Notes on the Lie symmetry exact explicit solutions for nonlinear Burgers' equation

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In light of Liu's original works, this paper revisits the solution of general Burgers's nonlinear equation. We obtain two exact and explicit solutions for group G_4 and G_6, and a most general solution as well. As applications, a numerical example is carried out.

Keywords: Lie group, Burgers equation, exact solution, general solution, elementary function

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I. INTRODUCTION

Most of the problems in science and engineering are nonlinear in nature, and their corresponding governing equations are nonlinear equations. Solving them is an important task. It is well known that nonlinear partial differential equations have infinitely number solutions, and most of them are variable coefficient equations. Therefore, it is very difficult to obtain their accurate solutions. Although there are few methods for solving nonlinear partial differential equations, but for variable-coefficient nonlinear partial differential equations, the current research methods only stay in numerical solution or approximate solution. Around 1875, Norwegian mathematician Sophus Lie [1–3] proposed a method of continuous transform group, which can simplify the original equation to obtain an exact solution through Lie group or symmetry group transformation. This method does not require special transformation techniques. It is a systematic method that is applicable not only to ordinary differential equations but also to partial differential equations. It is currently the most powerful analytical tool. Cantwell [4] even initiates: students in science and engineering should learn both dimension analysis and Lie group (symmetric analysis), the former is for establishing a physical model (equation), and the latter is for solving or analyzing the structure of the solution [27].

In mathematics, Lie group is a group and also a differential manifold with the property of maintaining a smooth structure under group operation. There is a relatively rich literature on Lie group analysis. Lie began to create a single-parameter continuous group theory (later known as the Lie group) around 1875, and completed the three-step continuation of the famous monographs about Lie group [1–3]. After publication of Lie's three volume monographs, the first textbook on the Lie group was completed in 1897 by American scholar J.M. Page [5], who was one of the first students to attend Lie's lecture on the Lie group. American scholar A. Cohen published the second textbook in 1911 [6]. Ince, a scholar at the American University of Egypt in 1926, introduced the Lie group symmetry method in his ordinary differential equation book [7]. After a long period of silence, in 1960 American scholar Birkhoff published a monograph on the study of fluid mechanics with the Lie group. Since the high-level study presented by Birkhoff, the Lie group symmetry method has begun to attract more attention and enter a high-speed application. development stage [8]. In the application development of the Lie group symmetry method, it must especially mention that the Lie

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group analysis monograph [9] of the differential equation of the former Soviet scholar Ovsiannikov, who leaded Soviet school have promoted the development of Lie group application and computer computing. Ovsiannikov's monograph was translated into English by Bluman in 1967. Hansen's book in 1964 discusses fluid mechanics and heat transfer problems. In 1965, Ames' monograph studied nonlinear problems in engineering [10]. Ibragimov and Bluman later made an influential contribution to the education and application of the Lie group [11–21, 24–27].

Liu, *et al.* [23] applied the Lie group to study the general nonlinear Burgers equation [22]:

$$u_t = a(u_x)^2 + bu_{xx}. \tag{1}$$

where $u = u(x,t)$ is a real function, and parameters $a, b \in R$ with $ab \neq 0$, and $u_t = \frac{\partial}{\partial t}$, $u_x = \frac{\partial}{\partial x}$ and $u_{xx} = \frac{\partial^2}{\partial x^2}$.

For the general nonlinear Burgers equation, Liu, *et al.* [23] obtained six one-parameter Lie group G_i, $i = 1...6$, four exact explicit solutions corresponding the group G_1, G_2, G_3 and G_5. Unfortunately, Liu, *et al.* [23] could not find the exact and explicit solutions for both group G_4 and G_6. For group G_4, Liu, *et al.* [23] left an unfinished integration:

$$f(\xi) = \int \frac{e^{-\frac{\xi^2}{4b}}}{\int e^{-\frac{\xi^2}{4b^2}} + p_1} d\xi + p_2. \tag{2}$$

For the group G_6, Liu, *et al.* [23] derived a nonlinear ordinary differential equation:

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0. \tag{3}$$

Liu, *et al.* [23] states that "we cannot obtain the exact and explicit solutions for Eq.(3) by using the elementary functions.". To solve the Eq.(3), they cleverly proposed a generalized power series solution: $f(\xi) = \alpha \log|\xi| + \beta + \sum_{n=0}^{\infty} f_n \xi^{n+1}$, and took a quite effect to determine all coefficients.

General speaking, the results of Liu, *et al.* [23] are nice and will be more beautiful if the solutions for both group G_4 and G_6 can be obtained and expressed in elementary functions. We will make up this gap in this paper.

The paper is organized as follows: for self-contained purpose, following the introduction we summary some results of Liu, *et al.* [23], including Lie symmetric infinitesimal generator, Lie algebra; and in Section 3, we presents a most general solution for $u_t = a(u_x)^2 + bu_{xx}$; in Section 4, two exact nd explicit solutions will be proposed. In Section 5, as application, a numerical example has been carried out and results are plotted. Finally, the paper concludes with perspectives about the future development of research studies.

II. LIE INFINITESIMAL GENERATOR, LIE ALGEBRA AND LIE GROUP OF $u_t = a(u_x)^2 + bu_{xx}$

First, do an infinitesimal transformation on x, t, u

$$x \mapsto x + \epsilon\xi(x,t,u), \tag{4}$$
$$t \mapsto t + \epsilon\tau(x,t,u), \tag{5}$$
$$u \mapsto u + \epsilon\phi(x,t,u). \tag{6}$$

The corresponding infinitesimal generator X can be expressed as

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}, \tag{7}$$

and the 2nd order prolongation $X^{(2)}$ is given

$$X^{(2)} = X + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}}, \tag{8}$$

where

$$\phi^t = D_t\phi - u_x D_t\xi - u_t D_t\tau, \tag{9}$$
$$\phi^x = D_x\phi - u_x D_x\xi - u_t D_x\tau, \tag{10}$$
$$\phi^{xx} = D_x^2\phi - u_x D_x^2\xi - u_t D_t^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau. \tag{11}$$

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and D_x, D_t are the total deferential respect to x, t, respectively

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t}, \tag{12}$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t}. \tag{13}$$

The Lie group symmetry of the equation (1) is

$$X^{(2)}(u_t - a u_x^2 - b u_{xx}) = 0, \quad \text{ie.} \quad \phi^t - 2a u_x \phi^x - b\phi^{xx} = 0. \tag{14}$$

Substituting the relevant expression into the symmetry determining equation (14), paying attention to replacing u_t with $au_x^2 + bu_{xx}$, and comparing the coefficient functions that can get infinitesimal generators (you can also use Maple obtained)

$$\xi = 4ac_1xt + c_2x + 2ac_4t + c_5, \tag{15}$$

$$\tau = 4ac_1t^2 + 2c_2t + c_3, \tag{16}$$

$$\phi = ae^{-au/b} - c_1x^2 - c_4x - 2b_1t + c_6. \tag{17}$$

Where $c_i(i = 1, 2, \cdots, 6)$ is an arbitrary constant. $\alpha = \alpha(x, t)$ is a function that satisfies the following conduction equation

$$\alpha_t = b\alpha_{xx}. \tag{18}$$

So we have the problem of infinitesimal generators.

$$X = \sum_{i=1}^{6} c_i X_i + V_\alpha. \tag{19}$$

where the base vector are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u},$$
$$X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_5 = 2at\frac{\partial}{\partial x} - x\frac{\partial}{\partial u},$$
$$X_6 = 4axt\frac{\partial}{\partial x} + 4at^2\frac{\partial}{\partial t} - (x^2 + 2bt)\frac{\partial}{\partial u},$$
$$X_\alpha = \alpha(x,t)e^{-au/b}\frac{\partial}{\partial u}. \tag{20}$$

Where X_α is an infinite dimensional subalgebra。

Now verify that X_i satisfies the closure of Lie algebra, ie X_i, and $X_j \in \mathcal{L}$, then $[X_i, X_j] \in \mathcal{L}$. The vector field exchange sub-$[X_i, X_j]$ structure coefficient list 15.3 is as follows:

TABLE I: Burgers equation Lie algebra structure coefficient table

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_α
X_1	0	0	0	$-X_1$	X_3	$-2X_5$	$-X_{\alpha_x}$
X_2	0	0	0	$-2X_2$	$-2aX_1$	$2bX_3 - 4aX_4$	$-X_{\alpha_t}$
X_3	0	0	0	0	0	0	$\frac{a}{b}X_\alpha$
X_4	X_1	$2X_2$	0	0	$-X_5$	$-2X_6$	$-X_{\alpha'}$
X_5	$-X_3$	$2aX_1$	0	X_5	0	0	$-X_{\alpha''}$
X_6	$2X_5$	$-2bX_3 + 4aX_4$	0	$2X_6$	0	0	$-X_{\alpha'''}$
X_α	X_{α_x}	X_{α_t}	$-\frac{a}{b}X_\alpha$	$X_{\alpha'}$	$X_{\alpha''}$	$X_{\alpha'''}$	0

where $\alpha' = x\alpha_x + 2t\alpha_t$, $\alpha'' = 2at\alpha_x + \frac{a}{b}x\alpha$, $\alpha''' = 4atx\alpha_x + 4at^2\alpha_t + \frac{a}{b}(x^2 + 2bt)\alpha$.

The Lie group G_i generated by X_i can take advantage of the previously obtained transformation relationship $e^{\epsilon X_i}(x, t, u) = (\hat{x}, \hat{t}, \hat{u})$, such as $\hat{x}$ for G_1

$$\hat{x} = e^{\epsilon X_1}x = e^{\epsilon\frac{\partial}{\partial x}}x = [1 + \epsilon\frac{\partial}{\partial x} + \frac{1}{2}\epsilon^2(\frac{\partial}{\partial x})^2 \cdots]x = x + \epsilon\frac{\partial x}{\partial x} = x + \epsilon. \tag{21}$$

and G_3

$$\hat{u} = e^{\epsilon X_3}u = e^{\epsilon\frac{\partial}{\partial u}}u = [1+\epsilon\frac{\partial}{\partial u}+\frac{1}{2}\epsilon^2(\frac{\partial}{\partial u})^2\cdots]u = u+\epsilon\frac{\partial u}{\partial u} = u+\epsilon. \tag{22}$$

Other situations can be calculated the same. All Lie groups are listed below

$$G_1 : (x,t,u) \mapsto (x+\epsilon, t, u), \tag{23}$$
$$G_2 : (x,t,u) \mapsto (x, t+\epsilon, u), \tag{24}$$
$$G_3 : (x,t,u) \mapsto (x, t, u+\epsilon), \tag{25}$$
$$G_4 : (x,t,u) \mapsto (e^{\epsilon}x, e^{2\epsilon}t, u), \tag{26}$$
$$G_5 : (x,t,u) \mapsto (x+2\epsilon at, t, u-\epsilon x-\epsilon^2 t), \tag{27}$$
$$G_6 : (x,t,u) \mapsto (\frac{x}{1-4\epsilon at}, \frac{t}{1-4\epsilon at}, u-\frac{\epsilon x^2}{1-4\epsilon at}+\frac{b}{a}\ln\sqrt{1-4\epsilon at}), \tag{28}$$
$$G_\alpha : (x,t,u) \mapsto (x, t, \frac{b}{a}\ln(e^{\frac{au}{b}}+\epsilon\frac{a}{b}\alpha)). \tag{29}$$

It can be seen from the observation that the G_1 group is spatial translation, G_2 is time translation, G_3 is the translation of the dependent variable, G_4 is the extension transformation, G_5 is the Galileo (speed) transformation, G_6 is a true local transform group whose physical meaning is ambiguous but very important. It can be used to derive similarity solutions.

III. EXACT GENERAL SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

With Lie symmetry we can construct exact solutions. Since G_i is a symmetric group, if $u = f(x,t)$ is a solution of the equation (1), then the solution $u^{(i)}$ corresponding to G_i can be expressed. Reach

$$u^{(1)} = f(x-\epsilon, t), \tag{30}$$
$$u^{(2)} = f(x, t-\epsilon), \tag{31}$$
$$u^{(3)} = f(x,t)+\epsilon, \tag{32}$$
$$u^{(4)} = f(e^{-\epsilon}x, e^{-2\epsilon}t), \tag{33}$$
$$u^{(5)} = f(x-2\epsilon at, t)-\epsilon x+\epsilon^2 at, \tag{34}$$
$$u^{(6)} = f(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon at})-\frac{\epsilon x^2}{1+4\epsilon at}-\frac{b}{a}\ln\sqrt{1+4\epsilon at}, \tag{35}$$
$$u^{(\alpha)} = \frac{b}{a}\ln(e^{\frac{af(x,t)}{b}}+\epsilon\frac{a}{b}\alpha(x,t)). \tag{36}$$

The most common symmetry group case is the linear combination $X = c_1X_1 + ... + c_6X_6 + X_\alpha$. The explicit representation of this group is very complicated. We can construct an exponential map $\mathfrak{g}$: $\mathfrak{g} \mapsto G$, ie

$$\mathfrak{g} = \exp(X_\alpha)\prod_{i=1}^{6}\exp(\epsilon_i X_i) = \exp(X_\alpha)\cdot\exp(\epsilon_1 X_1)\cdot ... \cdot\exp(\epsilon_6 X_6). \tag{37}$$

If $u = f(x,t)$ is known to be a solution to the equation (1), under the $\mathfrak{g}$ group transformation, the most general solution can be obtained as follows

$$u = f\left(e^{-\epsilon_4}\frac{x-2\epsilon_5 at}{1+4\epsilon_6 at}-\epsilon_1, \frac{t\epsilon^{-2\epsilon_4}}{1+4\epsilon_6 at}-\epsilon_2\right)\epsilon_3-\epsilon_5\frac{x-\epsilon_5 at}{1+4\epsilon_6 at}-\frac{\epsilon_6 x^2}{1+4\epsilon_6 at}$$
$$-\frac{b}{a}\ln\sqrt{1+4\epsilon_6 at}+\frac{b}{a}\ln\left[e^{\frac{a}{b}f(x,t)}+\epsilon_6\frac{a}{b}\alpha(x,t)\right]. \tag{38}$$

This general solution is obtained by this paper and has never been apparent in the literature.

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IV. EXACT SIMILARITY SOLUTIONS OF $u_t = a(u_x)^2 + bu_{xx}$

With an infinitesimal generator X_i, you can determine the corresponding invariant, and you can use the invariant to simplify the equation (1) to find its solution.

Notice that the combination of X_1 and X_2 produces a moving wave solution. Let the new variable $\mu = x - ct$, so $u = u(x,t) = f(\mu)$, where c is the wave speed. Thus the equation (1) becomes

$$bf'' + af'^2 + cf' = 0. \tag{39}$$

Where $f' = \frac{df}{d\mu}$. The solution of this equation can be integrated

$$u(x,t) = \frac{b}{a}\ln[c_1 e^{\frac{c}{b}(x-ct)} - \frac{a}{c}] - \frac{c}{a}(x-ct) + c_2. \tag{40}$$

In the formula, c_1, c_2 are integral constants.

Corresponding to G_3, obviously, if $u = f(x,t)$ is a solution of the equation $u_t = a(u_x)^2 + bu_{xx}$, then $u = f(x,t) + c$ is also the equation.

For the stretch group of G_4, you can derive invariants and similar variables

$$\theta = xt^{-1/2}, \quad \omega = u. \tag{41}$$

The Lie group invariant solution is $\omega = f(\theta) = f(xt^{-1/2})$, ie

$$u = f(xt^{-1/2}). \tag{42}$$

Substituting this formula into (1) can be simplified into ordinary differential equations.

$$bf'' + af'^2 + \frac{1}{2}\theta f' = 0. \tag{43}$$

where $f' = \frac{df}{d\theta}$ and

$$f(\theta) = \int \frac{exp(\frac{-\theta^2}{4b})}{\int exp(\frac{-\theta^2}{4b})d\theta + p_1} d\theta + p_2. \tag{44}$$

Liu at el. [23] did not complete the above integration and left as it is.

If you check the integration carefully, it is not hard to recognise that $\int exp(\frac{-\theta^2}{4b})d\theta$ should be linkage with error function. After simple cancellation, the solution of the equation (43) can be expressed by error function [25].

$$f(\theta) = \frac{b}{a}\ln\{\frac{a}{b}[c_1\sqrt{\pi}\sqrt{b}\cdot \mathrm{erf}(\frac{1}{2}\frac{\theta}{\sqrt{b}}) + c_2]\}. \tag{45}$$

where $\mathrm{erf}(\theta) = \frac{2}{\pi}\int_0^\theta e^{-\theta^2}d\theta$ is error function.

For the Galileo group G_5, we have similar variables

$$\theta = t, \quad \omega = \frac{1}{2}x^2 + 2atu. \tag{46}$$

This group invariant solution is $\omega = f(\theta)$, ie

$$u = \frac{1}{2at}f(t) - \frac{1}{4at}x^2. \tag{47}$$

Substituting (47) into (1) to obtain ordinary differential equations

$$tf' - f + bt = 0. \tag{48}$$

where $f' = \frac{df}{d\theta}$。 By integration, we obtain its solution as follows

$$u(x,t) = \frac{c_1}{2a} - \frac{b}{2a}\ln t - \frac{1}{4at}x^2. \tag{49}$$

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For the group G_6, the first set of similar variables can be obtained.

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln t. \tag{50}$$

This Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln t. \tag{51}$$

Substituting (51) into (1) gives an ordinary differential equation

$$bf'' + af'^2 = 0. \tag{52}$$

by integration

$$\omega = \frac{b}{a}\ln|\frac{a}{b}\theta + c_1| + c_2. \tag{53}$$

So, in this case corresponding to G_6, there is a solution.

$$u(x,t) = \frac{b}{a}\ln|\frac{a}{b}xt^{-1} + c_1| - \frac{x^2}{4at} - \frac{b}{2a}\ln t + c_2. \tag{54}$$

Corresponding to G_6, another similar variable, the $\ln t$ in the expression (50) is changed to $\ln x$, and the second set of similar variables is

$$\theta = xt^{-1}, \quad \omega = u + \frac{x^2}{4at} + \frac{b}{2a}\ln x. \tag{55}$$

At this time, the Lie group invariant solution is

$$u = f(xt^{-1}) - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{56}$$

Substituting the expression (56) into the original equation (1) yields the equation for the function $f(\theta)$

$$4ab\theta^2 f'' + 4a^2\theta^2 f'^2 - 4ab\theta f' + 3b^2 = 0. \tag{57}$$

where $f' = \frac{df}{d\theta}$. Liu at el. [23] pointed out that the exact and explicit solutions of equation (57) cannot be obtained and be expressed by elementary functions, so they propose a creative series solution.

In fact, this equation can be solved and has the following exact and explicit elementary solution:

$$f(\theta) = \frac{b}{2a}\ln[(\frac{a}{b})^2\theta(c_1\theta - c_2)^2]. \tag{58}$$

So there is a second form of exact solution

$$u(x,t) = \frac{b}{2a}\ln[(\frac{a}{b})^2\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4at} - \frac{b}{2a}\ln x. \tag{59}$$

The exact solution of the equations (58) and (59) are obtained by this paper [25].

V. NUMERICAL CALCULATIONS OF $u_t = (u_x)^2 + u_{xx}$ AND DISCUSSIONS

For equation $u_t = (u_x)^2 + u_{xx}$, its solution is $u(x,t) = \frac{1}{2}\ln[\frac{x}{t}(c_1\frac{x}{t} - c_2)^2] - \frac{x^2}{4t} - \frac{1}{2}\ln x$. As an example, we have initial conditions: $u(1,1) = 1$ and $u(1,2) = 1$. Hence, $c_1 = e^2 - e^{3/2}$ and $c_2 = e^2 - 2e^{5/4}$. The solution is plotted in Figure 1 and 1. It is worth mentioning that the above solutions have never been apparent in the literature.

The numerical simulations indicate that the solution is in a rapid oscillation and tends to zero in a long-time.

In summary, the three new solutions have been obtained in this paper are listed in the Table below.

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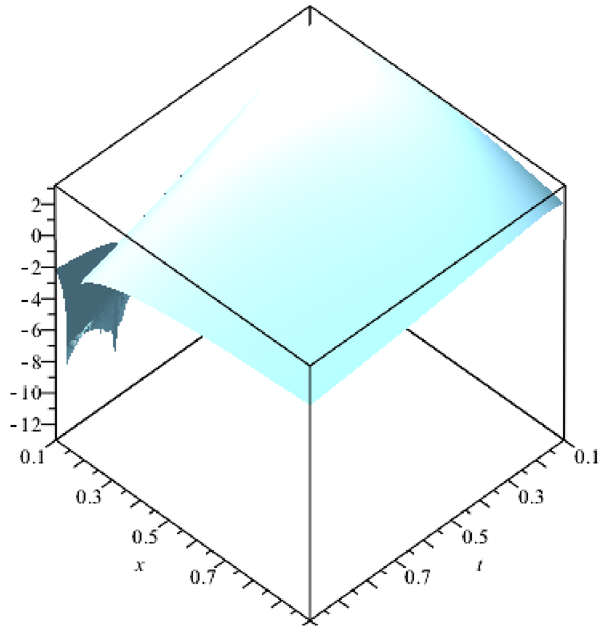

FIG. 1: The solution is quite oscillation near the origin $(0, 0)$.

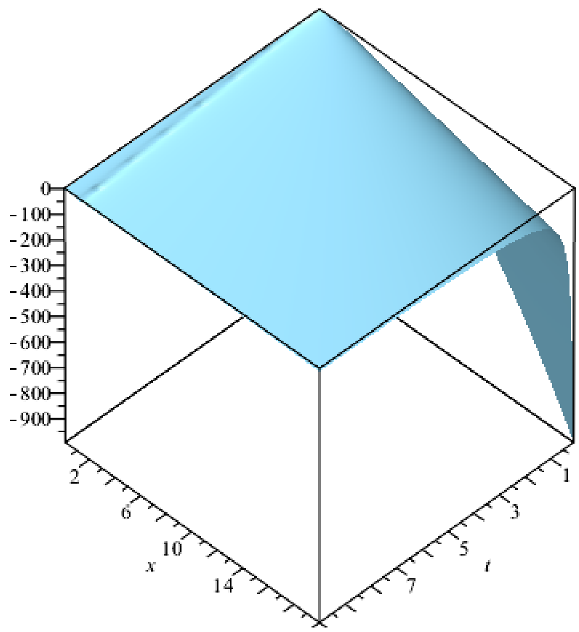

FIG. 2: The solution tends to zero when it is far from the origin $(0, 0)$.

TABLE II: Three new solutions obtained in this paper

General solution	Eq.(38)
Group G4	Eq.(45)
Group G6	Eq.(59)

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