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Notes on the Lie Symmetry Exact Explicit Solutions for Nonlinear Burgers' Equation

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In light of Liu *at el.*'s original works, this research article revisits the solution of Burgers's nonlinear equation. The researcher found two exact and explicit solutions for groups G_4 and G_6, as well as a general solution. Their applications were conducted by using numerical calculations.

Keywords: Lie group, Burgers equation, exact solution, general solution, elementary function

Contents

I. INTRODUCTION

Most science and engineering problem are nonlinear in nature, while their corresponding governing equations are nonlinear. Solving these is an important task. It is well known that nonlinear partial differential equations having infinite number solutions, and most of them are variable coefficient equations. Making it difficult to obtain accurate solutions. Although there are a few ways to solve nonlinear partial differential equations, for variable-coefficient nonlinear partial differential equations the current research methods used numerical or approximate solutions. Around 1875, Norwegian mathematician, Sophus Lie [1–3] proposed a method the continuous transformation group method, which simplified the original equation to obtain an exact solution by using the Lie group, or symmetry group transformation. This method does not require special transformation techniques. It is a systematic method that is applicable not only to ordinary differential equations, but also to partial differential equations. It is currently the most powerful analytical tool. Cantwell [4] even asserts that students in science and engineering should learn both dimension analysis and the Lie group (symmetric analysis) as the former establishes a physical model (equation), and the latter solves the solution's structure [27].

In mathematics the Lie group is a group, as well as a differential manifold with the property of maintaining a smooth structure under group operation. There is a relatively rich literature collection on Lie group analysis. Lie began to create a single-parameter continuous group theory (later known as the Lie group) around 1875, and completed the three-step continuation of famous monographs about the Lie group [1–3]. After publication of Lie's three volume monographs, the first textbook on the Lie group was completed in 1897 by American scholar, J.M. Page [5], who was one of the first students to attend Lie's lecture on the Lie group. American scholar, A. Cohen, published the second textbook in 1911 [6]. Ince, a scholar at the American University of Egypt in 1926, introduced the Lie group symmetry method in his ordinary differential equation book [7]. After a long period of silence, in 1960 American scholar Birkhoff, published a monograph on the study of fluid mechanics with the Lie group in 1960. Due to the high-level study of Birkhoff, the Lie group symmetry method has started to attract more attention [8]. In respect of the application development of the Lie group symmetry method, it must be mentioned that the Lie group analysis monograph of former Soviet scholar, Ovsiannikov [9] who led the Soviet school, promoted the development of the Lie group application and computer computing. Bluman translated Ovsiannikov's monograph in 1967. Hansen's 1964 book discusses fluid mechanics and heat transfer problems. In 1965, Ames' monograph studied nonlinear problems in engineering [10], while Ibragimov and Bluman made an influential contribution to the education and application of the Lie group later [11–21, 24–27].

Liu *et al.* [23] applied the Lie group to study the general nonlinear Burgers's equation [22]:

$$u_t = a(u_x)^2 + bu_{xx}. \tag{1}$$

where $u = u(x,t)$ is a real function, and parameters $a, b \in R$ with $ab \neq 0$, and $u_t = \frac{\partial}{\partial t}$, $u_x = \frac{\partial}{\partial x}$ and $u_{xx} = \frac{\partial^2}{\partial x^2}$.

For the general nonlinear Burgers' equation, Liu *et al.* [23] obtained six one-parameter Lie groups G_i, $i = 1...6$, and four exact explicit solutions, corresponding the group G_1, G_2, G_3 and G_5. However, Liu *et al.* [23] could not find the exact and explicit solutions for both groups G_4 and G_6. Liu *et al.* [23] did not finishing the integration below for group G_4:

$$f(\xi) = \int \frac{e^{-\frac{\xi^2}{4b}}}{\int e^{-\frac{\xi^2}{4b^2}} + p_1} d\xi + p_2. \tag{2}$$

Liu *et al.* [23] derived a nonlinear ordinary differential equation for group G_6:

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0. \tag{3}$$

Liu *et al.* [23] stated that "we cannot obtain the exact and explicit solutions for Eq.(3) by using the elementary functions.". To solve the Eq.(3), they cleverly proposed a generalized power series solution: $f(\xi) = \alpha \log|\xi| + \beta + \sum_{n=0}^{\infty} f_n \xi^{n+1}$.

In general, Liu *et al.*'s results were effective, and would be more so if the solutions for both groups G_4 and G_6 can be obtained and expressed in elementary functions. This research article addresses this shortcoming.

The research article is organized as follows: following an introduction, Section 2 gives a general solution for $u_t = a(u_x)^2 + bu_{xx}$ for a combined group; Section 3 propose two exact and explicit solutions. Section 4 provides a numerical calculation. Finally, the research article concludes with perspectives about the development of future research studies in this domain. For the sake of clarity, In appendix, for the sake of clarity, the researcher presents some Liu *et al.*'s [23] results, including the Lie symmetric infinitesimal generator and Lie algebra.

II. EXACT GENERAL SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

The Lie symmetry can construct exact solutions. Since G_i is a symmetric group, if $u = f(x,t)$ is a solution of the equation (1), then the solution $u^{(i)}$ corresponding to G_i can be expressed (see appendix). Liu *et al.* [23] obtained following solutions u_i, $i = 1, ..., 6$ as follows:

$$\begin{aligned}
u^{(1)} &= f(x-\epsilon, t), & (4)\\
u^{(2)} &= f(x, t-\epsilon), & (5)\\
u^{(3)} &= f(x,t) + \epsilon, & (6)\\
u^{(4)} &= f(e^{-\epsilon}x, e^{-2\epsilon}t), & (7)\\
u^{(5)} &= f(x - 2\epsilon at, t) - \epsilon x + \epsilon^2 at, & (8)\\
u^{(6)} &= f(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon at}) - \frac{\epsilon x^2}{1+4\epsilon at} - \frac{b}{a}\ln\sqrt{1+4\epsilon at}, & (9)\\
u^{(\alpha)} &= \frac{b}{a}\ln(e^{\frac{af(x,t)}{b}} + \epsilon\frac{a}{b}\alpha(x,t)). & (10)
\end{aligned}$$

Besides those solutions corresponding to each group G_i, it is necessary to combine them to form a general solution based on the general symmetry group $X = c_1X_1 + ... + c_6X_6 + X_\alpha$. An exponential map $\mathfrak{g}$: $\mathfrak{g} \mapsto G$ can be constructed as:

$$\mathfrak{g} = \exp(X_\alpha)\prod_{i=1}^{6}\exp(\epsilon_i X_i) = \exp(X_\alpha)\cdot\exp(\epsilon_1 X_1)\cdot ... \cdot\exp(\epsilon_6 X_6). \tag{11}$$

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If $u = f(x,t)$ is known as a solution for the equation (1) under the $\mathfrak{g}$ group transformation, the the most general solution can be obtained as follows:

$$u = f\left(e^{-\epsilon_4}\frac{x-2\epsilon_5 at}{1+4\epsilon_6 at} - \epsilon_1, \frac{t\epsilon^{-2\epsilon_4}}{1+4\epsilon_6 at} - \epsilon_2\right)\epsilon_3 - \epsilon_5\frac{x-\epsilon_5 at}{1+4\epsilon_6 at} - \frac{\epsilon_6 x^2}{1+4\epsilon_6 at}$$
$$-\frac{b}{a}\ln\sqrt{1+4\epsilon_6 at} + \frac{b}{a}\ln\left[e^{\frac{a}{b}f(x,t)} + \epsilon_6\frac{a}{b}\alpha(x,t)\right]. \qquad (12)$$

This research study obtained the general solution, which has not been reported in any literature to date [28].

III. EXACT SIMILARITY SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

With an infinitesimal generator X_i, one can determine the corresponding invariant,whilst using it to simplify the equation (1) and to find its solution.

It should be noted that the combination of X_1 and X_2 produces a moving wave solution. Hence, the new variable $\mu = x - ct$, therefore, $u = u(x,t) = f(\mu)$, where c is the wave speed. Thus, the equation (1) becomes

$$bf'' + af'^2 + cf' = 0. \qquad (13)$$

where $f' = \frac{df}{d\mu}$. The solution of this equation can be integrated as follows

$$u(x,t) = \frac{b}{a}\ln[c_1 e^{\frac{c}{b}(x-ct)} - \frac{a}{c}] - \frac{c}{a}(x-ct) + c_2. \qquad (14)$$

In the above formula, c_1, c_2 are integral constants.

Corresponding to group G_3, it is clear that if $u = f(x,t)$ is a solution for the equation $u_t = a(u_x)^2 + bu_{xx}$, then $u = f(x,t) + c$ is also the equation.

For the stretching group G_4, one can derive an invariant and similarity variable, namely

$$\theta = xt^{-1/2}, \quad \omega = u. \qquad (15)$$

The Lie group invariant solution is $\omega = f(\theta) = f(xt^{-1/2})$, namely

$$u = f(xt^{-1/2}). \qquad (16)$$

Substituting this formula into (1), which can be simplified into ordinary differential equation as follows

$$bf'' + af'^2 + \frac{1}{2}\theta f' = 0. \qquad (17)$$

where $f' = \frac{df}{d\theta}$ and

$$f(\theta) = \int \frac{exp(\frac{-\theta^2}{4b})}{\int exp(\frac{-\theta^2}{4b})d\theta + p_1}d\theta + p_2. \qquad (18)$$

Liu *at el.* [23] did not complete the above integration, and left it as is.

When one check the integration (18) carefully, it is not difficulty to recognise that the integration $\int exp(\frac{-\theta^2}{4b})d\theta$ is actually an error function. After simple calculation, the solution of the equation (17) can be expressed by the error function [25] as:

$$f(\theta) = \frac{b}{a}\ln\{\frac{a}{b}[c_1\sqrt{\pi}\sqrt{b}\cdot \mathrm{erf}(\frac{1}{2}\frac{\theta}{\sqrt{b}}) + c_2]\}. \qquad (19)$$

where $\mathrm{erf}(\theta) = \frac{2}{\pi}\int_0^\theta e^{-\theta^2}d\theta$ is error function.

There are similarity variable for the Galileo group G_5, namely

$$\theta = t, \quad \omega = \frac{1}{2}x^2 + 2atu. \qquad (20)$$

This group invariant solution is $\omega = f(\theta)$, namely

$$u = \frac{1}{2at}f(t) - \frac{1}{4at}x^2. \tag{21}$$

Substituting (21) into equation (1), which is reduced to an ordinary differential equation in the following

$$tf' - f + bt = 0. \tag{22}$$

where $f' = \frac{df}{d\theta}$. By integration, we obtain its solution as follows

$$u(x,t) = \frac{c_1}{2a} - \frac{b}{2a}\ln t - \frac{1}{4at}x^2. \tag{23}$$

For group G_6, results in the first set of similarity variables as follows:

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln t. \tag{24}$$

This Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln t. \tag{25}$$

Substituting (25) into equation (1) produces an ordinary differential equation

$$bf'' + af'^2 = 0. \tag{26}$$

by integration

$$f = \frac{b}{a}\ln|\frac{a}{b}\theta + c_1| + c_2. \tag{27}$$

Hence, in this case, corresponding to group G_6, the solution is as follows

$$u(x,t) = \frac{b}{a}\ln|\frac{a}{b}xt^{-1} + c_1| - \frac{x^2}{4at} - \frac{b}{2a}\ln t + c_2. \tag{28}$$

Corresponding to group G_6, another similarity variable, the $\ln t$ in the expression (24), is changed to $\ln x$, and the second set of similarity variables is

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln x. \tag{29}$$

Hence, the Lie group invariant solution is:

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{30}$$

Substituting the expression (30) into equation (1) yields the equation for the function $f(\theta)$

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0. \tag{31}$$

where $f' = \frac{df}{d\theta}$. Liu *at el.* [23] pointed out that the exact and explicit solutions of equation (31) cannot be obtained, and be able to be expressed by elementary functions, therefore they proposed a generalized series solution.

In fact, this equation can be solved with the following exact and explicit elementary solution:

$$f(\theta) = \frac{b}{2a}\ln[(\frac{a}{b})^2\theta(c_1\theta - c_2)^2]. \tag{32}$$

Hence, we have second exact solution, anmely

$$u(x,t) = \frac{b}{2a}\ln[(\frac{a}{b})^2\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{33}$$

The exact solution for the equations (32) and (33) were found by this research study [25]. It is worth mentioning that these solutions have never been reported in any literature to date [28].

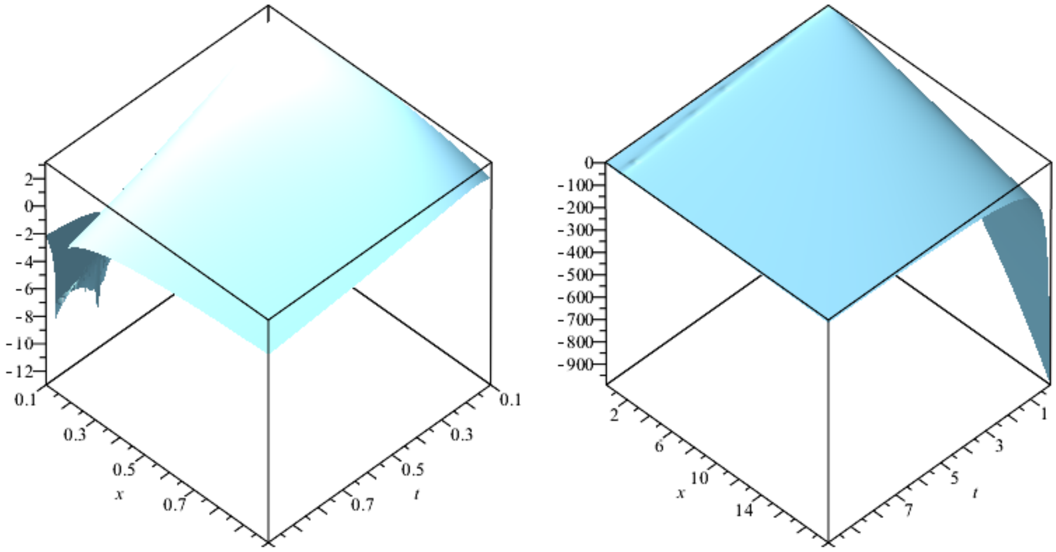

FIG. 1: (a) The solution is in quite oscillation state near the origin $(0, 0)$; (b) The solution tends to reach zero when it is far from the origin $(0, 0)$.

IV. NUMERICAL CALCULATIONS OF $u_t = (u_x)^2 + u_{xx}$ AND DISCUSSIONS

The solution for equation $u_t = (u_x)^2 + u_{xx}$ is $u(x,t) = \frac{1}{2}\ln[\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4t} - \frac{1}{2}\ln x$. For example, the initial-boundary conditions: $u(1, 1) = 1$ and $u(1, 2) = 1$. Hence, $c_1 = e^2 - e^{3/2}$ and $c_2 = e^2 - 2e^{5/4}$. The solution is plotted in Figure 1. The numerical simulations indicate that the solution is in a rapid oscillation state near the origin (0, 0, and tends to take a long-time to reach zero.

To conclude, the three new solutions, which this research study revealed, are listed listed in the Table below.

TABLE I: Three new solutions obtained in this research study

General solution	Eq.(38)
Group G4	Eq.(45)
Group G6	Eq.(59)

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Appendix: Lie infinitesimal generator, Lie algebra and Lie group of $u_t = a(u_x)^2 + bu_{xx}$

For the sake of clarity, it is our honor to summarize some results from Liu *et al.* [23] as follows:
The infinitesimal transformation of x, t, u

$$x \mapsto x + \epsilon\xi(x,t,u), \tag{34}$$
$$t \mapsto t + \epsilon\tau(x,t,u), \tag{35}$$
$$u \mapsto u + \epsilon\phi(x,t,u). \tag{36}$$

The corresponding infinitesimal generator X can be expressed as:

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}, \tag{37}$$

and the second order prolongation $X^{(2)}$ is

$$X^{(2)} = X + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}}, \tag{38}$$

where

$$\phi^t = D_t\phi - u_xD_t\xi - u_tD_t\tau, \tag{39}$$
$$\phi^x = D_x\phi - u_xD_x\xi - u_tD_x\tau, \tag{40}$$
$$\phi^{xx} = D_x^2\phi - u_xD_x^2\xi - u_tD_t^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau. \tag{41}$$

and D_x and D_t are the total differential derivative respect to x and t, respectively

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t}, \tag{42}$$
$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t}. \tag{43}$$

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The Lie group symmetry of the equation (1) is

$$X^{(2)}(u_t - au_x^2 - bu_{xx}) = 0, \quad \text{ie.} \quad \phi^t - 2au_x\phi^x - b\phi^{xx} = 0. \tag{44}$$

The relevant expression can be substituted into the symmetry determining equation (44), paying attention to replacing u_t with $au_x^2 + bu_{xx}$, and comparing the coefficient functions to obtain get infinitesimal generators (one may also use Maple to obtain):

$$\xi = 4ac_1xt + c_2x + 2ac_4t + c_5, \tag{45}$$
$$\tau = 4ac_1t^2 + 2c_2t + c_3, \tag{46}$$
$$\phi = ae^{-au/b} - c_1x^2 - c_4x - 2b_1t + c_6. \tag{47}$$

where $c_i(i = 1, 2, \cdots, 6)$ is an arbitrary constant, $\alpha = \alpha(x, t)$ is a function that satisfies the following conduction equation:

$$\alpha_t = b\alpha_{xx}. \tag{48}$$

Hence the problem of infinitesimal generators:

$$X = \sum_{i=1}^{6} c_iX_i + V_\alpha. \tag{49}$$

where the base vector is

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_5 = 2at\frac{\partial}{\partial x} - x\frac{\partial}{\partial u},$$
$$X_6 = 4axt\frac{\partial}{\partial x} + 4at^2\frac{\partial}{\partial t} - (x^2 + 2bt)\frac{\partial}{\partial u}, \quad X_\alpha = \alpha(x,t)e^{-au/b}\frac{\partial}{\partial u}. \tag{50}$$

and X_α is an infinite dimensional subalgebra。

This verifies that X_i satisfies the closure of Lie algebra, namely X_i, and $X_j \in \mathcal{L}$, then $[X_i, X_j] \in \mathcal{L}$. The vector field commutator-$[X_i, X_j]$ structure coefficient list, is as follows:

TABLE II: Burgers' equation Lie algebra structure coefficient table

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_α
X_1	0	0	0	$-X_1$	X_3	$-2X_5$	$-X_{\alpha_x}$
X_2	0	0	0	$-2X_2$	$-2aX_1$	$2bX_3 - 4aX_4$	$-X_{\alpha_t}$
X_3	0	0	0	0	0	0	$\frac{a}{b}X_\alpha$
X_4	X_1	$2X_2$	0	0	$-X_5$	$-2X_6$	$-X_{\alpha'}$
X_5	$-X_3$	$2aX_1$	0	X_5	0	0	$-X_{\alpha''}$
X_6	$2X_5$	$-2bX_3 + 4aX_4$	0	$2X_6$	0	0	$-X_{\alpha'''}$
X_α	X_{α_x}	X_{α_t}	$-\frac{a}{b}X_\alpha$	$X_{\alpha'}$	$X_{\alpha''}$	$X_{\alpha'''}$	0

where $\alpha' = x\alpha_x + 2t\alpha_t$, $\alpha'' = 2at\alpha_x + \frac{a}{b}x\alpha$, $\alpha''' = 4atx\alpha_x + 4at^2\alpha_t + \frac{a}{b}(x^2 + 2bt)\alpha$.

The Lie group, G_i, generated by X_i, can take advantage of the previously obtained transformation relationship $e^{\epsilon X_i}(x, t, u) = (\hat{x}, \hat{t}, \hat{u})$, such as $\hat{x}$ for G_1:

$$\hat{x} = e^{\epsilon X_1}x = e^{\epsilon\frac{\partial}{\partial x}}x = [1 + \epsilon\frac{\partial}{\partial x} + \frac{1}{2}\epsilon^2(\frac{\partial}{\partial x})^2 \cdots]x = x + \epsilon\frac{\partial x}{\partial x} = x + \epsilon. \tag{51}$$

and G_3:

$$\hat{u} = e^{\epsilon X_3}u = e^{\epsilon\frac{\partial}{\partial u}}u = [1 + \epsilon\frac{\partial}{\partial u} + \frac{1}{2}\epsilon^2(\frac{\partial}{\partial u})^2 \cdots]u = u + \epsilon\frac{\partial u}{\partial u} = u + \epsilon. \tag{52}$$

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Other situations can be calculated in the same manner. The Lie groups are listed below:

$$G_1 : (x,t,u) \mapsto (x+\epsilon, t, u); \tag{53}$$
$$G_2 : (x,t,u) \mapsto (x, t+\epsilon, u); \tag{54}$$
$$G_3 : (x,t,u) \mapsto (x, t, u+\epsilon); \tag{55}$$
$$G_4 : (x,t,u) \mapsto (e^{\epsilon}x, e^{2\epsilon}t, u); \tag{56}$$
$$G_5 : (x,t,u) \mapsto (x+2\epsilon at, t, u-\epsilon x-\epsilon^2 t); \tag{57}$$
$$G_6 : (x,t,u) \mapsto \left(\frac{x}{1-4\epsilon at}, \frac{t}{1-4\epsilon at}; u-\frac{\epsilon x^2}{1-4\epsilon at}+\frac{b}{a}\ln\sqrt{1-4\epsilon at}\right); \text{and} \tag{58}$$
$$G_\alpha : (x,t,u) \mapsto \left(x, t, \frac{b}{a}\ln(e^{\frac{au}{b}}+\epsilon\frac{a}{b}\alpha\right). \tag{59}$$

The above shows that the G_1 group is spatial translation, G_2 is time translation, G_3 is the translation of the dependent variable, G_4 is the extension transformation, G_5 is the Galileo (speed) transformation, and G_6 is a true local transformation group, whose physical meaning is ambiguous but very important, which can be used to find similarity solutions.