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Lie Symmetry Exact Explicit Solutions for Nonlinear Burgers' Equation

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In light of Liu *at el.*'s original works, this paper revisits the solution of Burgers's nonlinear equation $u_t = a(u_x)^2 + bu_{xx}$. The study found two exact and explicit solutions for groups G_4 and G_6, as well as a general solution. A numerical simulation is carried out. In the appendix a Maple code is provided.

Keywords: Lie group, Burgers equation, exact solution, general solution, elementary function

I. INTRODUCTION

It is well known that many, if not all, of the fundamental equations of physics are nonlinear and that linearity is achieved as an approximation. Slving the nonlinear equations is a difficulty task. Sophus Lie [1–3] studied mathematics systems from the perspective of those transformation groups which left the system invariant and developed his notion of continuous transformation groups and their role in the theory of differential equations. The continuous transformation groups can simplify the original differential equations to obtain an exact solution. The Lie group method is a systematic and powerful analytical tool, therefore Cantwell [4] even asserts that students in science and engineering should learn both dimensional analysis and the Lie group (symmetric analysis) as the former one establishes a physical model (equation), and the latter solves the solutions.

In mathematics the Lie group is a group, as well as a differential manifold with the property of maintaining a smooth structure under group operation. Lie began to create a single-parameter continuous group theory (later known as the Lie group) around 1875, and completed the famous monographs about the Lie group [1–3]. After publication of Lie's three volume monographs, the first textbook on the Lie group was completed in 1897 by American scholar, J.M. Page [5], who was one of the first students to attend Lie's lecture on the Lie group. American scholar, A. Cohen, published the second textbook in 1911 [6]. Ince, a scholar at the American University of Egypt in 1926, introduced the Lie group symmetry method in his ordinary differential equation book [7]. After a long period of silence, in 1960 American scholar Birkhoff, published a monograph on the study of fluid mechanics with the Lie group in 1960. Due to the high-level study of Birkhoff, the Lie group symmetry method has started to attract more attention [8]. In respect of the application development of the Lie group symmetry method, it must be mentioned that the Lie group analysis monograph of former Soviet scholar, Ovsiannikov [9] who led the Soviet school, promoted the development of the Lie group application and computer computing. Bluman translated Ovsiannikov's monograph in 1967. Hansen's 1964 book discusses fluid mechanics and heat transfer problems. In 1965, Ames' monograph studied nonlinear problems in engineering [10], while Ibragimov and Bluman made an influential contribution to the education and application of the Lie group later [11–26].

Liu *et al.* [23] applied the Lie group to study the following nonlinear Burgers' equation [27]:

$$u_t = a(u_x)^2 + bu_{xx}. \tag{1}$$

where $u = u(x,t)$ is a real function, and parameters $a, b \in R$ with $ab \neq 0$, and $u_t = \frac{\partial}{\partial t}$, $u_x = \frac{\partial}{\partial x}$ and $u_{xx} = \frac{\partial^2}{\partial x^2}$.

For the general nonlinear Burgers' equation, Liu *et al.* [23] obtained six one-parameter Lie groups G_i, $i = 1...6$, and four exact explicit solutions corresponding to the group G_1, G_2, G_3 and G_5. However, Liu *et al.* [23] could not find the exact and explicit solutions for both groups G_4 and G_6. Liu *et al.* [23] did not finishing the integration below for group G_4 as well and just keeping the solution in following unfinished integration:

$$f(\theta) = \int \frac{\exp(-\frac{\theta^2}{4b})}{\int \exp(-\frac{\theta^2}{4b^2}) + c_1} d\theta + c_2. \tag{2}$$

Liu *et al.* [23] derived a nonlinear ordinary differential equation for group G_6:

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0, \quad \text{where} \quad f' = \frac{df}{d\theta}, \tag{3}$$

and stated that "we cannot obtain the exact and explicit solutions for Eq.(3) by using the elementary functions.". To solve the Eq.(3), they cleverly proposed a generalized power series solution: $f(\xi) = \beta - \frac{b}{2a}\log t - \frac{x^2}{4at} + \sum_{n=0}^{\infty}(-1)^n \frac{a^n}{(n+1)b^n}\eta^{n+1}(xt^{-1})^{n+1}$, where β and η are arbitrary numbers.

General speaking, Liu *et al.*'s results were very good, however, it would be pity to leave some solutions that are not completed. To make up the shortcoming, namely did not find the solutions corresponding to groups G_4 and G_6 that can be expressed in elementary functions, this paper will following the footstep of Liu *et al.* [23] and to solve the problems.

The research article is organized as follows: following an introduction, Section 2 reviews Lie infinitesimal generator, Lie algebra obtained by Liu *et al.*; Section 3 gives exact solution in form for each group; Section 4 propose two exact and explicit solutions. Section 5 propose a general solution for $u_t = a(u_x)^2 + bu_{xx}$ for a combined group; Section 6 provides a numerical calculation. Finally, concludes with perspectives. In the appendix A, a Maple code is provided.

II. LIE INFINITESIMAL GENERATOR, LIE ALGEBRA AND LIE GROUP OF $u_t = a(u_x)^2 + bu_{xx}$

For the sake of clarity, it is our honor to summarize some results from Liu *et al.* [23] as follows: The infinitesimal transformation of x, t, u

$$\begin{aligned} x &\mapsto x + \epsilon\xi(x,t,u), &(4)\\ t &\mapsto t + \epsilon\tau(x,t,u), &(5)\\ u &\mapsto u + \epsilon\phi(x,t,u). &(6)\end{aligned}$$

The corresponding infinitesimal generator X can be expressed as:

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}, \tag{7}$$

and the second order prolongation $X^{(2)}$ is

$$X^{(2)} = X + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}}, \tag{8}$$

where

$$\begin{aligned} \phi^t &= D_t\phi - u_x D_t\xi - u_t D_t\tau, &(9)\\ \phi^x &= D_x\phi - u_x D_x\xi - u_t D_x\tau, &(10)\\ \phi^{xx} &= D_x^2\phi - u_x D_x^2\xi - u_t D_t^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau. &(11)\end{aligned}$$

and D_x and D_t are the total differential derivative respect to x and t, respectively

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t}, &(12)\\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t}. &(13)\end{aligned}$$

The Lie group symmetry of the equation (1) is

$$X^{(2)}(u_t - au_x^2 - bu_{xx}) = 0, \quad \text{ie.} \quad \phi^t - 2au_x\phi^x - b\phi^{xx} = 0. \tag{14}$$

The relevant expression can be substituted into the symmetry determining equation (14), paying attention to replacing u_t by $au_x^2 + bu_{xx}$, and comparing the coefficient functions to get infinitesimal generators (one may also use Maple to obtain, see Maple code in Appendix):

$$\begin{aligned} \xi &= 4ac_1xt + c_2x + 2ac_4t + c_5, &(15)\\ \tau &= 4ac_1t^2 + 2c_2t + c_3, &(16)\\ \phi &= ae^{-au/b} - c_1x^2 - c_4x - 2b_1t + c_6. &(17)\end{aligned}$$

where $c_i(i = 1, 2, \cdots, 6)$ is an arbitrary constant, $\alpha = \alpha(x,t)$ is a function that satisfies the following conduction equation:

$$\alpha_t = b\alpha_{xx}. \tag{18}$$

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Hence the problem of infinitesimal generators are

$$X = \sum_{i=1}^{6} c_i X_i + V_\alpha. \tag{19}$$

where the base vector are

$$\begin{aligned} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_5 = 2at\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \\ X_6 = 4axt\frac{\partial}{\partial x} + 4at^2\frac{\partial}{\partial t} - (x^2+2bt)\frac{\partial}{\partial u}, \quad X_\alpha = \alpha(x,t)e^{-au/b}\frac{\partial}{\partial u}. \end{aligned} \tag{20}$$

and X_α is an infinite dimensional subalgebra.

This verifies that X_i satisfies the closure of Lie algebra, namely X_i, and $X_j \in \mathcal{L}$, then $[X_i, X_j] \in \mathcal{L}$. The vector field commutator-$[X_i, X_j]$ structure coefficient list, is as follows:

TABLE I: Burgers' equation Lie algebra structure coefficient table

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_α
X_1	0	0	0	$-X_1$	X_3	$-2X_5$	$-X_{\alpha_x}$
X_2	0	0	0	$-2X_2$	$-2aX_1$	$2bX_3 - 4aX_4$	$-X_{\alpha_t}$
X_3	0	0	0	0	0	0	$\frac{a}{b}X_\alpha$
X_4	X_1	$2X_2$	0	0	$-X_5$	$-2X_6$	$-X_{\alpha'}$
X_5	$-X_3$	$2aX_1$	0	X_5	0	0	$-X_{\alpha''}$
X_6	$2X_5$	$-2bX_3 + 4aX_4$	0	$2X_6$	0	0	$-X_{\alpha'''}$
X_α	X_{α_x}	X_{α_t}	$-\frac{a}{b}X_\alpha$	$X_{\alpha'}$	$X_{\alpha''}$	$X_{\alpha'''}$	0

where $\alpha' = x\alpha_x + 2t\alpha_t$, $\alpha'' = 2at\alpha_x + \frac{a}{b}x\alpha$, $\alpha''' = 4atx\alpha_x + 4at^2\alpha_t + \frac{a}{b}(x^2+2bt)\alpha$.

The Lie group, G_i, generated by X_i, can take advantage of the previously obtained transformation relationship $e^{\epsilon X_i}(x,t,u) = (\hat{x}, \hat{t}, \hat{u})$, such as $\hat{x}$ for G_1:

$$\hat{x} = e^{\epsilon X_1}x = e^{\epsilon\frac{\partial}{\partial x}}x = [1 + \epsilon\frac{\partial}{\partial x} + \frac{1}{2}\epsilon^2(\frac{\partial}{\partial x})^2 \cdots]x = x + \epsilon\frac{\partial x}{\partial x} = x + \epsilon. \tag{21}$$

and G_3:

$$\hat{u} = e^{\epsilon X_3}u = e^{\epsilon\frac{\partial}{\partial u}}u = [1 + \epsilon\frac{\partial}{\partial u} + \frac{1}{2}\epsilon^2(\frac{\partial}{\partial u})^2 \cdots]u = u + \epsilon\frac{\partial u}{\partial u} = u + \epsilon. \tag{22}$$

Other situations can be calculated in the same manner.

The Lie groups are listed below:

$$G_1 : (x,t,u) \mapsto (x+\epsilon, t, u); \tag{23}$$
$$G_2 : (x,t,u) \mapsto (x, t+\epsilon, u); \tag{24}$$
$$G_3 : (x,t,u) \mapsto (x, t, u+\epsilon); \tag{25}$$
$$G_4 : (x,t,u) \mapsto (e^\epsilon x, e^{2\epsilon}t, u); \tag{26}$$
$$G_5 : (x,t,u) \mapsto (x + 2\epsilon at, t, u - \epsilon x - \epsilon^2 t); \tag{27}$$
$$G_6 : (x,t,u) \mapsto (\frac{x}{1-4\epsilon at}, \frac{t}{1-4\epsilon at}; u - \frac{\epsilon x^2}{1-4\epsilon at} + \frac{b}{a}\ln\sqrt{1-4\epsilon at}); \text{and} \tag{28}$$
$$G_\alpha : (x,t,u) \mapsto (x, t, \frac{b}{a}\ln(e^{\frac{au}{b}} + \epsilon\frac{a}{b}\alpha). \tag{29}$$

The above shows that the G_1 group is spatial translation, G_2 is time translation, G_3 is the translation of the dependent variable, G_4 is the extension transformation, G_5 is the Galileo (speed) transformation, and G_6 is a true local transformation group, whose physical meaning is ambiguous but very important, which can be used to find similarity solutions.

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III. EXACT SOLUTION OF $u_t = a(u_x)^2 + bu_{xx}$

Since G_i is a symmetric group, if $u = f(x,t)$ is one solution of the equation (1), then the solution $u^{(i)}$ corresponding to G_i can be obtained as well. Liu *et al.* [23] obtained following solutions u_i, $i = 1, ..., 6$ as follows:

$$u^{(1)} = f(x-\epsilon, t), \tag{30}$$
$$u^{(2)} = f(x, t-\epsilon), \tag{31}$$
$$u^{(3)} = f(x,t)+\epsilon, \tag{32}$$
$$u^{(4)} = f(e^{-\epsilon}x, e^{-2\epsilon}t), \tag{33}$$
$$u^{(5)} = f(x-2\epsilon at, t) - \epsilon x + \epsilon^2 at, \tag{34}$$
$$u^{(6)} = f(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon at}) - \frac{\epsilon x^2}{1+4\epsilon at} - \frac{b}{a}\ln\sqrt{1+4\epsilon at}, \tag{35}$$
$$u^{(\alpha)} = \frac{b}{a}\ln(e^{\frac{af(x,t)}{b}} + \epsilon\frac{a}{b}\alpha(x,t)). \tag{36}$$

With an infinitesimal generator X_i, one can determine the corresponding invariant,whilst using it to simplify the equation (1) and to find its solution.

It should be noted that the combination of X_1 and X_2 produces a moving wave solution. Hence, the new variable $\mu = x - ct$, therefore, $u = u(x,t) = f(\mu)$, where c is the wave speed. Thus, the equation (1) becomes

$$bf'' + af'^2 + cf' = 0. \tag{37}$$

where $f' = \frac{df}{d\mu}$. The solution of this equation can be integrated as follows

$$u(x,t) = \frac{b}{a}\ln[\frac{a}{b}c_2 - c_1\frac{a}{c}\exp(-\frac{c}{b}(x-ct)]. \tag{38}$$

In the above formula, c_1, c_2 are integral constants. The solution in Eq. 38 is different from the solution obtained by Liu *et al.* [23], whose solution is $u(x,t) = \frac{b}{a}\ln[c_1\exp(c(x-ct)/b) - a/c] - c(x-ct)/a + c_2$.

Corresponding to group G_3, it is clear that if $u = f(x,t)$ is a solution for the equation $u_t = a(u_x)^2 + bu_{xx}$, then $u = f(x,t) + c$ is also the equation.

For the stretching group G_4, one can derive an invariant and similarity variable, namely

$$\theta = xt^{-1/2}, \quad \omega = u. \tag{39}$$

The Lie group invariant solution is $\omega = f(\theta) = f(xt^{-1/2})$, namely

$$u = f(xt^{-1/2}). \tag{40}$$

Substituting this formula into (1), which can be simplified into ordinary differential equation as follows

$$bf'' + af'^2 + \frac{1}{2}\theta f' = 0. \tag{41}$$

where $f' = \frac{df}{d\theta}$ and

$$f(\theta) = \int \frac{\exp(\frac{-\theta^2}{4b})}{\int \exp(\frac{-\theta^2}{4b})d\theta + c_1} d\theta + c_2. \tag{42}$$

Liu *at el.* [23] did not complete the above integration, and left it as is.

If we check the integration (42) carefully, it is not difficulty to recognise that the integration $\int \exp(\frac{-\theta^2}{4b})d\theta$ is actually an error function. After simple calculation, the solution of the equation (41) can be expressed by the error function as:

$$f(\theta)|_{\theta=xt^{-1/2}} = \frac{b}{a}\ln\{\frac{a}{b}[c_1\sqrt{\pi}\sqrt{b}\cdot \mathrm{erf}(\frac{1}{2}\frac{xt^{-1/2}}{\sqrt{b}}) + c_2]\}. \tag{43}$$

where $\mathrm{erf}(\theta) = \frac{2}{\pi}\int_0^\theta e^{-\theta^2}d\theta$ is error function.

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There are similarity variable for the Galileo group G_5, namely

$$\theta = t, \quad \omega = \frac{1}{2}x^2 + 2atu. \tag{44}$$

This group invariant solution is $\omega = f(\theta)$, namely

$$u = \frac{1}{2at}f(t) - \frac{1}{4at}x^2. \tag{45}$$

Substituting (45) into equation (1), which is reduced to an ordinary differential equation in the following

$$tf' - f + bt = 0. \tag{46}$$

where $f' = \frac{df}{d\theta}$. By integration, we obtain its solution as follows

$$u(x,t) = \frac{c_1}{2a} - \frac{b}{2a}\ln t - \frac{1}{4at}x^2. \tag{47}$$

The solution in Eq. 47 was obtained by Liu *et al.* [23].

For group G_6, results in the first set of similarity variables as follows:

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln t. \tag{48}$$

This Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln t. \tag{49}$$

Substituting (49) into equation (1) produces an ordinary differential equation

$$bf'' + af'^2 = 0. \tag{50}$$

by integration

$$f = \frac{b}{a}\ln|\frac{a}{b}\theta + c_1| + c_2. \tag{51}$$

Hence, in this case, corresponding to group G_6, the solution is as follows

$$u(x,t) = \frac{b}{a}\ln|\frac{a}{b}xt^{-1} + c_1| - \frac{x^2}{4at} - \frac{b}{2a}\ln t + c_2. \tag{52}$$

The solution in Eq. 52 was obtained by Liu *et al.* [23].

Corresponding to group G_6, another similarity variable, the $\ln t$ in the expression (48), is changed to $\ln x$, and the second set of similarity variables is

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln x. \tag{53}$$

Hence, the Lie group invariant solution is:

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{54}$$

Substituting the expression (54) into equation (1) yields the equation for the function $f(\theta)$

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0. \tag{55}$$

where $f' = \frac{df}{d\theta}$. Liu *at el.* [23] pointed out that the exact and explicit solutions of equation (55) cannot be obtained, therefore they proposed a series solution.

In fact, this equation can be solved with the following exact and explicit elementary solution:

$$f(\theta) = \frac{b}{2a}\ln[(\frac{a}{b})^2\theta(c_1\theta - c_2)^2]. \tag{56}$$

Hence, we have second exact solution, namely

$$u(x,t) = \frac{b}{2a}\ln[(\frac{a}{b})^2\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{57}$$

The exact solution for the equations (56) and (57) were found by this author [26**?**].

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IV. COMBINED EXACT SOLUTION OF $u_t = a(u_x)^2 + bu_{xx}$

Besides those solutions corresponding to each group G_i given in the section 3, it is necessary to combine them to form a general solution based on the general symmetry group $X = c_1X_1 + ... + c_6X_6 + X_\alpha$. An exponential map $\mathfrak{g}$: $\mathfrak{g} \mapsto G$ can be constructed as:

$$\mathfrak{g} = \exp(X_\alpha)\prod_{i=1}^{6}\exp(\epsilon_i X_i) = \exp(X_\alpha)\cdot\exp(\epsilon_1X_1)\cdot ...\cdot\exp(\epsilon_6X_6). \tag{58}$$

If $u = f(x,t)$ is known as a solution for the equation (1) under the $\mathfrak{g}$ group transformation, the most general solution can be obtained as follows:

$$\begin{aligned} u = f&\left(e^{-\epsilon_4}\frac{x-2\epsilon_5 at}{1+4\epsilon_6 at} - \epsilon_1, \frac{t\epsilon^{-2\epsilon_4}}{1+4\epsilon_6 at} - \epsilon_2\right)\epsilon_3 - \epsilon_5\frac{x-\epsilon_5 at}{1+4\epsilon_6 at} - \frac{\epsilon_6 x^2}{1+4\epsilon_6 at} \\ &- \frac{b}{a}\ln\sqrt{1+4\epsilon_6 at} + \frac{b}{a}\ln\left[e^{\frac{a}{b}f(x,t)} + \epsilon_6\frac{a}{b}\alpha(x,t)\right]. \end{aligned} \tag{59}$$

This solution has not been reported by Liu *et al.* [23].

V. NUMERICAL SIMULATIONS OF $u_t = (u_x)^2 + u_{xx}$

The solution for equation $u_t = (u_x)^2 + u_{xx}$ is $u(x,t) = \frac{1}{2}\ln[\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4t} - \frac{1}{2}\ln x$. For example, the initial-boundary conditions: $u(1,1) = 1$ and $u(1,2) = 1$. Hence, $c_1 = e^2 - e^{3/2}$ and $c_2 = e^2 - 2e^{5/4}$. The solution is plotted in Figure 1. The numerical simulations indicate that the solution is in a rapid oscillation state near the origin $(0,0)$, and tends to take a long-time to reach zero.

VI. SUMMARY AND REMARKS

We have fixed the problems left by Liu *at el.*. Although we have found the above two similarity solutions, it would be honor to say that the similarity solutions of the nonlinear Burgers' equation in Eq. (1) can be easily obtained by Maple code in the appendix A. Regarding the further investigation, this study may be useful to deal with high order nonlinear PDE such as nonlinear Schrödinger equations in [28, 29].

Appendix: Maple code

```
restart;with(PDEtools);
with(PDEtools, InfinitesimalGenerator, declare);
U := diff_table(u(x, t));
pde := U[t] = a*U[x]*U[x] + b*U[x, x];
declare(U[ ]);
Infinitesimals(pde);
Infinitesimals(pde, specialize_Cn = false);
SimilaritySolutions(pde)
```


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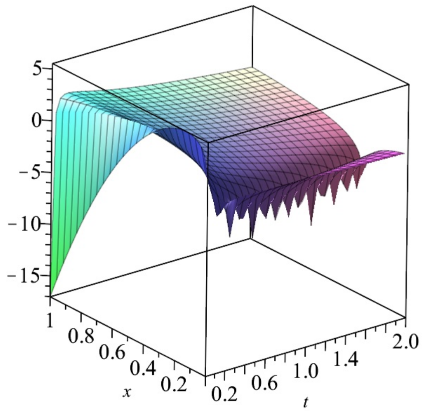

FIG. 1: The solution is in quite oscillation state near the origin $(0, 0)$ and tends to reach zero when it is far from the origin $(0, 0)$.