Notes on the Lie symmetry exact explicit solutions for nonlinear Burgers’ equation

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In light of Liu’s original works, this paper revisits the solution of general Burgers’s nonlinear equation. We obtain two exact and explicit solutions for group $G_4$ and $G_6$, and a most general solution as well. As applications, a numerical example is carried out.

Keywords: Lie group, Burgers equation, exact solution, general solution, elementary function

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I. INTRODUCTION

Most of the problems in science and engineering are nonlinear in nature, and their corresponding governing equations are nonlinear equations. Solving them is an important task. It is well known that nonlinear partial differential equations have infinitely number solutions, and most of them are variable coefficient equations. Therefore, it is very difficult to obtain their accurate solutions. Although there are few methods for solving nonlinear partial differential equations, but for variable-coefficient nonlinear partial differential equations, the current research methods only stay in numerical solution or approximate solution. Around 1875, Norwegian mathematician Sophus Lie [1–3] proposed a method of continuous transform group, which can simplify the original equation to obtain an exact solution through Lie group or symmetry group transformation. This method does not require special transformation techniques. It is a systematic method that is applicable not only to ordinary differential equations but also to partial differential equations. It is currently the most powerful analytical tool. Cantwell [4] even initiates: students in science and engineering should learn both dimension analysis and Lie group (symmetric analysis), the former is for establishing a physical model (equation), and the latter is for solving or analyzing the structure of the solution [27].

In mathematics, Lie group is a group and also a differential manifold with the property of maintaining a smooth structure under group operation. There is a relatively rich literature on Lie group analysis. Lie began to create a single-parameter continuous group theory (later known as the Lie group) around 1875, and completed the three-step continuation of the famous monographs about Lie group [1–3]. After publication of Lie’s three volume monographs, the first textbook on the Lie group was completed in 1897 by American scholar J.M. Page [5], who was one of the first students to attend Lie’s lecture on the Lie group. American scholar A. Cohen published the second textbook in 1911 [6]. Ince, a scholar at the American University of Egypt in 1926, introduced the Lie group symmetry method in his ordinary differential equation book [7]. After a long period of silence, in 1960 American scholar Birkhoff published a monograph on the study of fluid mechanics with the Lie group. Since the high-level study presented by Birkhoff, the Lie group symmetry method has begun to attract more attention and enter a high-speed application development stage [8]. In the application development of the Lie group symmetry method, it must especially mention that the Lie
where the 2nd order prolongation

\[
X^{(2)} = X + \phi' \frac{\partial}{\partial u_t} + \phi'' \frac{\partial}{\partial u_x} + \phi''' \frac{\partial}{\partial u_{xx}},
\]

and the 2nd order prolongation \(X^{(2)}\) is given

\[
\phi' = D_t \phi - u_0 D_t \xi - u_1 D_t \tau,
\]

\[
\phi'' = D_x \phi - u_0 D_x \xi - u_1 D_x \tau,
\]

\[
\phi''' = D_x^2 \phi - u_0 D_x^2 \xi - u_1 D_x^2 \tau - 2u_{xx} D_x \xi - 2u_{x} D_x \tau.
\]

II. LIE INFINITESIMAL GENERATOR, LIE ALGEBRA AND LIE GROUP OF \(u_t = a(u_x)^2 + bu_{xx}\)

First, do an infinitesimal transformation on \(x, t, u\)

\[
x \mapsto x + \epsilon \xi(x, t, u),
\]

\[
t \mapsto t + \epsilon \tau(x, t, u),
\]

\[
u \mapsto u + \epsilon \phi(x, t, u).
\]

The corresponding infinitesimal generator \(X\) can be expressed as

\[
X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}.
\]
and $D_x$, $D_t$ are the total deferential respect to $x$, $t$, respectively

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_xt \frac{\partial}{\partial u_t},$$  \hspace{1em} (12) $$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_xt \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t}.$$  \hspace{1em} (13)

The Lie group symmetry of the equation (1) is

$$X^{(2)}(u_t - au_x^2 - bu_{xx}) = 0, \text{ i.e. } \phi^t = 2au_x\phi^x - b\phi^{xx} = 0.$$  \hspace{1em} (14)

Substituting the relevant expression into the symmetry determining equation (14), paying attention to replacing $u_t$ with $au_x^2 + bu_{xx}$, and comparing the coefficient functions that can get infinitesimal generators (you can also use Maple obtained)

$$\xi = 4ac_1xt + c_2x + 2ac_4t + c_5,$$

$$\tau = 4ac_1t^2 + 2c_3t + c_3,$$

$$\phi = a e^{-au/b} - c_1x^2 - c_4x - 2b_1t + c_0.$$  \hspace{1em} (15) (16) (17)

Where $c_i(i = 1, 2, \cdots, 6)$ is an arbitrary constant. $\alpha = \alpha(x, t)$ is a function that satisfies the following conduction equation

$$\alpha_t = ba\alpha_{xx}.$$  \hspace{1em} (18)

So we have the problem of infinitesimal generators.

$$X = \sum_{i=1}^{6} c_iX_i + V_\alpha.$$  \hspace{1em} (19)

where the base vector are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u},$$

$$X_4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}, \quad X_5 = 2at \frac{\partial}{\partial x} - x \frac{\partial}{\partial u},$$

$$X_6 = 4axt \frac{\partial}{\partial x} + 4at^2 \frac{\partial}{\partial t} - (x^2 + 2bt) \frac{\partial}{\partial u},$$

$$X_\alpha = \alpha(x, t) e^{-au/b} \frac{\partial}{\partial u}.$$  \hspace{1em} (20)

Where $X_\alpha$ is an infinite dimensional subalgebra.

Now verify that $X_i$ satisfies the closure of Lie algebra, ie $X_i$, and $X_j \in \mathcal{L}$, then $[X_i, X_j] \in \mathcal{L}$. The vector field exchange sub-$[X_i, X_j]$ structure coefficient list 15.3 is as follows:

**TABLE I: Burgers equation Lie algebra structure coefficient table**

<table>
<thead>
<tr>
<th>$[X_i, X_j]$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-X_3$</td>
<td>$-X_\alpha$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{a}X_\alpha$</td>
</tr>
<tr>
<td>$X_4$</td>
<td>$X_3$</td>
<td>$2X_2$</td>
<td>0</td>
<td>0</td>
<td>$-X_5$</td>
<td>$-2X_6$</td>
<td>$-X_\alpha''$</td>
</tr>
<tr>
<td>$X_5$</td>
<td>$-X_3$</td>
<td>$2aX_1$</td>
<td>$X_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-X_\alpha''$</td>
</tr>
<tr>
<td>$X_6$</td>
<td>$2X_5$</td>
<td>$-2bX_3 + 4aX_4$</td>
<td>0</td>
<td>$2X_6$</td>
<td>0</td>
<td>0</td>
<td>$-X_\alpha'''$</td>
</tr>
<tr>
<td>$X_\alpha$</td>
<td>$X_\alpha$</td>
<td>$X_\alpha$</td>
<td>$X_\alpha$</td>
<td>$X_\alpha'''$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $\alpha' = x\alpha + 2t\alpha_1$, $\alpha'' = 2at\alpha_2 + \frac{2}{a}x\alpha$, $\alpha''' = 4at\alpha_3 + 4at^2\alpha_1 + \frac{4}{a^2}(x^2 + 2bt)\alpha$.

The Lie group $G_i$ generated by $X_i$ can take advantage of the previously obtained transformation relationship $e^{X_i}(x, t, u) = (\hat{x}, \hat{t}, \hat{u})$, such as $\hat{x}$ for $G_1$

$$\hat{x} = e^{X_i} x = e^{X_i} x = [1 + \epsilon \frac{\partial}{\partial x} + \frac{1}{2} \epsilon^2 \left( \frac{\partial}{\partial x} \right)^2 \cdots] x = x + \frac{\partial x}{\partial x} x = x + \epsilon.$$  \hspace{1em} (21)
and $G_3$

$$\dot{u} = e^{\epsilon X_3} u = e^{\frac{\epsilon}{\partial x}} u = [1 + \epsilon \frac{\partial}{\partial u} + \frac{1}{2} \epsilon^2 (\frac{\partial}{\partial u})^2, \ldots] u = u + \epsilon \frac{\partial u}{\partial u} = u + \epsilon.$$  \hspace{1cm} (22)

Other situations can be calculated the same. All Lie groups are listed below

$$G_1 : (x, t, u) \rightarrow (x + \epsilon, t, u),$$  \hspace{1cm} (23)
$$G_2 : (x, t, u) \rightarrow (x, t + \epsilon, u),$$  \hspace{1cm} (24)
$$G_3 : (x, t, u) \rightarrow (x, t, u + \epsilon),$$  \hspace{1cm} (25)
$$G_4 : (x, t, u) \rightarrow (e^{\epsilon x}, e^{2\epsilon t}, u),$$  \hspace{1cm} (26)
$$G_5 : (x, t, u) \rightarrow (x + 2\epsilon a t, t, u - \epsilon x - \epsilon^2 t),$$  \hspace{1cm} (27)
$$G_6 : (x, t, u) \rightarrow \left( \frac{x}{1 - 4\epsilon at}, \frac{t}{1 - 4\epsilon at}, u - \frac{\epsilon x^2}{1 - 4\epsilon at} + \frac{b}{a} \ln \sqrt{1 - 4\epsilon at} \right),$$  \hspace{1cm} (28)
$$G_\alpha : (x, t, u) \rightarrow (x, t, \frac{b}{a} \ln (e^{\frac{x}{\alpha}} + \epsilon \frac{a}{b})),$$  \hspace{1cm} (29)

It can be seen from the observation that the $G_1$ group is spatial translation, $G_2$ is time translation, $G_3$ is the translation of the dependent variable, $G_4$ is the extension transformation, $G_5$ is the Galileo (speed) transformation, $G_6$ is a true local transform group whose physical meaning is ambiguous but very important. It can be used to derive similarity solutions.

### III. EXACT GENERAL SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

With Lie symmetry we can construct exact solutions. Since $G_i$ is a symmetric group, if $u = f(x, t)$ is a solution of the equation (1), then the solution $u^{(i)}$ corresponding to $G_i$ can be expressed. Reach

$$u^{(1)} = f(x - \epsilon, t),$$  \hspace{1cm} (30)
$$u^{(2)} = f(x, t - \epsilon),$$  \hspace{1cm} (31)
$$u^{(3)} = f(x, t + \epsilon),$$  \hspace{1cm} (32)
$$u^{(4)} = f(e^{-\epsilon x}, e^{-2\epsilon t}),$$  \hspace{1cm} (33)
$$u^{(5)} = f(x - 2\epsilon a t, t) - \epsilon x + \epsilon^2 at,$$  \hspace{1cm} (34)
$$u^{(6)} = f(\frac{x}{1 + 4\epsilon at}, \frac{t}{1 + 4\epsilon at}) - \frac{\epsilon x^2}{1 + 4\epsilon at} + \frac{b}{a} \ln \sqrt{1 - 4\epsilon at},$$  \hspace{1cm} (35)
$$u^{(\alpha)} = \frac{b}{a} \ln (e^{\frac{x}{\alpha b}} + \epsilon \frac{a}{b} f(x, t)).$$  \hspace{1cm} (36)

The most common symmetry group case is the linear combination $X = \epsilon_1 X_1 + \ldots + \epsilon_6 X_6 + X_\alpha$. The explicit representation of this group is very complicated. We can construct an exponential map $g$: $g \mapsto G_i$, ie

$$g = \exp(X_\alpha) \prod_{i=1}^{6} \exp(\epsilon_i X_i) = \exp(X_\alpha) \cdot \exp(\epsilon_1 X_1) \cdot \ldots \cdot \exp(\epsilon_6 X_6).$$  \hspace{1cm} (37)

If $u = f(x, t)$ is known to be a solution to the equation (1), under the $g$ group transformation, the most general solution can be obtained as follows

$$u = f \left( e^{-\epsilon_1 x} - 2\epsilon_5 at \right) - \epsilon_1, \quad \frac{te^{-2\epsilon_1 x}}{1 + 4\epsilon_6 at} - \epsilon_2 \right) \epsilon_3 - \epsilon_4 \left( \frac{x - 5\epsilon_1 at}{1 + 4\epsilon_6 at} - \frac{\epsilon_6 x^2}{1 + 4\epsilon_6 at} \right) - \frac{\epsilon}{a} \ln \sqrt{1 + 4\epsilon_6 at} + \frac{\epsilon}{a} \ln \left[ e^{\frac{a}{b} f(x, t)} + \epsilon_6 \frac{a}{b} f(x, t) \right].$$  \hspace{1cm} (38)

This general solution is obtained by this paper and has never been apparent in the literature.
IV. EXACT SIMILARITY SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

With an infinitesimal generator $X_i$, you can determine the corresponding invariant, and you can use the invariant to simplify the equation (1) to find its solution.

Notice that the combination of $X_1$ and $X_2$ produces a moving wave solution. Let the new variable $\mu = x - ct$, where $c$ is the wave speed. Thus the equation (1) becomes

$$bf'' + af'^2 + cf' = 0.$$  

(39)

Where $f' = \frac{df}{d\theta}$. The solution of this equation can be integrated

$$u(x, t) = \frac{b}{a} \ln[c_1 e^{\frac{\mu}{c}} - \frac{a}{c} (x - ct) + c_2].$$  

(40)

In the formula, $c_1$, $c_2$ are integral constants.

Corresponding to $G_3$, obviously, if $u = f(x, t)$ is a solution of the equation $u_t = a(u_x)^2 + bu_{xx}$, then $u = f(x, t) + c$ is also the equation.

For the stretch group of $G_4$, you can derive invariants and similar variables

$$\theta = xt^{-1/2}, \quad \omega = u.$$  

(41)

The Lie group invariant solution is

$$\omega = f(\theta) = f(xt^{-1/2}),$$  

ie

$$u = f(xt^{-1/2}).$$  

(42)

Substituting this formula into (1) can be simplified into ordinary differential equations,

$$bf'' + af'^2 + \frac{1}{2} \theta f' = 0.$$  

(43)

where $f' = \frac{df}{d\theta}$ and

$$f(\theta) = \int \frac{\exp(-\theta^2)}{\int \exp(-\frac{\theta^2}{4b})d\theta + p_1} d\theta + p_2.$$  

(44)

Liu at el. [23] did not complete the above integration and left as it is.

If you check the integration carefully, it is not hard to recognise that $\int \exp(-\frac{\theta^2}{4b})d\theta$ should be linkage with error function. After simple cancellation, the solution of the equation (43) can be expressed by error function [25].

$$f(\theta) = \frac{b}{a} \ln\{\frac{a}{b} [c_1 \sqrt{\pi} \sqrt{b} \cdot \text{erf}(\frac{1}{2} \sqrt{b} \cdot \theta) + c_2]\}.$$  

(45)

where erf$(\theta) = \frac{2}{\pi} \int_0^\theta e^{-\theta^2} d\theta$ is error function.

For the Galileo group $G_5$, we have similar variables

$$\theta = t, \quad \omega = \frac{1}{2} t^2 + 2atu.$$  

(46)

This group invariant solution is $\omega = f(\theta)$, ie

$$u = \frac{1}{2at} f(t) - \frac{1}{4at} x^2.$$  

(47)

Substituting (47) into (1) to obtain ordinary differential equations

$$tf' - f + bt = 0.$$  

(48)

where $f' = \frac{df}{d\theta}$. By integration, we obtain its solution as follows

$$u(x, t) = \frac{c_1}{2a} - \frac{b}{2a} \ln t - \frac{1}{4at} x^2.$$  

(49)
For the group $G_6$, the first set of similar variables can be obtained.

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a} \ln t. \quad (50)$$

This Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a} \ln t. \quad (51)$$

Substituting (51) into (1) gives an ordinary differential equation

$$bf'' + af'^2 = 0. \quad (52)$$

by integration

$$\omega = \frac{b}{a} \ln |\frac{a}{b} \theta + c_1| + c_2. \quad (53)$$

So, in this case corresponding to $G_6$, there is a solution.

$$u(x,t) = \frac{b}{a} \ln |\frac{a}{b} xt^{-1} + c_1| - \frac{x^2}{4at} - \frac{b}{2a} \ln t + c_2. \quad (54)$$

Corresponding to $G_6$, another similar variable, the ln $t$ in the expression (50) is changed to ln $x$, and the second set of similar variables is

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a} \ln x. \quad (55)$$

At this time, the Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a} \ln x. \quad (56)$$

Substituting the expression (56) into the original equation (1) yields the equation for the function $f(\theta)$

$$4ab\theta^2 f'' + 4a^2\theta f'^2 - 4ab\theta f' + 3b^2 = 0. \quad (57)$$

where $f' = \frac{df}{d\theta}$. Liu et al. [23] pointed out that the exact and explicit solutions of equation (57) cannot be obtained and be expressed by elementary functions, so they propose a creative series solution.

In fact, this equation can be solved and has the following exact and explicit elementary solution:

$$f(\theta) = \frac{b}{2a} \ln[\frac{a}{b} \theta (c_1 \theta - c_2)^2]. \quad (58)$$

So there is a second form of exact solution

$$u(x,t) = \frac{b}{2a} \ln[\frac{a}{b} \frac{x^2}{t} (c_1 \frac{x}{t} - c_2)^2] - \frac{x^2}{4at} - \frac{b}{2a} \ln x. \quad (59)$$

The exact solution of the equations (58) and (59) are obtained by this paper [25].

V. NUMERICAL CALCULATIONS OF $u_t = (u_x)^2 + u_{xx}$ AND DISCUSSIONS

For equation $u_t = (u_x)^2 + u_{xx}$, its solution is $u(x,t) = \frac{1}{3} \ln[\frac{1}{3} (c_1 x^2 - c_2)^2] - \frac{x^2}{3t} - \frac{1}{3} \ln x$. As an example, we have initial conditions: $u(1,1) = 1$ and $u(1,2) = 1$. Hence, $c_1 = e^2 - e^{3/2}$ and $c_2 = e^2 - 2e^{5/4}$. The solution is plotted in Figure 1 and 1. It is worth mentioning that the above solutions have never been apparent in the literature.

The numerical simulations indicate that the solution is in a rapid oscillation and tends to zero in a long-time.

In summary, the three new solutions have been obtained in this paper and are listed in the Table below.

VI. ACKNOWLEDGEMENT

It is my great pleasure to have shared and discussed some thoughts of this paper with Michael Sun from Bishops Diocesan College, whose pure and direct scientific sense inspired me. This paper is dedicated to the memory my beloved father, Zhong-Chuan Sun.
FIG. 1: The solution is quite oscillation near the origin (0, 0).

FIG. 2: The solution tends to zero when it is far from the origin (0, 0).

<table>
<thead>
<tr>
<th>General solution</th>
<th>Eq.(38)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group G4</td>
<td>Eq.(45)</td>
</tr>
<tr>
<td>Group G6</td>
<td>Eq.(59)</td>
</tr>
</tbody>
</table>