses using the shot noise process and the Esscher transform. *Insurance: Mathematics and Economics*, 35(1):97–111.

Lando, D. (2004). *Credit Risk Modeling: Theory and Applications*. Princeton University Press, New Jersey.

Léveillé, G. and Garrido, J. (2001). Moments of compound renewal sums with discounted claims. *Insurance: Mathematics and Economics*, 28(2):217–231.

Ogata, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *Journal of the American Statistical Association*, 83(401):9–27.

Ozaki, T. (1979). Maximum likelihood estimation of Hawkes' self-exciting point processes. *Annals of the Institute of Statistical Mathematics*, 31(1):145–155.

Stabile, G. and Torrisi, G. L. (2010). Risk processes with non-stationary Hawkes claims arrivals. *Methodology and Computing in Applied Probability*, 12(3):415–429.

Vere-Jones, D. (1975). Stochastic models for earthquake sequences. *Geophysical Journal International*, 42(2):811–826.

Vere-Jones, D. (1978). Earthquake prediction – a statistician's view. *Journal of Physics of the Earth*, 26(2):129–146.

Vere-Jones, D. and Ozaki, T. (1982). Some examples of statistical estimation applied to earthquake data. *Annals of the Institute of Statistical Mathematics*, 34(1):189–207.

Zhu, L. (2013a). Central limit theorem for nonlinear Hawkes processes. *Journal of Applied Probability*, 50(3):760–771.

Zhu, L. (2013b). Ruin probabilities for risk processes with non-stationary arrivals and subexponential claims. *Insurance: Mathematics and Economics*, 53(3):544–550.

Zhu, L. (2014). Limit theorems for a Cox-Ingersoll-Ross process with Hawkes jumps. *Journal of Applied Probability*, 51(3):699–712.

Zhu, L. (2015). Large deviations for Markovian nonlinear Hawkes processes. *Annals of Applied Probability*, 25(2):548–581.