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Unified Degenerate Central Bell Polynomials

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Abstract

In this paper, we firstly consider extended degenerate central factorial numbers of the second kind and provide some properties of them. We then introduce unified degenerate central Bell polynomials and numbers and investigate many relations and formulas including summation formula, explicit formula and derivative property. Moreover, we derive several correlations for the fully degenerate central Bell polynomials associated with the degenerate Bernstein polynomials and the degenerate Bernoulli, Euler and Genocchi numbers.

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1. Introduction

The classical Bell polynomials $Bel_n(x)$ (called also Touchard polynomials and exponential polynomials) and central Bell polynomials $B_n^{(c)}(x)$ (called also central exponential polynomials) are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} = e^{x\left(e^t-1\right)} \quad (cf.\ [2\text{-}4, 9, 12, 13, 16, 20]) \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!} = e^{x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (cf.\ [8, 11, 17]). \tag{1.2}$$

The classical Bell numbers Bel_n and central Bell numbers $B_n^{(c)}$ are acquired by choosing $x=1$ in (1.1) and (1.2), that is $Bel_n(1) := Bel_n$ and $B_n^{(c)}(1) := B_n^{(c)}$, which are given by the following exponential generating function:

$$\sum_{n=0}^{\infty} Bel_n\frac{t^n}{n!} = e^{\left(e^t-1\right)} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n^{(c)}\frac{t^n}{n!} = e^{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}. \tag{1.3}$$

The Bell polynomials extensively studied by Bell [2] appear as a standard mathematical tool and arise in combinatorial analysis. The familiar Bell polynomials and the central Bell polynomials have been intensely studied by many mathematicians, *cf.* [2-4, 8, 9, 11-13, 16, 17, 20] and see also the references cited therein. For example, Bouroubi [3] provides a novel and interesting approach to the determination of new formulas for the Bell polynomials, based on the Lagrange inversion formula, and the binomial sequences which gives the easy recovery of known relations and deduction of several new formulas covering these polynomials. Carlitz [4] investigate diverse formulas for the Bell numbers including correlations with the Stirling numbers of the

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second kind, combinatorial interpretation and derivative property. Kim *et al.* [8] considered the central Bell polynomials and numbers and presented several relations and identities for these polynomials and numbers. Kim *et al.* [9] analyses properties of the Bell polynomials by using and without using umbral calculus and proved several representations for multifarious known families of polynomials such as Cauchy polynomials, Bernoulli polynomials, poly-Bernoulli polynomials and falling factorials by means of the Bell polynomials. Mihoubi [20] gave some results tied the Bell polynomials and the binomial type sequences, which were used to derive some novel formulas for the Bell polynomials. These large investigations of the Bell polynomials and numbers yields a motivation to improve this mathematical tool.

For non-negative integer n, the central factorial numbers of the second kind $T(n,m)$ are defined by the following exponential generating function

$$\sum_{n=0}^{\infty} T(n,m) \frac{t^n}{n!} = \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^m}{m!} \quad (cf.\ [8,11,17]) \tag{1.4}$$

or by recurrence relation for a fixed non-negative integer n,

$$x^n = \sum_{m=0}^{n} T(n,m)\, x^{[m]}, \tag{1.5}$$

where the notation $x^{[m]}$ called as the central factorial equals to $x\left(x+\frac{m}{2}-1\right)\left(x+\frac{m}{2}-2\right)\cdots\left(x-\frac{m}{2}+1\right)$ with initial condition $x^{[0]}=1$, *cf.* $[8,11,17]$ and see also references cited therein.

The central Bell polynomials and central factorial numbers of the second kind satisfy the following relation (*cf.* $[8,11,17]$)

$$B_n^{(c)}(x) = \sum_{m=0}^{n} T(n,m)\, x^m. \tag{1.6}$$

The Stirling numbers of the first kind $S_1(n,m)$ are defined as follows (*cf.* [7-9, 11-14, 17, 20])

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)\, x^m, \tag{1.7}$$

where the notation $(x)_n$ called as the falling factorial equals to $x(x-1)\cdots(x-n+1)$, *cf.* [7-9, 11-14, 17, 20] and see also references cited therein.

The following sections are planned as follows: The second section includes the definition of the fully degenerate central Bell polynomials and numbers and provides several formulas and relations including the unified degenerate central factorial numbers of the second kind and Stirling numbers of the first kind. The third section covers diverse correlations for the fully degenerate central Bell polynomials associated with the degenerate Bernstein polynomials, the degenerate Bernoulli, Euler and Genocchi numbers. The last section of this paper analyses the results acquired in this paper.

2. Unified Degenerate Central Bell Polynomials

In this section, we perform to analyze and investigate degenerate forms of some special polynomials and numbers. We focus on the unified degenerate central factorial numbers of the second kind and the unified degenerate central Bell polynomials and numbers. We then derive several properties and formulas for these polynomials.

In the theory of special polynomials and special functions, the degenerate forms for polynomials and functions have been studied and investigated by many mathematicians for more than a century, *cf.* [4-6, 9-19] and see also the references cited therein. Carlitz [5] introduced higher order degenerate Euler polynomials and provided several properties. Carlitz [6] gave the degenerate Staudt-Clausen theorem and illustrated it for the degenerate Bernoulli numbers. Howard [7] proved some explicit formulas for degenerate Bernoulli polynomials. Kim *et al.* [10] considered the degenerate Bernstein polynomials and attained their generating function, recurrence relations, symmetric identities, and some connections with generalized falling factorial polynomials, higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second

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kind. Kim *et al.* [11] studied on the degenerate central Bell numbers and polynomials and derived some properties, identities, and recurrence relations. Kim *et al.* [12] considered degenerate Bell numbers and polynomials and presented several novel formulas for those numbers and polynomials associated with special numbers and polynomials by using the notion of composita. Kim *et al.* [13] acquired diverse properties, recurrence relations, and identities associated with the degenerate r-Stirling numbers of the second kind and the degenerate r-Bell polynomials by means of umbral calculus. Kim *et al.* [14] presented various explicit formulas and recurrence relationships for the degenerate Mittag-Leffler polynomials and also gave several connections between Mittag-Leffler polynomials and other known families of polynomials. Kim *et al.* [15] introduced the degenerate Laplace transform and degenerate gamma function and obtained several interesting formulas related to this transform and this gamma function. Kim *et al.* [16] considered partially degenerate Bell polynomials numbers and developed their properties and identities. Kim *et al.* [17] defined and studied on the extended degenerate r-central factorial numbers of the second kind and the extended degenerate r-central Bell polynomials. Kwon *et al.* [18] considered degenerate Changhee polynomials and proved several relations and formulas for these polynomials. Lim [19] defined higher order degenerate Genocchi polynomials and gave some identities and formulas for these polynomials.

For non-negative integer n, the degenerate central factorial numbers of the second kind $T_{2,\lambda}(n,m)$ are defined by the following exponential generating function

$$\sum_{n=0}^{\infty} T_{2,\lambda}(n,m)\frac{t^n}{n!} = \frac{\left(e_{\lambda}^{\frac{1}{2}}(t) - e_{\lambda}^{-\frac{1}{2}}(t)\right)^m}{m!} \quad (cf.\ [11]), \tag{2.1}$$

where the notation $e_{\lambda}^{x}(t)$ denotes the degenerate exponential function for a real number λ, given by

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{\frac{x}{\lambda}} \text{ and } e_{\lambda}^{1}(t) = e_{\lambda}(t). \tag{2.2}$$

It is readily seen that $\lim_{\lambda\to 0} e_{\lambda}^{x}(t) = e^{xt}$, *cf.* [11] and [17].

Remark 1. *When λ approaches to 0, the degenerate central factorial numbers of the second kind (2.1) reduces to the central factorial numbers of the second kind (1.4), namely $\lim_{\lambda\to 0} T_{2,\lambda}(n,m) = T(n,m)$.*

We are now ready to give the definition of the unified degenerate central factorial numbers of the second kind.

Definition 1. *Let λ and ω be real numbers. The unified degenerate central factorial numbers of the second kind $T_{2,\lambda;\omega}(n,m)$ are introduced by means of the following generating function*

$$\sum_{n=0}^{\infty} T_{2,\lambda;\omega}(n,m)\frac{t^n}{n!} = \frac{\left(e_{\lambda}^{\omega}(t) - e_{\lambda}^{-\omega}(t)\right)^m}{m!}. \tag{2.3}$$

We here analyze some circumstances of the unified degenerate central factorial numbers of the second kind $T_{2,\lambda;\omega}(n,m)$ as follows.

Remark 2.

(1) *When $\omega = \frac{1}{2}$, we obtain the degenerate central factorial numbers of the second kind $T_{2,\lambda}(n,m)$ in (2.1), cf.* [11].

(2) *When $\lambda \to 0$, the unified degenerate central factorial numbers of the second kind $T_{2,\lambda;\omega}(n,m)$ reduce to the ω-analogue of the central factorial numbers of the second kind denoted by $T_{2;\omega}(n,m)$, which is also new generalization of the factorial numbers of the second kind $T(n,m)$ in (1.4), given by*

$$\sum_{n=0}^{\infty} T_{2;\omega}(n,m)\frac{t^n}{n!} = \frac{\left(e^{\omega t} - e^{-\omega t}\right)^m}{m!}. \tag{2.4}$$

(3) *When $\omega = \frac{1}{2}$ and $\lambda \to 0$, we obtain the usual central factorial numbers of the second kind $T(n,m)$ in (1.4), cf. [8, 11, 17].*

We now investigate some properties of the unified degenerate central factorial numbers of the second kind $T_{2,\lambda;\omega}(n,m)$. Hence, we give the following Theorem 1.

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Theorem 1. *For non-negative integers k and m, we have*

$$T_{2,\lambda;\omega}(n,k+m)=\frac{k!m!}{(k+m)!}\sum_{u=0}^{n}\binom{n}{u}T_{2,\lambda;\omega}(u,k)\,T_{2,\lambda;\omega}(n-u,m)\,. \tag{2.5}$$

Proof. In view of Definition 1, we write

$$\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^k}{k!}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}=\sum_{n=0}^{\infty}T_{2;\omega}(n,k)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{2;\omega}(n,m)\frac{t^n}{n!}$$

and then we get

$$\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^{k+m}}{(k+m)!}=\frac{k!m!}{(k+m)!}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\binom{n}{u}T_{2;\omega}(u,k)\,T_{2;\omega}(n-u,m)\frac{t^n}{n!},$$

which implies the asserted result (2.5). □

We here give the following correlation.

Theorem 2. *The following correlation*

$$T_{2,\lambda;\omega}(n,m)=\sum_{l=m}^{n}T_{2;\omega}(l,m)\,\lambda^{n-l}S_1(n,l) \tag{2.6}$$

is valid for real numbers λ and ω.

Proof. By Definition 1 and the identity (2.2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,m)\frac{t^n}{n!} &= \frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}\\ &= \frac{1}{m!}\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)^m\\ &= \frac{1}{m!}\left(e^{\frac{\omega}{\lambda}\log(1+\lambda t)}-e^{-\frac{\omega}{\lambda}\log(1+\lambda t)}\right)^m\\ &= \sum_{l=0}^{\infty}T_{2;\omega}(l,m)\,\lambda^{-l}\frac{\left(\log(1+\lambda t)\right)^l}{l!}\\ &= \sum_{l=0}^{\infty}T_{2;\omega}(l,m)\,\lambda^{-l}\sum_{n=l}^{\infty}S_1(n,l)\,\lambda^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}T_{2;\omega}(l,m)\,S_1(n,l)\,\lambda^{n-l}\frac{t^n}{n!},\end{aligned}$$

which provides the desired result (2.6). □

The degenerate classical Bell polynomials and the degenerate central Bell polynomials are given by the following Taylor series expansion at $t=0$ as follows:

$$\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}=e^{x(e_\lambda(t)-1)}\quad (cf.\ [10,16]) \tag{2.7}$$

and

$$\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x)\frac{t^n}{n!}=e^{x\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)}\quad (cf.\ [11]), \tag{2.8}$$

When $x=1$ in (2.7) and (2.8), the mentioned polynomials ($B_{n,\lambda}(x)$ and $B_{n,\lambda}^{(c)}(x)$) reduces to the corresponding numbers

$$B_{n,\lambda}(1):=B_{n,\lambda}\text{ and }B_{n,\lambda}^{(c)}(1):=B_{n,\lambda}^{(c)}, \tag{2.9}$$

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termed as the degenerate Bell numbers and the degenerate central Bell numbers, respectively.

Remark 3. *We note that using (2.2), the degenerate classical Bell polynomials (2.7) and the degenerate central Bell polynomials (2.8) reduce the classical Bell polynomials (1.1) and the central Bell polynomials (1.2) in the following limit cases:*

$$\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x) \text{ and } \lim_{\lambda\to 0} B^{(c)}_{n,\lambda}(x) = B^{(c)}_n(x). \tag{2.10}$$

For non-negative integer n and a real number λ, λ-extension of falling factorial is defined by (*cf.* [10, 16])

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda). \tag{2.11}$$

From (2.2) and (2.11), we obtain the following relation (*cf.* [11, 17])

$$e^x_\lambda(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}. \tag{2.12}$$

The degenerate central Bell polynomials and the degenerate central factorial numbers of the second kind satisfy the following relation (*cf.* [11])

$$B^{(c)}_{n,\lambda}(x) = \sum_{m=0}^{n} T_{2,\lambda}(n,m)\,x^m. \tag{2.13}$$

We are now ready to define the unified degenerate central Bell polynomials and numbers by the following Definition 2.

Definition 2. *Let α,λ and ω be real numbers. The unified degenerate central Bell polynomials $B^{(c)}_{n,\lambda,\alpha}(x:\omega)$ and the unified degenerate central Bell numbers $B^{(c)}_{n,\lambda,\alpha}(\omega)$ are respectively defined by the following generating functions*

$$G(x:\lambda,\alpha;\omega) = \sum_{n=0}^{\infty} B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!} = e^x_\alpha\left(e^\omega_\lambda(t) - e^{-\omega}_\lambda(t)\right) \tag{2.14}$$

and

$$G(\lambda,\alpha;\omega) = \sum_{n=0}^{\infty} B^{(c)}_{n,\lambda,\alpha}(\omega)\frac{t^n}{n!} = e_\alpha\left(e^\omega_\lambda(t) - e^{-\omega}_\lambda(t)\right). \tag{2.15}$$

The immediate relation for the unified degenerate central Bell polynomials and numbers is $B^{(c)}_{n,\lambda,\alpha}(1:\omega) := B^{(c)}_{n,\lambda,\alpha}(\omega)$.

We now examine some special cases of the unified degenerate central Bell polynomials as follows.

Remark 4.

(1) *When $\omega = \frac{1}{2}$, the unified degenerate central Bell polynomials $B^{(c)}_{n,\lambda,\alpha}(x:\omega)$ and numbers $B^{(c)}_{n,\lambda,\alpha}(\omega)$ in (2.14) and (2.15) reduce to the fully degenerate central Bell polynomials $B^{(c)}_{n,\lambda,\alpha}(x)$ and numbers $B^{(c)}_{n,\lambda,\alpha}$ in (2.16), which are also new generalizations of the central Bell polynomials $B^{(c)}_n(x)$ and numbers $B^{(c)}_n$ in (1.2) and (1.3), given by*

$$\sum_{n=0}^{\infty} B^{(c)}_{n,\lambda,\alpha}(x)\frac{t^n}{n!} = e^x_\alpha\left(e^{\frac{1}{2}}_\lambda(t) - e^{-\frac{1}{2}}_\lambda(t)\right) \text{ and } \sum_{n=0}^{\infty} B^{(c)}_{n,\lambda,\alpha}\frac{t^n}{n!} = e_\alpha\left(e^{\frac{1}{2}}_\lambda(t) - e^{-\frac{1}{2}}_\lambda(t)\right). \tag{2.16}$$

(2) *When $\alpha\to 0$, the unified degenerate central Bell polynomials $B^{(c)}_{n,\lambda,\alpha}(x:\omega)$ and numbers $B^{(c)}_{n,\lambda,\alpha}(\omega)$ in (2.14) and (2.15) reduce to the extended degenerate central Bell polynomials $B^{(c)}_{n,\lambda}(x:\omega)$ and numbers $B^{(c)}_{n,\lambda}(\omega)$ in (2.17), which are also novel extensions of the central Bell polynomials $B^{(c)}_n(x)$ and numbers $B^{(c)}_n$ in (1.2) and (1.3), given by*

$$\sum_{n=0}^{\infty} B^{(c)}_{n,\lambda}(x:\omega)\frac{t^n}{n!} = e^{x\left(e^\omega_\lambda(t) - e^{-\omega}_\lambda(t)\right)} \text{ and } \sum_{n=0}^{\infty} B^{(c)}_{n,\lambda}(\omega)\frac{t^n}{n!} = e^{\left(e^\omega_\lambda(t) - e^{-\omega}_\lambda(t)\right)}. \tag{2.17}$$

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(3) *When $\omega=\frac{1}{2}$ and $\alpha\to 0$, we get the degenerate central Bell polynomials and numbers denoted by $B^{(c,)}_{n,\lambda}(x)$ and $B^{(c)}_{n,\lambda}$ in (2.8) and (2.9) (cf.* [11]*).*

(4) *When $\omega=\frac{1}{2}$ and $\lambda\to 0$, we attain the degenerate central Bell polynomials and numbers denoted by $\mathcal{B}^{(c)}_{n,\lambda}(x)$ and $\mathcal{B}^{(c)}_{n,\lambda}$, which is different from the polynomials and numbers in (2.8) and (2.9) given by Kim et al.* [11]*:*

$$\sum_{n=0}^{\infty}\mathcal{B}^{(c)}_{n,\alpha}(x)\frac{t^n}{n!}=e^x_\alpha\left(e^{\frac{1}{2}}(t)-e^{-\frac{1}{2}}(t)\right)\text{ and }\sum_{n=0}^{\infty}\mathcal{B}^{(c)}_{n,\alpha}\frac{t^n}{n!}=e_\alpha\left(e^{\frac{1}{2}}(t)-e^{-\frac{1}{2}}(t)\right) \tag{2.18}$$

(5) *When $\alpha\to 0$, $\lambda\to 0$ and $\omega=\frac{1}{2}$, we reach the central Bell polynomials and numbers in (1.2) and (1.3) (cf. [8, 11, 17]).*

We now investigate the properties of the unified degenerate central Bell polynomials $B^{(c)}_{n,\lambda,\alpha}(x:\omega)$. Thus, we firstly give the following theorem that includes a formula which is the generalization of the relations (1.6) and (2.13).

Theorem 3. *The following relation*

$$B^{(c)}_{n,\lambda,\alpha}(x:\omega)=\sum_{m=0}^{n}T_{2,\lambda;\omega}(n,m)(x)_{m,\alpha} \tag{2.19}$$

holds true for real numbers α,λ and ω.

Proof. By Definition 2 and formulas (2.2) and (2.12), we get

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\frac{\left(e^{\omega}_{\lambda}(t)-e^{-\omega}_{\lambda}(t)\right)^m}{m!}\\ &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,m)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(x)_{m,\alpha}T_{2,\lambda;\omega}(n,m)\frac{t^n}{n!},\end{aligned}$$

which gives the claimed result (2.19). □

We now state a summation formula for $B^{(c)}_{n,\lambda,\alpha}(x:\omega)$ as follows.

Theorem 4. *The following summation formula*

$$B^{(c)}_{n,\lambda,\alpha}(x+y:\omega)=\sum_{m=0}^{n}\binom{n}{m}B^{(c)}_{n-m,\lambda,\alpha}(x:\omega)B^{(c)}_{m,\lambda,\alpha}(y:\omega) \tag{2.20}$$

is valid for real numbers α,λ and ω.

Proof. By Definition 2 and the identity (2.12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x+y:\omega)\frac{t^n}{n!} &= e^{x+y}_{\alpha}\left(e^{\omega}_{\lambda}(t)-e^{-\omega}_{\lambda}(t)\right)\\ &= e^{x}_{\alpha}\left(e^{\omega}_{\lambda}(t)-e^{-\omega}_{\lambda}(t)\right)e^{y}_{\alpha}\left(e^{\omega}_{\lambda}(t)-e^{-\omega}_{\lambda}(t)\right)\\ &= \sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(y:\omega)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B^{(c)}_{n-m,\lambda,\alpha}(x:\omega)B^{(c)}_{m,\lambda,\alpha}(y:\omega)\frac{t^n}{n!},\end{aligned}$$

which provides the desired result (2.20). □

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We now provide a correlation as follows.

Theorem 5. *The following formula*

$$B_{n,\lambda,\alpha}^{(c)}(x:\omega)=\sum_{m=0}^{n}\sum_{l=m}^{n}T_{2;\omega}(n,m)\,S_1(n,l)\,\lambda^{n-1}(x)_{m,\alpha} \tag{2.21}$$

holds true for real numbers α,λ *and* ω.

Proof. By Definition 2 and Theorem 2, we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}\\ &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\sum_{n=0}^{\infty}\sum_{l=0}^{n}T_{2;\omega}(l,m)\,S_1(n,l)\,\lambda^{n-l}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n}(x)_{m,\alpha}\,T_{2;\omega}(l,m)\,S_1(n,l)\,\lambda^{n-l}\frac{t^n}{n!}\end{aligned}$$

which gives the claimed result (2.21). □

We here provide an explicit formula for $B_{n,\lambda,\alpha}^{(c)}(x:\omega)$ as follows.

Theorem 6. *The following explicit formula*

$$B_{n,\lambda,\alpha}^{(c)}(x:\omega)=\sum_{j=0}^{n}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{n}{j}\binom{m}{k}(x)_{m,\alpha}\frac{(-1)^{m-k}}{m!}(\omega k)_{j,\lambda}\frac{t^n}{n!}(\omega(k-m))_{n-j,\lambda} \tag{2.22}$$

holds true for real numbers α,λ *and* ω.

Proof. By Definition 2 and formulas (2.2) and (2.12), we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\binom{m}{k}e_\lambda^{\omega k}(t)\,e_\lambda^{-\omega(m-k)}(t)\,(-1)^{m-k}\\ &= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(1+\lambda t)^{\frac{\omega k}{\lambda}}(1+\lambda t)^{-\frac{\omega(m-k)}{\lambda}}\\ &= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\sum_{n=0}^{\infty}(\omega k)_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}(\omega(k-m))_{n,\lambda}\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(\omega k)_{j,\lambda}\frac{t^n}{n!}(\omega(k-m))_{n-j,\lambda}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{n}{j}\binom{m}{k}(x)_{m,\alpha}\frac{(-1)^{m-k}}{m!}(\omega k)_{j,\lambda}\frac{t^n}{n!}(\omega(k-m))_{n-j,\lambda}\frac{t^n}{n!}\end{aligned}$$

which gives the asserted result (2.22). □

We give the following theorem.

Theorem 7. *The following relation*

$$B_{n+1,\lambda,\alpha}^{(c)}(x:\omega)=x\omega\sum_{j=0}^{n}\binom{n}{j}B_{n-j,\lambda,\alpha}^{(c)}(x-\alpha:\omega)\left((\omega-\lambda)_{j,\alpha}+(-\omega-\lambda)_{j,\alpha}\right) \tag{2.23}$$

holds true for real numbers α,λ *and* ω.

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Proof. Differentiating with respect to t to both sides of Definition 2 and in terms of formulas (2.2) and (2.12), we get

$$\begin{aligned}
\frac{d}{dt}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= \frac{d}{dt}e_{\alpha}^{x}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\\
&= \frac{d}{dt}\left(1+\alpha\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)\right)^{\frac{x}{\alpha}}\\
&= \frac{x}{\alpha}\left(1+\alpha\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)\right)^{\frac{x}{\alpha}-1}\alpha\left(\omega(1+\lambda t)^{\frac{\omega}{\lambda}-1}+\omega(1+\lambda t)^{-\frac{\omega}{\lambda}-1}\right)\\
&= x\omega\left(1+\alpha\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)\right)^{\frac{x-\alpha}{\alpha}}\left(e_{\lambda}^{\omega-\lambda}(t)+e_{\lambda}^{-\omega-\lambda}(t)\right)\\
&= x\omega\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x-\alpha:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left((\omega-\lambda)_{n,\alpha}+(-\omega-\lambda)_{n,\alpha}\right)\frac{t^n}{n!}\\
&= x\omega\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}B_{n-j,\lambda,\alpha}^{(c)}(x-\alpha:\omega)\left((\omega-\lambda)_{j,\alpha}+(-\omega-\lambda)_{j,\alpha}\right)\frac{t^n}{n!}
\end{aligned}$$

which implies the desired result (2.23). □

We now present a derivative property for $B_{n,\lambda,\alpha}^{(c)}(x:\omega)$ as follows.

Theorem 8. *The following relation*

$$\frac{d}{dx}B_{n,\lambda,\alpha}^{(c)}(x:\omega)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}B_{n-k,\lambda,\alpha}^{(c)}(x:\omega)T_{2,\lambda;\omega}(k,m) \tag{2.24}$$

holds true for real numbers α,λ and ω.

Proof. By Definition 2 and formulas (2.2) and (2.12), we get

$$\begin{aligned}
\frac{d}{dx}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= \frac{d}{dx}e_{\alpha}^{x}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\\
&= \frac{d}{dx}\left(1+\alpha\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\right)^{\frac{x}{\alpha}}\\
&= \left(1+\alpha\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\right)^{\frac{x}{\alpha}}\ln\left(\left(1+\alpha\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\right)^{\alpha^{-1}}\right)\\
&= \alpha^{-1}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\alpha^{m}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}\\
&= \sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}\frac{\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)^{m}}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,m)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=1}^{\infty}(m-1)!(-\alpha)^{m-1}B_{n-k,\lambda,\alpha}^{(c)}(x:\omega)T_{2,\lambda;\omega}(k,m)\frac{t^n}{n!}
\end{aligned}$$

which means the claimed result (2.24). □

3. Connections with Some Known Degenerate Polynomials and Numbers

The main aim of this section is to derive diverse connections with some earlier degenerate polynomials such as Bernstein, Bernoulli, Genocchi and Euler polynomials for the fully degenerate central Bell polynomials. Thanks to this purpose, we acquire multifarious correlations in the family of the degenerate polynomials.

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We firstly perform to obtain a relation with the degenerate Bernstein polynomials. We start with the following computations

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}(x)_{m,\alpha}\frac{\left(e_\lambda^{\omega}(t)-e_\lambda^{-\omega}(t)\right)^m}{m!}\\
&= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\binom{m}{k}e_\lambda^{\omega k}(t)\,e_\lambda^{-\omega(m-k)}(t)(-1)^{m-k}\\
&= \sum_{m=0}^{\infty}\frac{(x)_{m,\alpha}}{m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}(-1)^{m-k}(1+\lambda t)^{\frac{\omega k}{\lambda}}(1+\lambda t)^{-\frac{\omega(m-k)}{\lambda}}\\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}}{t^k(1-\omega(m-2k))_{k,\lambda}}\frac{(-1)^{m-k}}{(m-k)!}\frac{(1+\omega(m-2k))_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{\omega(2k-m)}{\lambda}}\\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}}{(1+\omega(m-2k))_{k,\lambda}}\frac{(-1)^{m-k}}{(m-k)!}\sum_{n=0}^{\infty}\mathfrak{B}_{k,n}(1+\omega(m-2k):\lambda)\frac{t^{n-k}}{n!},
\end{aligned}$$

where the degenerate Bernstein polynomials are defined by

$$\sum_{n=0}^{\infty}\mathfrak{B}_{k,n}(x:\lambda)\frac{t^n}{n!}=\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}}, \tag{3.1}$$

cf. [10]. Thus, we obtain the following theorem.

Theorem 9. *The following relation*

$$B_{n,\lambda,\alpha}^{(c)}(x:\omega)=n!\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(x)_{m,\alpha}(-1)^{m-k}}{(1+\omega(m-2k))_{k,\lambda}}\frac{\mathfrak{B}_{k,n+k}(1+\omega(m-2k):\lambda)}{(m-k)!(n+k)!} \tag{3.2}$$

is valid.

Let

$$I=(x)_{k,\alpha}\frac{\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)^k}{k!}\left(1+\alpha\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)\right)^{\frac{1-x}{\alpha}}.$$

Therefore, from (2.3) and (3.1), we obtain

$$\begin{aligned}
I &= \sum_{j=0}^{\infty}\mathfrak{B}_{k,j}(x:\alpha)\frac{\left((1+\lambda t)^{\frac{\omega}{\lambda}}-(1+\lambda t)^{-\frac{\omega}{\lambda}}\right)^j}{j!}\\
&= \sum_{j=0}^{\infty}\mathfrak{B}_{k,j}(x:\alpha)\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,j)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\mathfrak{B}_{k,j}(x:\alpha)T_{2,\lambda;\omega}(n,j)\right)\frac{t^n}{n!}
\end{aligned}$$

and by (2.3) and (2.14), similarly

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}B_{n,\lambda,\alpha}^{(c)}(1-x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_{2,\lambda;\omega}(n,k)\frac{t^n}{n!}(x)_{k,\alpha}\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}B_{j,\lambda,\alpha}^{(c)}(1-x:\omega)T_{2,\lambda;\omega}(n-j,k)(x)_{k,\alpha}\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we arrive at the following theorem.

Theorem 10. *The following summation correlation*

$$\sum_{j=0}^{n}\mathfrak{B}_{k,j}\left(x:\alpha\right)T_{2,\lambda;\omega}\left(n,j\right)=\sum_{j=0}^{n}\binom{n}{j}\left(x\right)_{k,\alpha}B_{j,\lambda,\alpha}^{(c)}\left(1-x:\omega\right)T_{2,\lambda;\omega}\left(n-j,k\right) \tag{3.3}$$

holds true.

In the special cases of the Theorems 9 and 10, we get new formulas for the usual central Bell polynomials and familiar Bernstein polynomials as follows.

Corollary 1. *We have*

$$B_{n}^{(c)}\left(x\right)=n!\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\left(x\right)_{m}\left(-1\right)^{m-k}}{\left(1+\frac{1}{2}\left(m-2k\right)\right)_{k,\lambda}}\frac{\mathfrak{B}_{k,n+k}\left(1+\frac{1}{2}\left(m-2k\right)\right)}{\left(m-k\right)!\left(n+k\right)!}$$

and

$$\sum_{j=0}^{n}\mathfrak{B}_{k,j}\left(x\right)T_{2}\left(n,j\right)=\sum_{j=0}^{n}\binom{n}{j}\left(x\right)_{k}B_{j}^{(c)}\left(1-x\right)T_{2}\left(n-j,k\right),$$

where the Bernstein polynomials are defined by the following generating function (cf. [5]*):*

$$\sum_{n=0}^{\infty}\mathfrak{B}_{k,n}\left(x\right)\frac{t^{n}}{n!}=\frac{\left(tx\right)^{k}}{k!}e^{(1-x)t}.$$

The classical Bernoulli, Euler and Genocchi numbers are defined by the following generating functions (*cf.* [5, 19]):

$$\sum_{n=0}^{\infty}B_{n}\frac{t^{n}}{n!}=\frac{t}{e^{t}-1}\quad\left(\left|t\right|<2\pi\right),\ \sum_{n=0}^{\infty}E_{n}\frac{t^{n}}{n!}=\frac{2}{e^{t}+1}\quad\left(\left|t\right|<\pi\right)$$

and

$$\sum_{n=0}^{\infty}G_{n}\frac{t^{n}}{n!}=\frac{2t}{e^{t}+1}\quad\left(\left|t\right|<\pi\right).$$

The degenerate Bernoulli, Euler and Genocchi numbers are given by the following Taylor series expansions at $t=0$ (*cf.* [5]):

$$\sum_{n=0}^{\infty}B_{n,\lambda}\frac{t^{n}}{n!}=\frac{t}{e_{\lambda}\left(t\right)-1},\quad\sum_{n=0}^{\infty}E_{n,\lambda}\frac{t^{n}}{n!}=\frac{2}{e_{\lambda}\left(t\right)+1} \tag{3.4}$$

and (*cf.* [19]):

$$\sum_{n=0}^{\infty}G_{n,\lambda}\frac{t^{n}}{n!}=\frac{2t}{e_{\lambda}\left(t\right)+1}. \tag{3.5}$$

We here give a formula including the degenerate Bernoulli numbers and the fully degenerate central Bell polynomials as follows.

Theorem 11. *The following formula*

$$\begin{aligned}B_{n,\lambda,\alpha}^{(c)}\left(x:\omega\right) &= \frac{1}{n+1}\sum_{k=0}^{n+1}\sum_{m=0}^{k}\binom{n+1}{k}\binom{k}{m}\left(1\right)_{n+1-k,\lambda}B_{k-m,\lambda,\alpha}^{(c)}\left(x:\omega\right)B_{m,\lambda} \\ &\quad -\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+1-k,\lambda,\alpha}^{(c)}\left(x:\omega\right)B_{k,\lambda}\end{aligned} \tag{3.6}$$

holds true.

Proof. By (2.14) and (3.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= e_{\alpha}^{x}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\frac{t}{e_{\lambda}(t)-1}\frac{e_{\lambda}(t)-1}{t}\\ &= \sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty} B_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^{n-1}}{n!}\\ &\quad -\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty} B_{n,\lambda}\frac{t^{n-1}}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(1)_{n-k,\lambda}B_{k-m,\lambda,\alpha}^{(c)}(x:\omega)B_{m,\lambda}\frac{t^{n-1}}{n!}\\ &\quad -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda,\alpha}^{(c)}(x:\omega)B_{k,\lambda}\frac{t^{n-1}}{n!},\end{aligned}$$

which implies the desired result (3.6). □

A relation including the degenerate Euler numbers and the fully degenerate central Bell polynomials is given by the following theorem.

Theorem 12. *The following summation formula*

$$\begin{aligned}B_{n,\lambda,\alpha}^{(c)}(x:\omega) &= \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(1)_{n-k,\lambda}}{2}B_{k-m,\lambda,\alpha}^{(c)}(x:\omega)E_{m,\lambda} \\ &\quad +\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda,\alpha}^{(c)}(x:\omega)E_{k,\lambda}\end{aligned} \tag{3.7}$$

is valid.

Proof. By (2.14) and (3.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!} &= e_{\alpha}^{x}\left(e_{\lambda}^{\omega}(t)-e_{\lambda}^{-\omega}(t)\right)\frac{2}{e_{\lambda}(t)+1}\frac{e_{\lambda}(t)+1}{2}\\ &= \frac{1}{2}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty} E_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^{n-1}}{n!}\\ &\quad +\frac{1}{2}\sum_{n=0}^{\infty} B_{n,\lambda,\alpha}^{(c)}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty} E_{n,\lambda}\frac{t^{n-1}}{n!}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(1)_{n-k,\lambda}B_{k-m,\lambda,\alpha}^{(c)}(x:\omega)E_{m,\lambda}\frac{t^{n}}{n!}\\ &\quad +\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda,\alpha}^{(c)}(x:\omega)E_{k,\lambda}\frac{t^{n}}{n!},\end{aligned}$$

which means the purposed result (3.7). □

A relationship for the degenerate Genocchi numbers and the fully degenerate central Bell polynomials is stated in the following theorem.

Theorem 13. *The following relation*

$$B^{(c)}_{n,\lambda,\alpha}(x:\omega) = \frac{1}{n+1}\sum_{k=0}^{n+1}\sum_{m=0}^{k}\binom{n+1}{k}\binom{k}{m}\frac{(1)_{n+1-k,\lambda}}{2}B^{(c)}_{k-m,\lambda,\alpha}(x:\omega)G_{m,\lambda} + \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{B^{(c)}_{n+1-k,\lambda,\alpha}(x:\omega)G_{k,\lambda}}{2(n+1)} \tag{3.8}$$

holds true.

Proof. In terms of (2.14) and (3.5), we derive

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!} &= e^x_\alpha\left(e^\omega_\lambda(t)-e^{-\omega}_\lambda(t)\right)\frac{2t}{e_\lambda(t)+1}\frac{e_\lambda(t)+1}{2t}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}G_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^{n-1}}{n!}\\ &\quad+\frac{1}{2}\sum_{n=0}^{\infty}B^{(c)}_{n,\lambda,\alpha}(x:\omega)\frac{t^n}{n!}\sum_{n=0}^{\infty}G_{n,\lambda}\frac{t^{n-1}}{n!}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(1)_{n-k,\lambda}B^{(c)}_{k-m,\lambda,\alpha}(x:\omega)G_{m,\lambda}\frac{t^{n-1}}{n!}\\ &\quad+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B^{(c)}_{n-k,\lambda,\alpha}(x:\omega)G_{k,\lambda}\frac{t^{n-1}}{n!},\end{aligned}$$

which implies the desired result (3.8). □

4. Conclusion

In this paper, we have firstly generalized the central factorial function termed as extended degenerate central factorial numbers of the second kind and have given some identities for the mentioned numbers. We then have defined unification of the degenerate central Bell polynomials and numbers. We have analyzed multifarious properties and formulas for the aforesaid polynomials and numbers. We have provided several correlations for the fully degenerate central polynomials related to the degenerate Bernstein polynomials and the degenerate Bernoulli, Euler and Genocchi numbers.

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