A topological perspective for interval type-2 fuzzy hedges

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Abstract

Type-2 fuzzy sets were introduced by L. Zadeh aiming at modelling some settings in which fuzzy sets (usually called type-1 fuzzy sets) are not sufficient to reflect certain uncertainty degrees - loosely speaking, they are fuzzy sets whose membership degrees are ordinary fuzzy sets. On the other hand, fiber bundles are topological entities of extreme importance in Mathematics itself and many other scientific areas, like Physics (General Relativity, Field Theory etc.), finance modelling, and statistical inference. The present work introduces a conceptual link between the two ideas and conjectures about the potential mutual benefits that can be obtained from this viewpoint.As an objective and usable product of the presented ideas, it is described a framework for defining type-2 fuzzy hedges, proper to operate on interval type-2 fuzzy sets.

Keywords: Type-2 fuzzy sets; Fiber bundles; Differential Topology.

1. Introduction

1.1. Type-2 fuzzy sets

Type-2 fuzzy sets were conceived as an extension of the concept of fuzzy set (also known as type-1 fuzzy set) [11]. Such entities may be regarded as fuzzy sets whose membership degrees are ordinary fuzzy sets, being very useful in settings in which it is not feasible to establish traditional membership functions. In this fashion, they may be helpful when dealing with more severe modelling uncertainties. Besides, they may be used to describe higher level uncertainties than those typically found in typical membership functions associated to type-1 sets, due to the intrinsic descriptional limitation of (type-1) membership functions.

This kind of problem can happen when obtaining information from experts or automatic training processes, due to semantic or numerical noise,

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respectively. In any circumstance, information about the linguistic or numerical imprecision can be incorporated by means of this more general paradigm [4].

Following [4], we establish that a type-2 fuzzy set in the universe of discourse X is to be denoted by $\bar{\bar{A}}$ and the membership degree of $x \in X$ in $\bar{\bar{A}}$ is $\mu_{\bar{\bar{A}}}(x)$, which is a type-1 fuzzy set in [0,1]. Also, for each $x \in X$, the elements of the domain of $\mu_{\bar{\bar{A}}}(x)$ are called primary memberships of x in $\bar{\bar{A}}$, and the membership grades of the primary memberships in $\mu_{\bar{\bar{A}}}(x)$ are called secondary memberships of x in $\bar{\bar{A}}$. In particular, whenever the secondary memberships are constant, we call them interval type-2 fuzzy sets. In this case, it means that the uncertainty is equally distributed.

1.2. Basic aspects of fiber bundles

Loosely speaking, a fiber bundle is a topological space that looks like a product of two topological spaces but in a local manner, not being necessarily a global product. In its differential flavor (topological spaces being also differentiable manifolds) this magnificent concept finds great importance in theoretical physics and other relevant scientific fields [3, 10]. To get started, an easy example may be useful.

Informally, it is possible to consider a fiber bundle as a certain manifold with a copy of a generic fiber attached to every point. It differs from a product manifold, that is equivalent to a trivial bundle, in that the fibers can be deformed, so as to make the global structure "richer" than a mere product. The possible deformations are expressed by means of transition functions, which hold the fibers tied in a coherent and precise structure.

For example, the Möbius strip (Fig.1) is the total space of a specific fiber bundle, a circle is the base space, and the generic fiber is a line segment. The structure group is the discrete group $\mathbb{Z}_2$.

Before proceeding, some information about manifolds is in order: a topological manifold of dimension n is a Hausdorff topological space that is second countable and locally Euclidean of dimension n, that is, every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.
However, since topological manifolds were designed only for studying topological properties, there are no adequate devices for making quantitative operations, and to add the necessary elements that will provide the technical basis for the definition of derivatives of functions, curves, or maps defined on manifolds, it is essential to introduce a new kind of manifold, namely, a smooth (or C^∞) manifold. Unfortunately, it is not sufficient to define a smooth manifold simply as a topological manifold with special attributes because smoothness is not always invariant under homeomorphisms. Hence, a sound definition for the concept of smooth manifold should include extra structures beyond its topology, making it possible to decide which functions defined on it are differentiable, for instance [8].

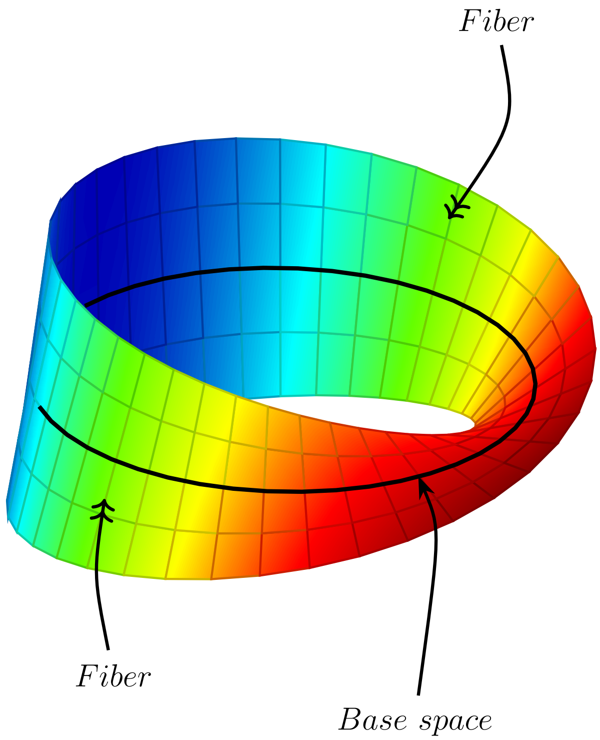

Figure 1: Möbius strip as a fiber bundle

That said, another very well-known example of fiber bundle (more specifically, a vector bundle) is the tangent bundle, associated to differentiable manifolds. It is defined as the disjoint union of all tangent spaces to a given manifold, being the tangent space at each point x in the manifold a vector space T_xM.

Hence, the tangent bundle TM is formed by the family of vector spaces $\{T_xM \mid x \in M\}$ of all tangent spaces of M. The base manifold is M, and the fiber, R^m, where m is the dimension of M. Its structure group is a subgroup of GL(m,$\mathbb{R}$).

For a general definition of fiber bundle, we have:

Definition 1. Let M, B and F be topological spaces (B connected).
A fiber bundle with fiber F may be defined as the structure formed by M, B, F, and π, which is a continuous and surjective map $\pi : M \to B$ with $\pi^{-1}(b)$ homeomorphic to F, for all $b \in B$, satisfying the local triviality condition, that is, for each $b \in B$ there exists an open neighborhood $U_b \ni b$ and a homeomorphism $\psi_b : \pi^{-1}(U_b) \to U_b \times F$ such that the diagram below (Figure 2) commutes:

It is not difficult to notice that fiber bundles are entirely determined by the projection map, subject to a collection of strict requisites. The projection is fundamental when determining the other objects which compose the

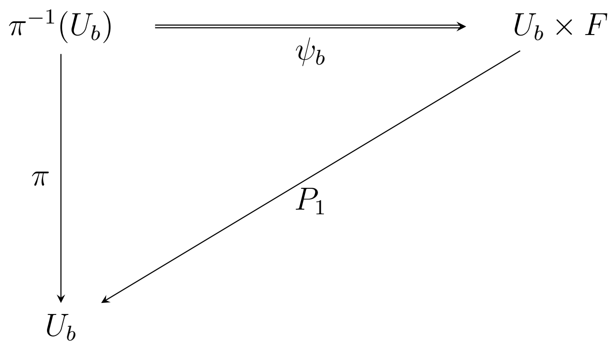

Figure 2: Local triviality condition

bundle. In this fashion, E and M are its domain and range and, assuming that we are dealing with differential fiber bundles, it is required that they are differential manifolds. Since it is required that E be a differentiable manifold, it is assumed that its differential structure is previously established, but this object is subject to all conditions in the proper definition. Also, the generic fiber must be homeomorphic/diffeomorphic to the preimages of each point in M. Local trivializations are required to be compatible between themselves, playing a similar role to that of coordinate charts on M - they are also linked with the projection. Since the total space is required to have a specific topology and differential structure, local trivializations are nothing more than all possible maps which are compatible with this structure. Once all trivializations are determined, this condition tacitly establishes the set of all transition functions, and hence the associated structure group. In this fashion, all components fundamentally depend on the projection map, and consequently it is often referred to as *the* fiber bundle.

In addition, it is worth to state that there are several ways to quantify the degree of nontriviality of a given bundle. However, before quantifying deviations from triviality, it seems highly interesting to know whether it is non-trivial or not. Fortunately, there are some equivalent ways to detect whether a bundle is a global product or not. Before proceeding, one important notion needs a clear definition.

Definition 2. Let B and E be topological manifolds, and $\pi : E \to B$ a fiber bundle.
A global (cross) section σ is a continuous map $\sigma : B \to E$ such that $\pi \circ \sigma(x) = x$, for all $x \in B$. If a given section is only defined on a proper subset U of B, it is called a local section, and the corresponding map is $\sigma : U \to E$.

Perhaps the best known example of the concept of section is the one of a vector field over a given manifold M, that may be defined as a section of the tangent bundle TM. In general, it is not very difficult to synthesize local sections on open subsets of manifolds. However, the construction of global secions over entire manifolds is not usually a simple task, being important to know how to decide whether a certain bundle is trivial or not. The following theorems shed some light in this direction, for vector and principal bundles.

Theorem 1. *A vector bundle of rank n is trivial if and only if it admits n pointwise linearly independent sections, namely, a global frame.*

Theorem 2. *[3]*
A principal bundle is trivial if and only if it admits a global section.

2. Interval type-2 fuzzy sets viewed as fiber bundles

In order to understand the connection between interval type-2 fuzzy sets and fiber bundles, it is advisable, at first, to observe the FOU (Footprint Of Uncertainty) of a typical interval type-2 fuzzy set, as shown in Figure 3. FOUs represent a good resource to qualitatively estimate the uncertainty in the primary membership values of type-2 fuzzy sets and consists of all primary membership domains. Intuitively, they delimit for each $x \in X$ what membership values are possible, and the secondary memberships establish the respective possibility degrees. In interval type-2 fuzzy sets all secondary membership values are equal, and could be disregarded for our purposes here.

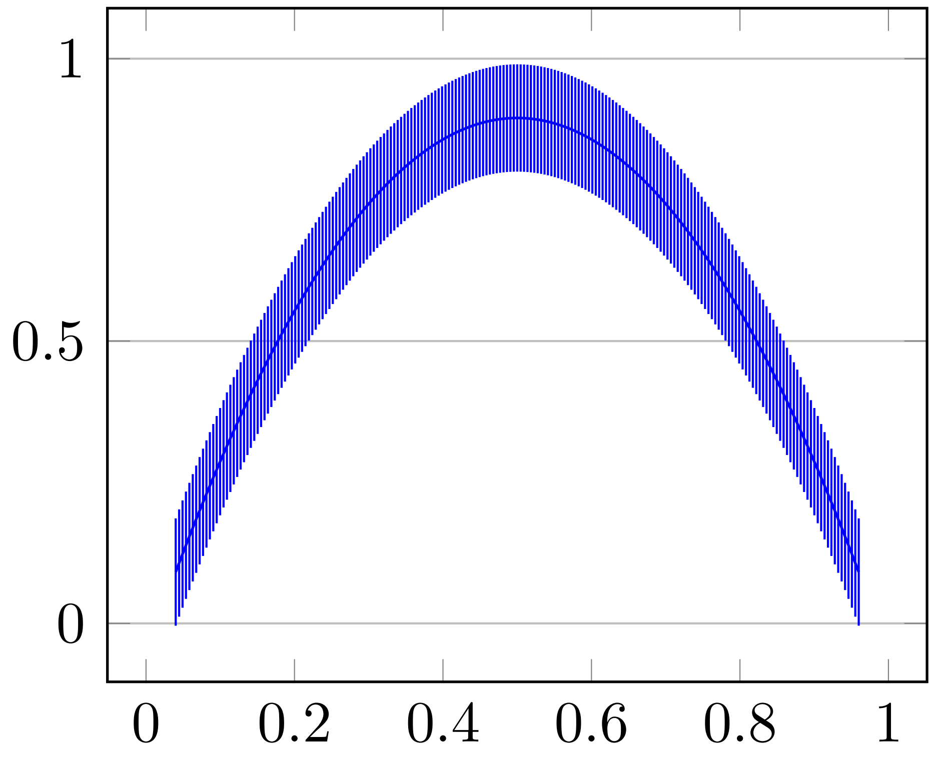

Figure 3: Footprint of uncertainty

For the sake of introducing the central idea of the paper relatively to interval type-2 fuzzy sets, set M (the total space) as the union of all intervals corresponding to primary memberships, and B (base space) as the support of the interval type-2 fuzzy set under study, that is. elements of the universe of discourse X with nonempty primary memberships. The projection $\pi : M \to B$ associates to each element of the primary membership set the corresponding element of the universe of discourse, as illustrates Figure 4.

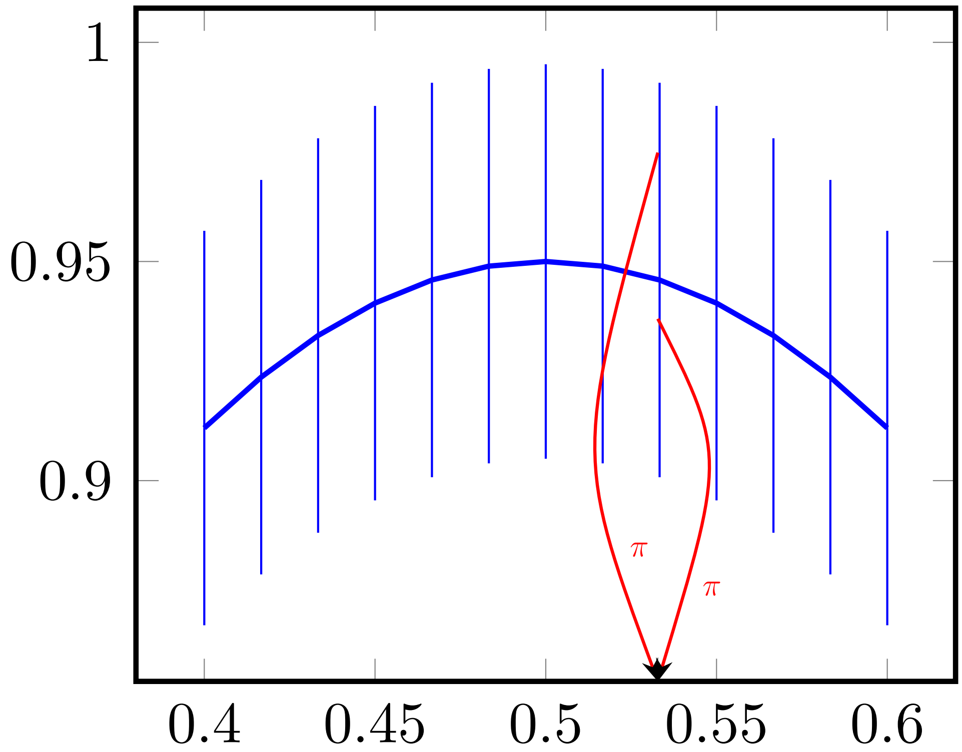

Figure 4: Projection map

Assuming that M and B are endowed with the subspace topology and B is a real open interval, we take as the generic fiber F the interval (0,1). Observing that each fiber $\pi^{-1}(b)$, $b \in B$, is an interval, it is naturally homeomorphic to F, as desired. At last, the map $\psi_b : \pi^{-1}(U_b) \to U_b \times F$ may be defined by associating to each element $x \in \pi^{-1}(U_b)$ the pair $(\pi(x), \phi(x)) \in U_b \times F$, where ϕ is the homeomorphism between $\pi^{-1}(\pi(x))$ and F. Of course, U_b is an open neighborhood of b.

In this fashion, it is possible to associate to each interval type-2 fuzzy set a trivial fiber bundle.

Proceeding with the investigation, it is possible to figure out that the sections of the respective fiber bundle correspond to "realizations" of type-1 fuzzy sets, that is, whenever the uncertainty (of type 2) is eliminated in a given model, the primary memberships collapse, giving rise to "conventional" fuzzy sets, with real numbers in [0,1] as membership grades and "conventional" membership functions - these functions represent the global cross sections of the corresponding bundle. Such a situation is illustrated in Figure 5.

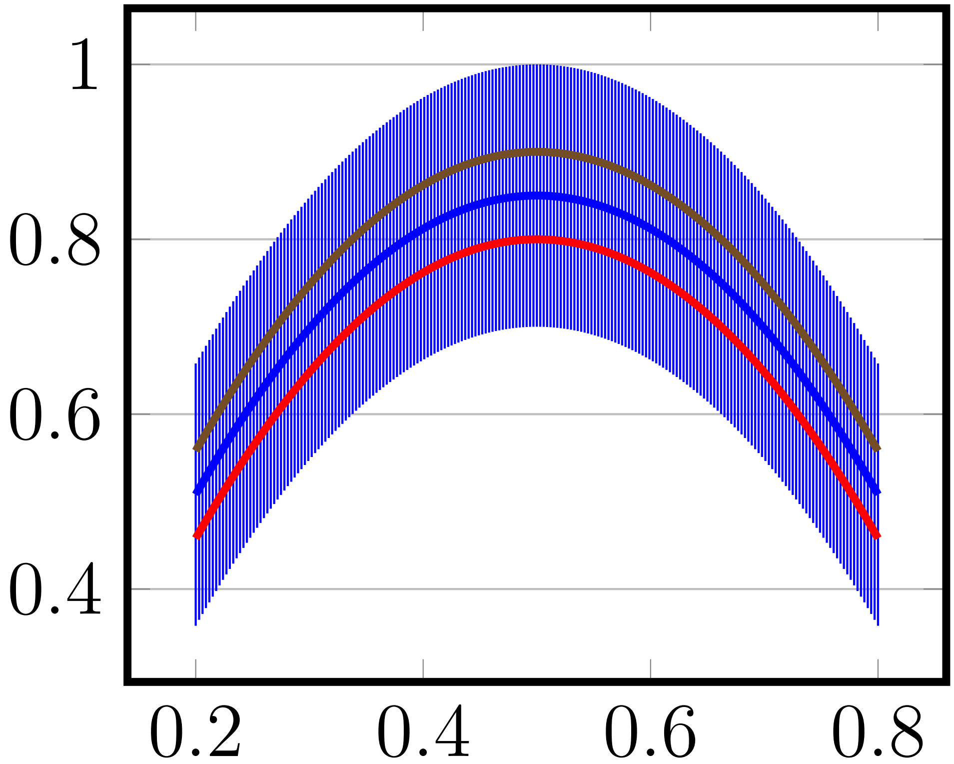

Figure 5: Sections of the fiber bundle

This happens because when a specific value inside a primary membership set is associated to a point in the universe of discourse, its projection back is nothing else than the initial point, showing that the defined map is a bundle section, as initially stated. Figure 6 illustrates the convenient scenario.
Going even further, it is worth highlighting that some well-known operations between interval type-2 fuzzy sets, like AND, OR, for instance, give rise to new fiber bundles, because they produce interval type-2 fuzzy sets as well [6], which may be considered (or are) trivial fiber bundles, as illustrated in Figures 7, 8 and 9. In this fashion, there is a subclass of fiber bundles that is closed under some fuzzy operations, like AND, OR and complement Therefore, considering that the theory of fiber bundles is very rich and features many deep results, it seems natural that the presented correspondence may be beneficial for the development of new ideas about type-2 fuzzy sets and systems.

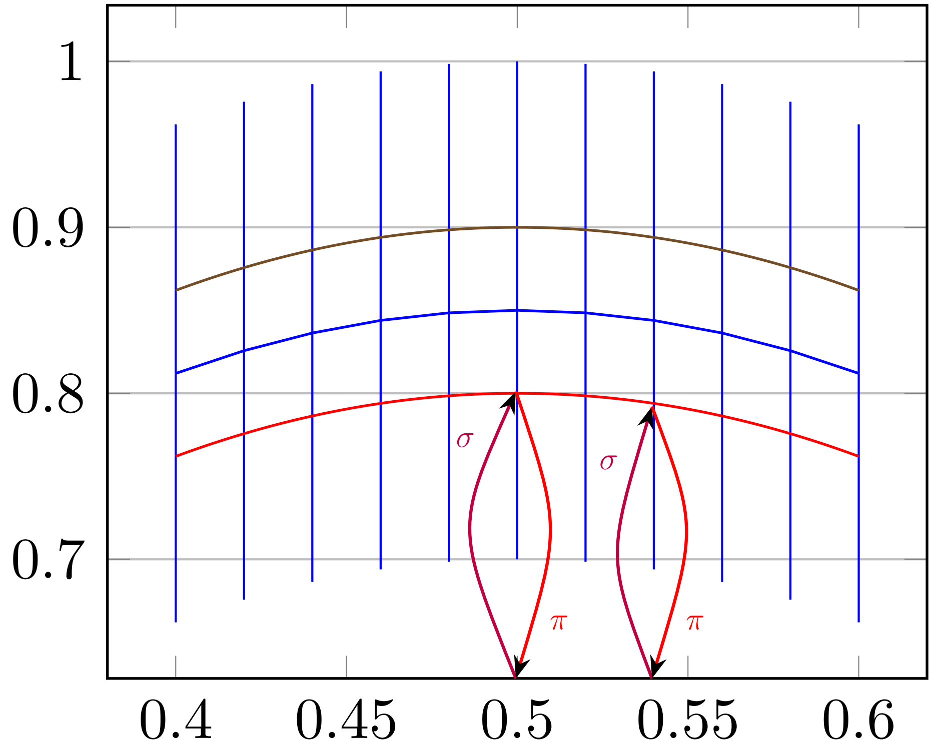

Figure 6: Details about the sections of the fiber bundle

3. Further discussion about possible connections between Fuzzy Logic, Differential Geometry and Topology

In this section we offer some ideas about possible ways of using the aforementioned facts and make some considerations on how certain topological and geometrical concepts could help in fuzzy modelling tasks.

As a further application of fiber bundles to interval type-2 fuzzy sets, we can describe an arbitrary transformation between them by means of the so-called bundle morphism concept, whose definition is given by

Definition 3. [3]
Let (E, p, B) and (E', p', B) be two bundles over B. A bundle morphism over B (or B-morphism) $u : (E, p, B) \to (E', p', B)$ is defined as a map $u : E \to E'$ such that $p = p' \circ u$, that is, the diagram in Figure 10 commutes.

Accordingly, by using a bundle morphism u, it is possible to change the uncertainty degree of a whole (interval type-2) fuzzy set by specifying, for example, an expansion of the primary membership intervals. The universe of discourse is preserved due to the condition $p = p' \circ u$. E would correspond to the original FOU and E' to the transformed one, as illustrated in Figure 11. Going even further, we can propose a kind of ***fuzzy hedging*** for interval type-2 fuzzy sets. This could be realized by using modulating morphisms that

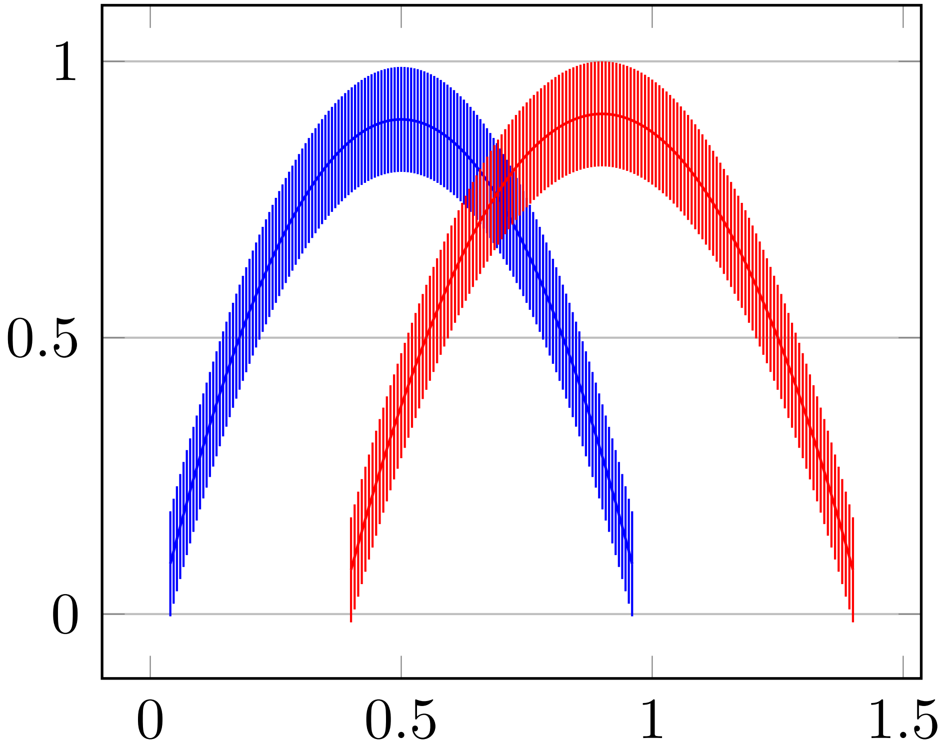

Figure 7: Type-2 fuzzy sets

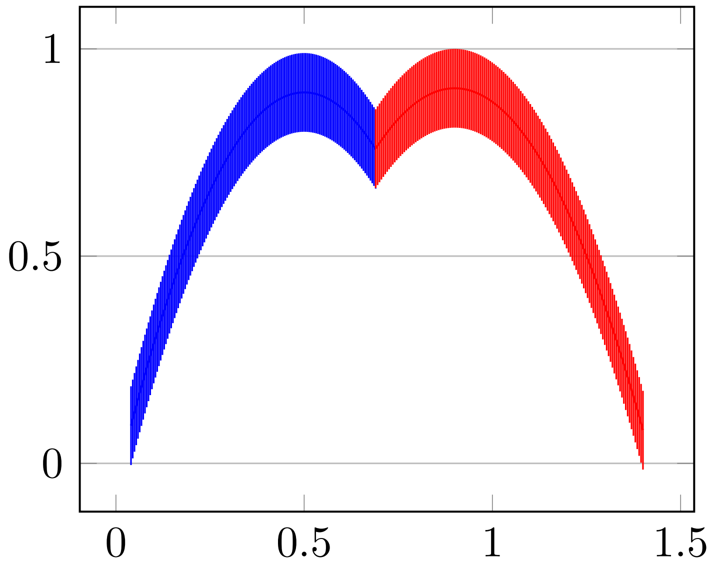

Figure 8: After OR operation

would transform the amplitude of primary membership intervals according to the needs of a certain modelling task.

In another dimension of the question, it seems sensible to consider, for Fuzzy Logic, what was (and continues to be) constructed for probability theory by Rao, Amari and others [1, 5, 9], relatively to its geometrization

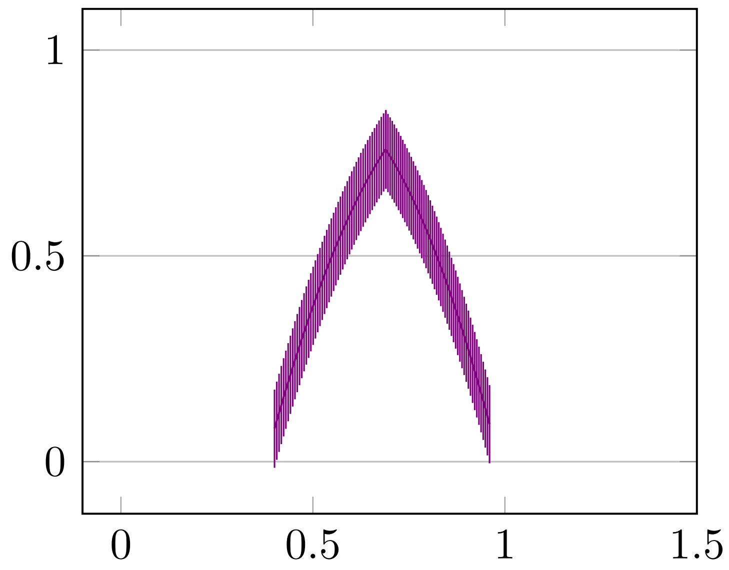

Figure 9: After AND operation

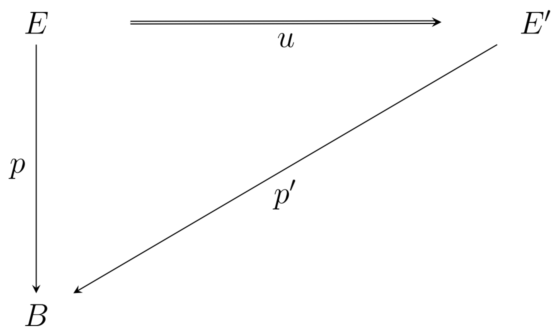

Figure 10: Bundle morphism condition

by means of manifolds and associated concepts. From the point of view of Information Geometry, parametric families of probability density functions (PDFs) may be faced as manifolds, whose dimension is given by the number of defining parameters [1]. Hence, for instance, the set of Gaussian distributions is considered to be a two-dimensional manifold, and a point on it corresponds to a particular PDF with coordinates given by the respective mean and standard deviation. Consequently, and following that line of reasoning, it is possible to consider the family of normal triangular fuzzy sets as a 3-dimensional manifold because, in principle, it is possible to specify this type of entity with three real numbers. Another example, analogous to the probabilistic one, is the class of Gaussian membership functions.

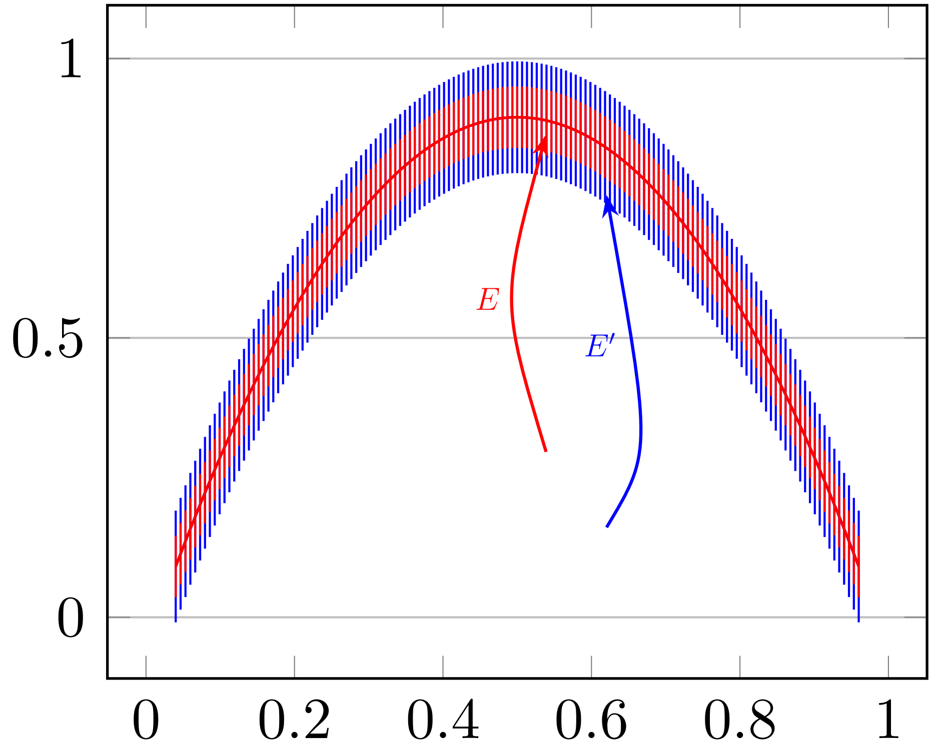

Figure 11: Bundle morphism as an interval type-2 fuzzy set shaping mechanism

4. A proposal for type-2 fuzzy hedges

In this section we propose a new approach for fuzzy hedging of interval type-2 fuzzy sets, based on the fiber bundle formalism. Before that, however, let us make some considerations about type-1 fuzzy linguistic hedges.

4.1. Type-1 fuzzy hedges

In general terms, hedges are linguistic transformers, whose function is to extend the reach of certain basic grammatical terms, like adjectives, for instance. In this subsection we briefly examine some kinds of fuzzy set shape modifiers, known as fuzzy hedges, that play the same role in the fuzzy modelling context as adverbs do in natural languages, changing the meaning of a given fuzzy set by shaping its associated membership function.

Considered as another type of hedge, the approximators may, in addition, convert scalars into fuzzy sets, the so-called fuzzy numbers. Their importance in fuzzy system modelling is mainly related to their power as a tool to approximate the semantics of underlying knowledge present in many approximate reasoning environments. By modifying the shape of a fuzzy set's membership function, a change in the underlying meaning of the corresponding fuzzy expressions is provoked. In this fashion, a hedge operation produces a new fuzzy set from the original one, and can be thought of as a functional operator. Typically, fuzzy reasoning systems use various distinct classes of hedge operations, each of which represented by linguistic-like constructs. Some hedges intensify the meaning of fuzzy sets, like **very** or **extremely**, for example. On the other hand, certain hedges may weaken, or dissolve, the membership function (**somewhat**), others aim at logically complementing fuzzy sets, (like **not**, for instance). There are also the ones that approximate fuzzy regions or "fuzzify" scalars, like **near** or **approximately**.

Considering that they play a similar role in fuzzy rules as adjectives or adverbs do in natural languages, their effect is alike, that is, fuzzy hedge's actions change the meaning of sentences in knowledge bases in the same way that adjectives and other modifiers alter the meaning and intention of ordinary sentences. Fuzzy hedge operations are generally synthesized with basis in heuristic directives, that is, the degree to which fuzzy membership functions are transformed and the nature of their transformations are not typically based on mathematical theories, but oriented by the perceptible psychological adequacy of the resulting mappings. In this fashion, the original definition of the hedge **very**, for instance, *intensifies* a given fuzzy set, so to speak, by squaring its membership function at each point of its domain.

Therefore, it is natural to wonder why do we square the membership function for this hedge instead of using, say, a power of three, or something else. The truth is that the squaring operation results in a good approximation for the corresponding concepts in a wide range of fuzzy sets and is very satisfactory in practical terms. In summary, hedge operations shape fuzzy sets in a very flexible but sensible way, being directed by the semantic needs of the model at hand - the choices are based upon specific linguistic characteristics that the designer is trying to implement.

4.2. Proposed framework for interval type-2 fuzzy hedges

Considering that interval type-2 fuzzy sets may be said to be, in a certain sense, higher dimensional objects than (type-1) fuzzy sets are, it seems sensible to say that type-2 fuzzy hedges also possess greater freedom degree in order to be able to deal with the corresponding additional flexibility needs. In this fashion, they must be capable of transforming not only parameters corresponding to the fundamental universe of discourse, but also the ones relative to the amplitude of intervals that delimit primary memberships, viewed as fibers in this work. Therefore, the fuzzy set depicted in Figure 12 could be transformed into the one shown in Figure 13 after applying the type-2 hedge **very**. In addition, Figure 14 displays the effect of the fuzzy hedge **somewhat** on the original FOU. In the latter example, the lower and upper membership functions were modified by taking their square roots, as usual in type-1 fuzzy operations. In [6] a related functional transformation was recently proposed, with similar expressions for upper and lower membership functions.

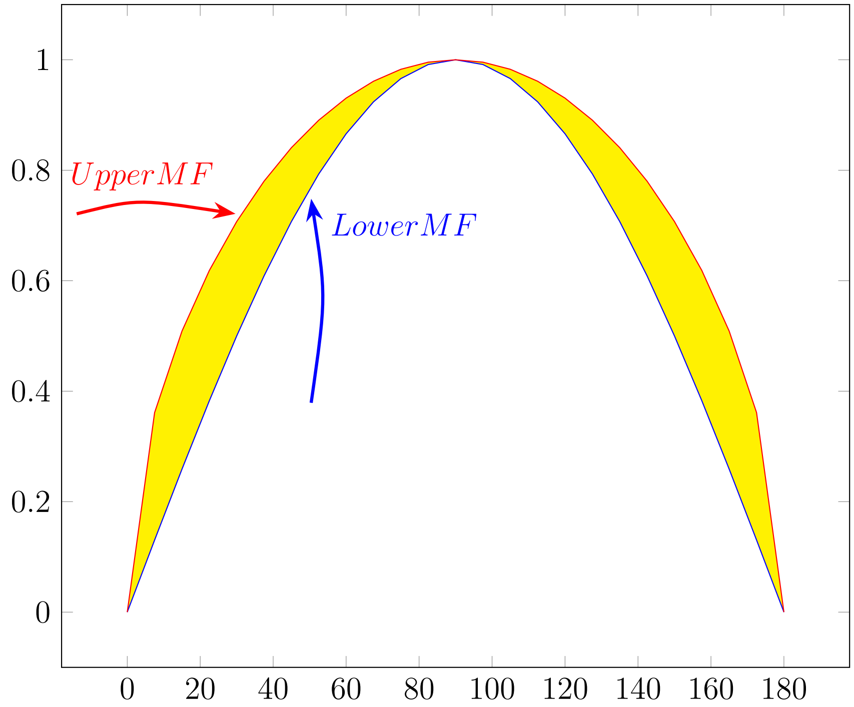

Figure 12: Original FOU

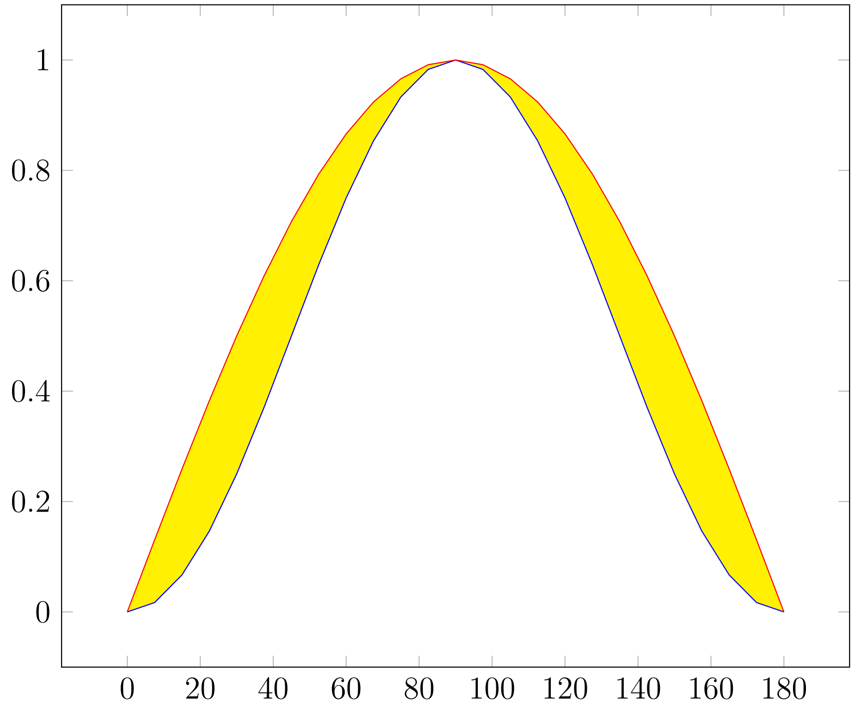

Figure 13: FOU after hedging with the VERY modifier

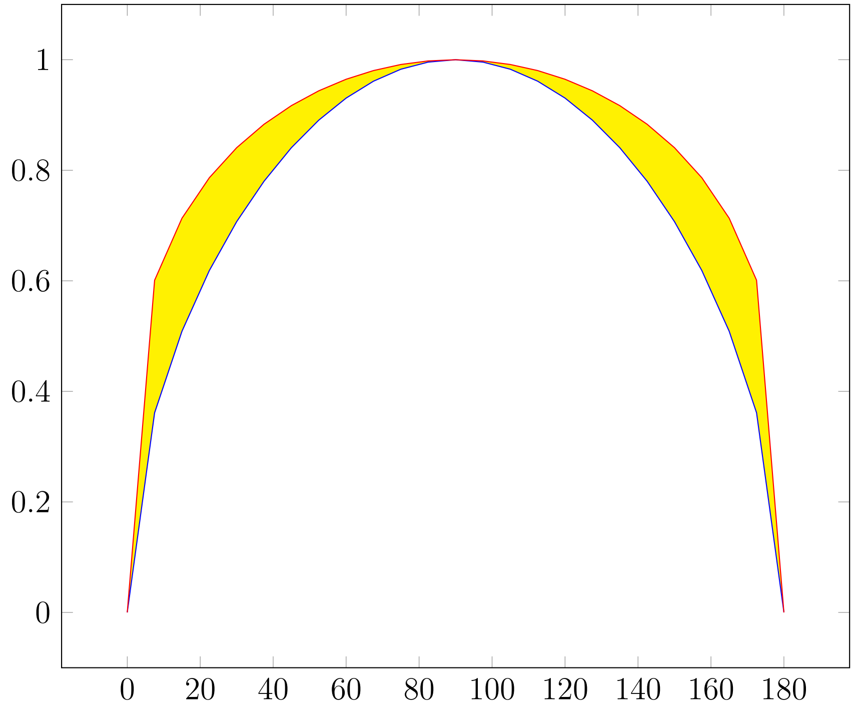

Figure 14: FOU after applying the SOMEWHAT hedge

5. Conclusion

This paper established a conceptual connection between fiber bundles and interval type-2 fuzzy sets, also indicating how certain constructs, like bundle cross sections, for example, may be regarded as the type-1 fuzzy sets resulting from the reduction of uncertainties in type-2 fuzzy sets. As stated above, the intention is to introduce a link between two important areas of knowledge and incentivate further studies, aiming at exploring the full potential of both concepts and their relationships.

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