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Hyperbolic Numbers in Modeling Genetic Phenomena

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Abstract. The article is devoted to applications of 2-dimensional hyperbolic numbers and their algebraic 2^n-dimensional extensions in modeling some genetic phenomena. Mathematical properties of hyperbolic numbers and their matrix representations are described in a connection with alphabets of DNA nucleobases and with inherited phyllotaxis phenomena. Known data on using hyperbolic rotations, which are particular cases of hyperbolic numbers, in physics and in some biological phenomena, including phyllotaxis laws and structural features of locomotions, are discussed. Applications of hyperbolic numbers reveal hidden interrelations between structures of different biological and physical phenomena. They lead to new approaches in mathematical modeling genetic phenomena.

Keywords: hyperbolic numbers, genetics, tensor product, Fibonacci numbers, phyllotaxis.

1 Introduction

The main task of the mathematical natural sciences is the creation of mathematical models of natural systems. Development of models and formalized theories depends highly on those mathematical notions and instruments, on which they are based. Modern science knows that different natural systems could possess their own individual geometries and their own individual arithmetic [Kline, 1982]. Various kinds of multi-dimensional numbers – complex numbers, hyperbolic numbers, dual numbers, quaternions and other hypercomplex numbers – are used in different branches of modern science. They have played the role of the magic tool for development of theories and calculations in problems of heat, light, sounds, fluctuations, elasticity, gravitation, magnetism, electricity, current of liquids, quantum-mechanical phenomena, special theory of relativity, nuclear physics, etc. For example, in physics thousands of works - only in XX century – were devoted to quaternions of Hamilton (their bibliography is in [Gsponer, Hurni, 2008].

The idea about special mathematical peculiarities of living matter exists long ago. For example V.I. Vernadsky put forward the hypothesis on a non-Euclidean geometry of living nature [Vernadsky, 1965]. It seems an important task to investigate what systems of multi-dimensional numbers are connected or can be connected with ensembles of parameters of the

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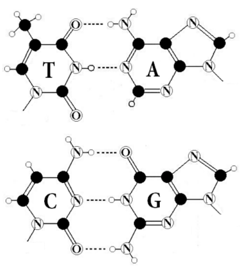
genetic code and inherited biological peculiarities. Some results of such investigation are presented in this article. They are connected with hyperbolic numbers and their algebraic extensions, matrix forms of which give a new class of mathematical models in genetics and in inherited physiological phenomena. Some results of such investigation are presented in this article. They are connected with hyperbolic numbers and their algebraic extensions, matrix forms of which give a new class of mathematical models in biology. Author's results described in this article are related in particularly to works by O. Bodnar who noted that ontogenetic transformations of phyllotaxis lattices in plants can be formaly modelled by hyperbolic rotations, which are particular cases of hyperbolic numbers and are well known in the special theory of relativity (Lorentz transformations) [Bodnar, 1992, 1994]. On this basis he stated that geometry of living bodies has structural relations with the Minkovsky geometry. Another evidence in favor of structural relaltions of inherited biological phenomena with hyperbolic rotations was shown in the work [Smolyaninov, 2000], which analyzed problems of locomotion control and put forward ideas of the "locomotor theory of relativity".

It is obvoius that all physiological systems must be argued with a genetic coding system in order to be genetically encoded for their survival and inheritance into next generations. For this reason, the structural organization of physiological systems can bear the imprint of the structural features of molecular genetic systems. Our study aims to identify such relationships of inherited physiological structures with the molecular genetic system.

2 Matrix representations of DNA alphabets and hyperbolic numbers

In DNA molecules DNA genetic information is written in sequences of 4 kinds of nucleobases: adenine A, cytosine C, guanine G and thymine T. They form a DNA alphabet of 4 monoplets. In addition, DNA alphabets of 16 doublets and 64 triplets also exist. It is known [Fimmel, Danielli, Strüngmann, 2013; Petoukhov, 2008; Petoukhov, He, 2010; Stambuk, 1999] that these four nucleobases A, C, G and T are interrelated due to their symmetrical peculiarities into the united molecular ensemble with its three pairs of binary-oppositional traits or indicators (Fig. 1):

1) Two letters are purines (A and G), and the other two are pyrimidines (C and T). From the standpoint of these binary-oppositional traits one can denote C = T = 0, A = G = 1;
2) Two letters are amino-molecules (A and C) and the other two are keto-molecules (G and T). From the standpoint of these traits one can designate A = C = 0, G = T = 1;
3) The pairs of complementary letters, A-T and C-G, are linked by 2 and 3 hydrogen bonds, respectively. From the standpoint of these binary traits, one can denote C = G = 0, A = T = 1.



№	Binary Symbols	C	A	G	T/U
1	0_1 — pyrimidines 1_1 — purines	0_1	1_1	1_1	0_1
2	0_2 — amino 1_2 — eto	0_2	0_2	1_2	1_2
3	0_3 — three hydrogen bonds; 1_3 — two hydrogen bonds	0_3	1_3	0_3	1_3

Fig. 1. Left: the four nitrogenous bases of DNA: adenine A, guanine G, cytosine C, and thymine T. Right: three binary sub-alphabets of the genetic alphabet on the basis of three pairs of binary-oppositional traits or indicators.

Taking into account the phenomenological fact that each of DNA-letters C, A, T and G is uniquely defined by any two kinds of mentioned binary-oppositional indicators (Fig. 1), these genetic letters can be represented by means of corresponding pairs of binary symbols, for example, from the standpoint of two first binary-oppositional indicators. It is convenient for us - for the further description - use at the first position of each of letters its binary symbol from the second pair of binary-oppositional indicators (the indicator "amino or keto": C=A=0, T=G=1) and at the second positions of each of letters its binary symbol from the first pair of binary-oppositional indicators (the indicator "pyrimidine or purine": C=T=0, A=G=1). In this case the letter C is represented by the binary symbol $0_2 0_1$ (that is as 2-bit binary number), A – by the symbol $0_2 1_1$, T – by the symbol $1_2 0_1$, G – by the symbol $1_2 1_1$. Using these representations of separate letters, each of 16 doublets is represented as the concatenation of the binary symbols of its letters (that is as 4-bit binary number): for example, the doublet CC is represented as 4-bit binary number $0_2 0_1 0_2 0_1$, the doublet CA – as 4-bit binary number $0_2 0_1 0_2 1_1$, etc. By analogy, each of 64 triplets is represented as the concatenation of the binary symbols of its letters (that is as 6-bit binary number): for example, the triplet CCC is represented as 6-bit binary number $0_2 0_1 0_2 0_1 0_2 0_1$, the triplet CCA – as 6-bit binary number $0_2 0_1 0_2 0_1 0_2 1_1$, etc. In general, each of n-plets is represented as the concatenation of the binary symbols of its letters (below we will not show these indexes 2 and 1 of separate letters in binary representations of n-plets but will remember that each of positions corresponds to its own kind of indicators from the first or from the second set of indicators in Fig. 1).

It is convenient to represent DNA-alphabets of 4 nucleotides, 16 doublets, 64 triplets, … 4^n n-plets in a form of appropriate square tables (Fig. 2), which rows and columns are numerated by binary symbols in line with the following principle. Entries of each column are numerated by binary symbols in line with the first set of binary-oppositional indicators in Fig. 1 (for example, the triplet CAG and all other triplets in the same column are the combination "pyrimidine-purin-purin" and so this column is correspondingly numerated 011). By contrast, entries of each of rows are numerated by binary numbers in line with the second set of indicators (for example, the same triplet CAG and all other triplets in the same row are the combination "amino-amino-keto" and so this row is correspondingly numerated 001). In such tables (Fig. 2), each of 4 letters, 16 doublets, 64 triplets, … takes automatically its own individual place and all components of the alphabets are arranged in a strict order.

It is essential that these 3 separate genetic tables form the joint tensor family of matrices since they are interrelated by the known operation of the tensor (or Kronecker) product of matrices [Bellman, 1960]. So they are not simple tables but matrices. By definition, under tensor multiplication of two matrices, each of entries of the first matrix is multiplied with the whole second matrix. The second tensor power of the (2*2)-matrix [C, A; T, G] of 4 DNA-letters gives automatically the (4*4)-matrix of 16 doublets; the third tensor power of the same (2*2)-matrix of 4 DNA-letters gives the (8*8)-matrix of 64 triplets with the same strict arrangement of entries as in Fig. 2. In this tensor construction of the tensor family of genetic matrices, data about binary-oppositional traits of genetic letters C, A, T and G are not used at

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all. So, the structural organization of the system of DNA-alphabets is connected with the algebraic operation of the tensor product. It is important since the operation of the tensor product is well known in mathematics, physics and informatics, where it gives a way of putting vector spaces together to form larger vector spaces. The following quotation speaks about the crucial meaning of the tensor product: «*This construction is crucial to understanding the quantum mechanics of multiparticle systems*» [Nielsen, Chuang, 2010, p. 71].

	0	1
0	C	A
1	T	G

	00	01	10	11
00	CC	CA	AC	AA
01	CT	CG	AT	AG
10	TC	TA	GC	GA
11	TT	TG	GT	GG

	000	001	010	011	100	101	110	111
000	CCC	CCA	CAC	CAA	ACC	ACA	AAC	AAA
001	CCT	CCG	CAT	CAG	ACT	ACG	AAT	AAG
010	CTC	CTA	CGC	CGA	ATC	ATA	AGC	AGA
011	CTT	CTG	CGT	CGG	ATT	ATG	AGT	AGG
100	TCC	TCA	TAC	TAA	GCC	GCA	GAC	GAA
101	TCT	TCG	TAT	TAG	GCT	GCG	GAT	GAG
110	TTC	TTA	TGC	TGA	GTC	GTA	GGC	GGA
111	TTT	TTG	TGT	TGG	GTT	GTG	GGT	GGG

Fig. 2. The square tables of DNA-alphabets of 4 nucleotides, 16 doublets and 64 triplets with a strict arrangement of all components. Each of tables is constructed in line with the principle of binary numeration of its column and rows on the basis of binary-oppositional traits of the nitrogenous bases (see explanations in the text).

In the DNA double helix, complementary nucleobases C and G are connected by 3 hydrogen bonds and complementary nucleobases A and T are connected by 2 hydrogen bonds. One can denote their typical connections with hydrogen bonds by expressions C=G=3 and A=T=2. Replacing in the (2*2)-matrix [C, A; T, G] (Fig. 2) symbols C, A, T and G by their numbers of hydrogen bonds 3 and 2, a numeric matrix [3, 2; 2, 3] appears (Fig. 3). The second and the third tensor powers of this matrix $[3, 2; 2, 3]^{(n)}$, where n = 2, 3, generate numeric (4*4)- and (8*8)-matrices in Fig. 3, which automatically represent symbolic matrices of 16 doublets and 64 triplets in Fig. 2 from the standpoint of the product of their numbers of hy-

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drogen bonds. For example the doublet CA is replaced by number 3*2=6 and the triplet AGT is replaced by number 2*3*2=12.

3	2
2	3

9	6	6	4
6	9	4	6
6	4	9	6
4	6	6	9

27	18	18	12	18	12	12	8
18	27	12	18	12	18	8	12
18	12	27	18	12	8	18	12
12	18	18	27	8	12	12	18
18	12	12	8	27	18	18	12
12	18	8	12	18	27	12	18
12	8	18	12	18	12	27	18
8	12	12	18	12	18	18	27

Fig. 3. Numeric representations of the tensor family of symbolic matrices (Fig. 2) of 4 monoplets, 16 doublets and 64 triplets from the standpoint of their numeric characteristics of hydrogen bonds C=G=3 and A=T=2.

Fig. 4 shows that the matrix [3, 2; 2, 3] is decomposed into sum of two sparse matrices, one of which is the identity matrix (j_0 = [1, 0; 0, 1]) and the second matrix j_1 = [0, 1; 1, 1]) represents imaginary unit of hyperbolic numbers since $j_1^2 = j_0$. The set of these matrices j_0 and j_1 is closed relative to multiplication and defines the multiplication table of algebra of hyperbolic numbers (Fig. 4, right).

$$\begin{vmatrix} 3, 2 \\ 2, 3 \end{vmatrix} = 3* \begin{vmatrix} 1, 0 \\ 0, 1 \end{vmatrix} + 2* \begin{vmatrix} 0, 1 \\ 1, 0 \end{vmatrix} = 3*j_0 + 2*j_1;$$

*	1	j_1
1	1	j_1
j_1	j_1	1

Fig. 4. The decomposition of the matrix [3, 2; 2, 3] into two sparse matrices, where matrices j_0 and j_1 are matrix representations of real and imaginary units of algebra of hyperbolic numbers with the shown multiplication table of these units.

Here we should remind that two-dimensional hyperbolic numbers are written in linear notation as $m_1 = a*\mathbf{1}+b*\mathbf{j}$ (where **1** is the real unit; **j** is the imaginary unit with the property

j ≠ ±**1** but $\mathbf{j}^2 = \mathbf{1}$; *a*, *b* are real coefficients). These numbers are used in physics and mathematics and they have also synonimical names: "split-complex numbers", "perplex numbers" and "double numbers". The collection of all hyperbolic numbers forms algebra over the field of real numbers [38 Harkin, Harkin, 2004; Kantor, Solodovnikov, 1989]. The algebra is not a division algebra or field since it contains zero divisors. Addition and multiplication of hyperbolic numbers are defined by (1):

$$(x+jy)+(u+jv)=(x+u)+j(y+v); \quad (x+jy)(u+jv)=(xu+yv)+j(xv+yu) \qquad (1)$$

This multiplication is commutative, associative and distributes over addition.

A hyperbolic number has its matrix form of representation: [a, b; b, a] = a*[1, 0; 0, 1] +b*[0, 1; 1, 0] where [1, 0; 0, 1] is the identity matrix representing real basic unit; [0, 1; 1, 0] represents imaginary basic unit. Fig. 4 shows the matrix representation of hyperbolic numbers *a****1**+*b****j** for the case $a = 3$ and $b = 2$.

If a^2-$b^2 = 1$, then the matrix [a, b; b, a] defines hyperbolic rotations known in the special theory of relativity as Lorentz transformations. Hyperbolic rotations are usually expressed by a symmetric matrix (2) through hyperbolic cosine «cosh» and hyperbolic sine «sinh» since $\cosh^2 x - \sinh^2 x = 1$ [Collins Concise Dictionary, 1999; Shervatov, 1954; Stakhov, 2009]:

$$\begin{vmatrix} \cosh^2 a, & \sinh^2 b \\ \sinh^2 b, & \cosh^2 a \end{vmatrix} \qquad (2)$$

Symmetric matrices that represent hyperbolic numbers have real eigenvalues and orthogonal eigenvectors (which distinguishes them from non-symmetic matrix repre-sentations of complex numbers). Such symmetric matrices form the basis of the theory of resonances of oscillatory systems with many degrees of freedom, and are also metric tensors from the point of view of Riemannian geometry.

The second tensor power of the bisymmetric matrix [*a*, *b*; *b*, *a*], which represents hyperbolic numbers, is decomposed into 4 sparse matrices e_0, e_1, e_2 and e_3 with real coefficients *aa, ab ba* and *bb* (Fig. 5). The set of matrices e_0, e_1, e_2 and e_3 is closed relative to multiplication and satisfies to the multiplication table in Fig. 5. The set of these (4x4)-matrices corresponds to algebra of 4-dimensional numbers $aa*e_0 + ab*e_1 + ba*e_2 + bb*e_3$, where the matrix e_0 represents the real unit 1 and matrices e_1, e_2 and e_3 represent imaginary units. These 4-dimensional numbers are algebraic extensions of 2-dimensional hyperbolic numbers and for simplicity they can be termed "4-dimensional hyperbolic numbers" (in our previous publications we termed them "hyperbolic matrions" [Petoukhov, 2008; Petoukhov, He, 2010]).

By comparing Fig. 3 and Fig. 5, one can see that the numeric (4*4)-matrix of hydrogen bonds in Fig. 3 represents 4-dimensional hyperbolic number $9+6e_1+6e_2+4e_3$. By analogy, the numeric (8*8)-matrix in Fig. 3 represents 8-dimensional hyperbolic number.

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$$\begin{vmatrix} a, b \\ b, a \end{vmatrix}^{(2)} = \begin{vmatrix} aa, ab, ba, bb \\ ab, aa, bb, ba \\ ba, bb, aa, ab \\ bb, ba, ab, aa \end{vmatrix} = aa \begin{vmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{vmatrix} + ab \begin{vmatrix} 0\,1\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{vmatrix} + ba \begin{vmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{vmatrix} +$$

$$+ bb \begin{vmatrix} 0\,0\,0\,1 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 1\,0\,0\,0 \end{vmatrix} = aa*\mathbf{1} + ab*e_1 + ba*e_2 + bb*e_3 \,.$$

*	**1**	e_1	e_2	e_3
1	**1**	e_1	e_2	e_3
e_1	e_1	**1**	e_3	e_2
e_2	e_2	e_3	**1**	e_1
e_3	e_3	e_2	e_1	**1**

Fig. 5. The decomposition of the matrix [a, b; b, a]$^{(2)}$, representing 4-dimensional hyperbolic numbers, into 4 sparse matrices, the set of which is closed relative to multiplication. The multiplication table for this set is shown at the right. The symbol **1** denotes the identity matrix e_0.

In general case, the higher tensor powers n = 3, 4, 5, … of the bisymmetric matrix [a, b; b, a] produce bisymmetric matrices, which can be also decomposed into 2^n sparse matrices, the set of which is closed relative to multiplication and which define appropriate multiplication tables of algebras of 2^n-dimensional hypercomplex numbers m_n (which were termed "hyperbolic matrions" of the order n in our previous publications [Petoukhov 2008; Petoukhov, He, 2010]).

3 Hyperbolic and Fibonacci numbers in phyllotaxis modelling

Fibonacci numbers F_n form an additive sequence such that each number is the sum of the two preceding ones: $F_n = F_{n-1} + F_{n-2}$ (Table 1).

Table 1. The Fibonacci sequence.

n	1	2	3	4	5	6	7	8	9	10	…
F_n	1	1	2	3	5	8	13	21	34	55	…

Fibonacci numbers are strongly related to the golden ratio $\varphi = (1+5^{0.5})/2$. Binet's formula (3) expresses the nth Fibonacci number in terms of n and the golden ratio, and implies that the ratio of two consecutive Fibonacci numbers tends to the golden ratio as n increases:

$$F_n = (\varphi^n - (-\varphi^{-1}))/5^{0.5} \qquad (3)$$

In biology, it has long been known that, for example, in many plant objects the spiral arrangement of their bioorganisms form ordered patterns (shoots of plants and trees, seeds in the heads of sunflowers, scales of coniferous cones and pineapples, etc.). These patterns are determined by overlapping left and right oriented spiral lines - parastichies. To characterize phyllotaxis of such botanical objects, usually indicate two parameters: number of left spirals and number of right spirals, which are observed on the surface of phyllotaxis objects. Phyllotaxis of structures with such patterns is described by ratios of neighboring Fibonacci numbers:

$$F_{n+1}/F_n: \quad 2/1,\ 3/2,\ 5/3,\ 8/5,\ 13/8,\ 21/13,\ 34/21,\ \ldots \qquad (4)$$

$$(F_{n+1}/F_n) \rightarrow (F_{n+2}/F_{n+1}):\ 2/1 \rightarrow 3/2 \rightarrow 5/3 \rightarrow 8/5 \rightarrow 13/8 \rightarrow 21/13 \rightarrow \ldots \qquad (5)$$

The sequence (4) is termed the "parastichic sequence" [Jean ,2006; Petoukhov, 1981]. It seems natural to use 2-dimensional hyperbolic numbers for modeling these 2-parametric patterns in phyllotaxis objects and their ontogenetic transformations. In this approach, proposed by the author, the sequence (4) of phyllotaxis ratios is transformed into additive sequences (6, 7) reflecting linear notation of appropriate hyperbolic numbers and their matrix representations (we term sequences (6, 7) as parastichic sequences of hyperbolic numbers):

$$F_{n+1} + jF_n: \ 2 + j,\ 3 + j2,\ 5 + 3j,\ 8 + 5j,\ 13 + 8j,\ 21 + 13j,\ 34 + 21j,\ \ldots. \qquad (6)$$

$$\begin{vmatrix} F_{n+1}, F_n \\ F_n, F_{n+1} \end{vmatrix} : \begin{vmatrix} 2, 1 \\ 1, 2 \end{vmatrix}, \begin{vmatrix} 3, 2 \\ 2, 3 \end{vmatrix}, \begin{vmatrix} 5, 3 \\ 3, 5 \end{vmatrix}, \begin{vmatrix} 8, 5 \\ 5, 8 \end{vmatrix}, \begin{vmatrix} 13, 8 \\ 8, 13 \end{vmatrix} \ldots \qquad (7)$$

In this approach, to define a hyperbolic number u+jv, which transforms a hyperbolic number $F_{n+1} + jF_n$ into its neighboring hyperbolic number $F_{n+2} + jF_{n+1}$ from the sequence (6), the following simple equation (8) should be solved:

$$(F_{n+1} + jF_n)(u + jv) = (F_{n+2} + jF_{n+1}) \qquad (8)$$

The solution to this equation (8) gives the following expressions (9) for components of the desired hyperbolic number u + jv:

$$u = F_{n+1}/F_n + (-1)^{n+1} {*} F_{n-1} / (F_n {*} (F_n^2 - F_{n-1}^2)), \quad v = (-1)^n / (F_n^2 - F_{n-1}^2) \qquad (9)$$

In the case of such components (9), $u^2 - v^2 \neq 1$ and the appropriate matrix [u, v; v, u] does not present a hyperbolic rotation in the sense of expression (2). But this matrix can be rewriting into the form (10) where the matrix of a hyperbolic rotation (in the sense of expression (2)) is multiplied by a coefficient $(u^2 - v^2)^{0.5}$:

$$[u, v; v, u] = (u^2 - v^2)^{0.5} [u(u^2 - v^2)^{-0.5}, v(u^2 - v^2)^{-0.5}; v(u^2 - v^2)^{-0.5}, u(u^2 - v^2)^{-0.5}] \qquad (10)$$

Now let us describe results of the author's study of eigenvalues of the symmetric matrices in the parastichic sequence (7). Each of these matrices $[F_{n+1}, F_n; F_n, F_{n+1}]$ has two eigenvalues, which are equal to two Fibonacci numbers again: F_{n+2} and F_{n-1}. One can noted that

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these eigenvalues are the sum and the difference of the Fibonacci components of the original hyperbolic number $F_{n+1}+ jF_n$ since $F_{n+2} = F_{n+1} + F_n$ and $F_{n-1} = F_{n+1} - F_n$. The ratio F_{n+2}/F_{n-1} of such eigenvalues defines a new sequence (11) of Fibonacci ratios, which tend to φ^3 as n increases:

$$F_{n+2}/F_{n-1}: \quad 3/1,\ 5/1,\ 8/2,\ 13/3,\ 21/5,\ 34/8,\ 55/13,\ \ldots. \qquad (11)$$

By analogy with expressions (4, 6, 7) such pair of eigenvalues F_{n+2} and F_{n-1} can be considered as components of a new hyperbolic number $F_{n+2} + jF_{n-1}$. In this case the sequence of ratios (11) is transformed into additive sequences (12, 13) reflecting linear notation of appropriate hyperbolic numbers and their matrix presentations:

$$F_{n+2} + jF_{n-1}: \quad 3 + j,\ 5 + j,\ 8 + j2,\ 13 + j3,\ 21 + j5,\ 34 + j8,\ 55 + j13,\ \ldots. \qquad (12)$$

$$\begin{vmatrix} F_{n+2}, F_{n-1} \\ F_{n-1}, F_{n+2} \end{vmatrix} : \begin{vmatrix} 3, 1 \\ 1, 3 \end{vmatrix} , \begin{vmatrix} 5, 1 \\ 1, 5 \end{vmatrix} , \begin{vmatrix} 8, 2 \\ 2, 8 \end{vmatrix} , \begin{vmatrix} 13, 3 \\ 3, 13 \end{vmatrix} , \begin{vmatrix} 21, 5 \\ 5, 21 \end{vmatrix} \ldots \qquad (13)$$

Each of symmetric matrices $[F_{n+2}, F_{n-1}; F_{n-1}, F_{n+2}]$ of the sequence (13) has two eigenvalues, which are again equal to two Fibonacci numbers multiplied by a factor 2 (twice the Fibonacci numbers): $2F_{n+1}$ and $2F_n$. Ratios $2F_{n+1}/2F_n$ of such eigenvalues form a sequence, which is identical to the initial parastichic sequence (4). Using the Binet's formula (3), all members of these sequences can be additionally expressed through the golden ratio φ in integer powers. This procedure of analysis of the eigenvalues of new and new sequences of symmetric matrices, representing hyperbolic numbers by analogy with sequences (6, 7, 12, 13), can be repeated as long as desired, obtaining a hierarchy of eigenvalues of the matrices based on Fibonacci numbers multiplied by a factor 2 at corresponding steps of the iterative procesure.

The following important point should be emphasized. In contrast to the traditional additive series of one-dimensional Fibonacci numbers, the author introduces an additive series of two-dimensional hyperbolic numbers and an additive series of (2*2)- matrices representing these numbers and defining an additional additive series of eigenvalues of these matrices (6, 7, 12, 13). As far as we know, such Fibonacci series of two-dimensional numbers have not been described in the literature by anyone, and therefore they can be considered new in the extensive subject matter of Fibonacci numbers and their applications (some of author's results of the study of additive series of 4-dimensional hyperbolic Fibonacci numbers will be presented below).

Similar results are obtained by considering the additive series of two-dimensional hyperbolic Lucas numbers and the additive series of their matrix representations, which determine the additive series of eigenvalues of these symmetric matrices (these results are been publishing in a separate article). Here one can remind that one-dimensional Lucas numbers form the series $L_{n+2} = L_n + L_{n+1}$: 2, 1, 3, 4, 7, 11, 18, ... , which is also known in phyllotaxis laws [8, 9]. A study of additive series of complex numbers, whose components are Fibonacci numbers, and of their ordinary representations by non-symmetric matrices gives also interesting additive series of their eigenvalues but in form of complex numbers.

It should be noted that the study of the eigenvalues of symmetric matrices has special meaning due to the fact that in the theory of oscillations symmetric matrices are matrix representations of oscillatory systems with many degrees of freedom. Moreover, the eigenvalues of such a matrix determine the resonant frequencies of the corresponding oscillatory system. The described results on the properties of inherited phyllotaxis phenomena with their Fibonacci ratios, represented by symmetric matrices and their matrix eigenvalues, are important, in particular, for the concept of multi-resonance genetics, which connects structural features of molecular-genetic systems with resonances of oscillatory systems [Petoukhov, 2016].

4 Fibonacci sequences of 2^n-dimensional hyperbolic numbers

This Section continues the theme of additive series of hyperbolic numbers, coordinates of which are Fibonacci numbers. Now we turn to algebraic extensions of hyperbolic numbers in forms of 2^n-dimensional hyperbolic numbers. Let us consider an additive sequence (14) of 4-dimensional hyperbolic numbers $F_{n+3}e_0+F_{n+2}e_1+F_{n+1}e_2+F_ne_3$ with Fibonacci coordinates from (Table 1). In this sequence, each member is equal to the sum of two previous members:

$$3e_0+2e_1+1e_2+1e_3;\ 5e_0+3e_1+2e_2+1e_3;\ 8e_0+5e_1+3e_2+2e_3;\ 13e_0+8e_1+5e_2+3e_3;\ \dots \qquad (14)$$

A corresponding matrix representation of each member from (14) has 4 eigenvalues, which can be considered again as coordinates of a new 4-dimensional hyperbolic number. The author reveals that these new 4-dimensional hyperbolic numbers form a new additive sequence (15):

$$1e_0+1e_1+3e_2+7e_3;\ 1e_0+3e_1+5e_2+11e_3;\ 2e_0+4e_1+8e_2+18e_3;\ 3e_0+7e_1+13e_2+29e_3;\dots \qquad (15)$$

The sequence (15) combines Fibonacci and Lucas sequences in the following sense. In its 4-dimensional hyperbolic numbers, coordinates of basic elements e_0 and e_2 are Fibonacci numbers and coordinates of basic elements e_1 and e_3 are Lucas numbers (3, 1, 4, 7, 11, 18, 29, …). Such aggregation of Fibonacci and Lucas numbers resembles a phyllotaxis-like locations of amino acid residues in the helices of polypeptides for various molecular chains - 11/3, 18/5, 29/8, 47/13; here fraction numerators are Lucas numbers and fraction denominators are Fibonacci numbers. These bio-molecular phenomena of polypeptides configurations are described in the fundamental book [Frey-Wissling, Muhlethaler, 1965].

A matrix representation of each member of the sequence (15) has 4 eigenvalues, which can be considered again as coordinates of a new 4-dimensional hyperbolic number. These 4-dimensional hyperbolic numbers form a new additive sequence (16):

$$-8e_0-4e_1+4e_2+12e_3;\ -12e_0-8e_1+4e_2+20e_3;\ -20e_0-12e_1+8e_2+32e_3;\ -32e_0-20e_1+12e_2+32e_3;..\ (16)$$

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Comparing sequences (14) and (16) reveals that a set of coordinates of each member of the sequence (16) repeats - with a factor 4 - a set of coordinates of the corresponding member of the sequence (14) with accuracy up to signs and a cyclic permutation of coordinates. For example, the first member of (14) contains coordinates 3, 2, 1, 1 and the first member of (16) contains coordinates -4*2, -4*1, 4*1, 4*3. This procedure of calculating repeating additive sequences of 4-dimensional hyperbolic numbers associated with Fibonacci and Lucas numbers can be repeated as long as desired. Similar results are received for additive sequences of 2^n-dimensional hyperbolic numbers with Fibonacci coordinates in cases $n = 3, 4, \ldots$.

Some concluding remarks

The development of modern mathematical natural sciences is based on the use of certain mathematical tools. Mathematical tools of theoretical research can be compared with glasses for a visually impaired person: adequate glasses provide a person with a clear and beautiful picture of reality, which he had previously seen as blurred and hidden by fog. This article attracts attention of researches to an important role of hyperbolic numbers and their matrix representations in modelling structural features of genetic phenomena.

The matrix form of presentation of hyperbolic numbers deserves special attention by the following reasons:

1) This presentation form is based on symmetric matrices, which are closely related with the theory of resonances of oscillatory systems, having many degrees of freedom, and also with Punnett squares from Mendelian genetics of inheritance of traits in living organisms [Petoukhov, 2011, 2016]. Symmetrical matrices are related with the theory of resonance of L. Pauling whose book [Pauling, 1940] about this theory in structural chemistry is the most quoted among scientific books of the XX century. The actual molecule, as Pauling proposed, is a sort of hybrid, a structure that resonates between the two alternative extremes; and whenever there is a resonance between the two forms, the structure is stabilized. Pauling claimed that living organisms are chemical in nature, and resonances in their molecules should be very essential for biological phenomena;

2) These symmetric matrices can be interpreted as metric tensors, which are main invariants in Riemanian geometry and which can be used in the theory of morpho-resonance morphogenesis [Petoukhov, 2008, 2016; Petoukhov, He, 2010];

3) These symmetric matrices are related with hyperbolic rotations [sh x, ch x; ch x, sh x], which are particular cases of hyperbolic numbers and are connected with the theory of biological phyllotaxis laws, with problems of locomotion control [Smolyaninov, 2000] and also with Lorenz transformations in the special theory of relativity;

4) These symmetric matrices are related with the theory of solitons of sine-Gordon equation. Such known solitons are the only relativistic type of soli-tons; they were put forward for the role of the fundamental type of solitons of living matter in the book [Petoukhov, 1999].

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