Reynolds' Turbulence Solution

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The study found an error in current literature, including numerous textbooks, about the number of independent unknowns in the Reynolds stress tensor and/or in Reynolds-averaged Navier-Stokes equations (RANS). Current literature claims that the Reynolds stress tensor has six unknowns; however, this article shows that the Reynolds stress tensor only has independent three unknowns, which are functions of the three components of fluctuation velocity. This research discovers that the misconception about the number of independent unknowns in the RANS could stem from misinterpreting the Reynolds stress tensor. The misconception has hampered the development of turbulence for longtime. In order to find a way out of this difficult situation, we return to the time of Reynolds in 1895 and revisit Reynolds' averaging formulation of turbulence. The present investigation can be considered as a renaissance of Reynolds' work in 1895, which might shed light on the well-known closure problem of turbulence, and help to understand the puzzle of the turbulence closure problem that has eluded scientists and mathematicians for more than a century.

Keywords: Turbulence, number of independent unknowns, Reynolds stress tensor, RANS, turbulence closure problem

I. INTRODUCTION

"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is relativity/quantum mechanics/quantum electrodynamics [in various versions], and the other is turbulent motion of fluids. About the former I am rather optimistic." –Horace Lamb [1]

You're on a airplane when you feel a sudden jolt. Outside your window nothing seems to be happening, as yet the plane continues to rattle you and your fellow passengers as it passes through turbulent air in the atmosphere, and yet, turbulence is ubiquitous, springing up in virtually any system that has moving fluids. That includes that airflow in your respiratory tract. The blood moving through your arteries. And the coffee in your cup as you stir it. Clouds are governed by turbulence, as are waves crashing along the shore and the gusts of plasma in our Sun. Understanding precisely how this phenomenon works would have a bearing on so many aspects of our lives [2–28, 30–42, 44–53, 55–65].

The turbulence phenomenon is one of the greatest prevailing unsolved mysteries of physics. After more than a century of studying turbulence, we've found a few answers on how it works, and affects the world around us. Most scientists argue that this will be by achieved relying on statistics and increased computing power. Extremely high-speed computer simulations of turbulent flows may help to identify patterns that lead to a theory that organizes and unifies predictions for different situations. Other scientists state that the phenomenon is so complex that such a full-fledged theory will not be possible

[2–28, 30–42, 44–53, 55–65].

In respect of the turbulence problem, a myriad of tentative theories have been proposed and each with its own doctrines and beliefs, which often focused on particular experiments; however, there is not much in the way of a coherent theoretical framework [6, 8–17, 39–41]. Turbulence is a unique subject, which engineers, mathematicians and physicists view differently [36]. Many engineers promote the use of semi-empirical models of turbulence [36], while mathematicians advocate the use of purely statistical models [5, 7, 30, 31, 45–48, 55, 56], the renomalization model [34], and the formalism of chaos theory and fractals [49–51].

In 1972 a new chapter was launched in turbulence theory: [52] demonstrated that it was possible to perform direct numerical simulation (DNS) of a fully turbulent flow. It is important to understand that DNS does not require any turbulence model to parameterize influence of the turbulent eddies. Rather, every eddy, from the largest to the smallest, is computed. From a technical perspective, the turbulence can be solved by DNS if computers have infinite speed. However, a huge chasm remains between what the engineer needs to know, and what can be realized by DNS, using current computers. Even if DNS can assist to solve turbulence issues and problems, one still requires turbulence modelling to acquire a physical understanding of it [37].

Although the above-mentioned professionals have different views about turbulence, there is consensus that the deterministic Navier-Stokes equation probably contains all the information about turbulence [6]. Turbulence prediction can be attained by understanding solutions to the Navier-Stokes equations [5, 6, 8–25, 27, 28, 30–42, 44–53, 55–62, 64, 65]. Navier [21] and Poisson [22] first obtained these equations, which were finally shown

by St. Venant and Sir Gabbriel Stokes [23, 24] on various considerations as to the mutual action of the ultimate molecules of fluids [19, 20]. Hence, numerous works have been reported on various aspects of the Navier-Stokes equations. [6, 8–19, 25, 28, 39–41].

The article adopts Reynolds's deterministic view on the turbulence [19], and revisits the Reynolds-averaged Navier-Stokes equations (RANS). This article deals with a basic problem in turbulence analysis, namely the number of unknowns in the Reynolds stress tensor. This is obviously a fundamental question in fluid mechanics as well. The research in this paper reveals that the RANS only has three independent unknowns instead of six, is stated in current literature, including textbooks [8–15, 18, 30–33, 39–41].

Although the Reynolds-averaged-Navier-Stokes-Equations (RANS) have been formulated for more than 120 years, much of current literature and standard turbulence textbooks have incorrect information about the number of unknowns and this prevented solving the turbulence problem. Hence, if the number of unknowns are incorrect, no solutions will be possible. This situation must be corrected immediately, else it will be harmful for turbulence studies.

In view of the fact that no essential result on turbulence definition has been made, it would probably be fitting to return to the beginning of turbulent research, that is, in 1895, to study Reynolds' seminal studies to rediscover certain useful information.

The aims of this paper are to revisit Reynolds' averaging formulation of turbulence, and to clarify the number of unknowns in the Reynolds stress tensor and/or in RANS. The paper is organized as follows: following an introduction, we introduce Reynolds velocity decomposition and reformulate Reynolds-averaged Navier-Stokes equations in tensorial form; we point out an important question about the number of independent unknowns in both the Reynolds stress tensor and the RANS; we provide three mathematical lemma and evidence about the number of independent unknowns. Finally, the paper concludes with perspectives about the future development of turbulence research.

II. REYNOLDS-AVERAGED NAVIER-STOKES EQUATIONS AND DETERMINISTIC NATURE

From the infinitesimal analysis of the Naviers-Stokes equation [43], there are 10 Lie groups have been founded, however there is no one group is suitable to the turbulence solution.

In ground-breaking research, Reynolds [19], considered turbulence from a different perspective. Assuming that turbulent motion already exists, he sought to establish a criterion, which decides whether the turbulent character will increase or diminish, or remain stationary [20].

From experiments, Reynolds [19], discovered the turbulence solution could be expressed in the sum of mean and fluctuation part of velocity field, which is the solution not been founded by the Lie group analysis [19]. Reynolds' solution method is a quite general approach for any nonlinear partial differential equations (PDE), which transfers the PDE to integro-differential equations.

In 1895 Reynolds proposed that flow velocity $\boldsymbol{u}$ and pressure p are decomposed into its time-averaged quantities, $\bar{\boldsymbol{u}}$, $\bar{p}$, and fluctuating quantities, $\boldsymbol{u}'$, p'; thus, the Reynolds' decompositions are:

$$\boldsymbol{u} = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{u}'(\boldsymbol{x},t), \qquad (1)$$

$$p(\boldsymbol{x},t) = \bar{p}(\boldsymbol{x}) + p'(\boldsymbol{x},t), \qquad (2)$$

where coordinates and time are $(\boldsymbol{x},t)$. According to Reynolds, $\bar{\boldsymbol{u}}$ represent a mean-motion at each point and $\boldsymbol{u}'$ a motion at the same point as the mean-motion at the point. Therefore, Reynolds called the $\bar{\boldsymbol{u}}$ mean-motion and $\boldsymbol{u}'$ relative-mean-motion [19]. Both velocity and velocity and pressure time-averaging definitions are defined as

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \boldsymbol{u}(\boldsymbol{x},t)dt, \qquad (3)$$

$$\bar{p}(\boldsymbol{x}) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} p(\boldsymbol{x},t)dt, \qquad (4)$$

where T is the period of time when the averaging takes place and must be sufficiently large to give meaningful averages to measure mean values. Naturally, the time-fluctuation velocity $\boldsymbol{u}' = \boldsymbol{u} - \bar{\boldsymbol{u}}(\boldsymbol{x})$, pressure $p' = p - \bar{p}(\boldsymbol{x})$; and their time-averaging vanish, namely $\bar{\boldsymbol{u}}' = \boldsymbol{0}$ and $\bar{p}' = 0$.

Decomposition causes the Navier-Stokes equation to transform into Reynolds-averaged Navier-Stokes equations (RANS), as follows

$$\rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}\bar{p} = \mu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}, \qquad (5)$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \qquad (6)$$

where dynamic viscosity μ, gradient operator $\boldsymbol{\nabla} = \boldsymbol{e}_i\nabla_i$, base vector in the i-coordinate $\boldsymbol{e}_i$, and tensor product $\otimes$, and the Reynolds stress tensor is given by

$$\boldsymbol{\tau}(\boldsymbol{x}) = -\rho\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'} = -\rho\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \ (7)$$

which reveals that Reynolds stress is apparent, depending on the fluctuating velocity field $\boldsymbol{u}'$.

In order to obtain more information about the Reynolds' stress tensor, Reynolds [19] derived velocity fluctuation equations, namely the equation (16) in [19]), called the equations of momentum of relative-mean-motion at each point. Chou [32] reformulated these equations into index-tensorial at form. The equations are presented in below bold-face tensorial form as follows

$$\rho\boldsymbol{u}'_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\boldsymbol{u}' + \boldsymbol{u}'\otimes\bar{\boldsymbol{u}} + \boldsymbol{u}'\otimes\boldsymbol{u}') + \boldsymbol{\nabla}p'$$
$$= \mu\boldsymbol{\nabla}^2\boldsymbol{u}' + \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \quad (8)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0. \qquad (9)$$

These are integral-differential equations governing $\bar{\boldsymbol{u}}$, $\boldsymbol{u}'$ and p'.

In above Eqs.5,6, 10 and 11 and relevant relations, why we express the problem in tensorial format instead of in cartesian components one, it is because that such formulations will not reply on any choice of coordinates, therefore, the results in this paper will be more general and universal. A basic philosophy of modern physics is that the universe does not come equipped with a coordinate system. While coordinate systems are necessary for doing specific calculations, the choice of the coordinate system to use is a matter of convenience, and there is often no "best" coordinate system. One should strive to write the laws of physics in a manifestly coordinate-independent manner, so one can see what they are really saying and not get distracted by things that might depend on the coordinates [54].

Denoting kinematic viscosity $\nu = \mu/\rho$, the Eqs.5,6 can be equivalently rewritten in conventional form, namely

$$\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\bar{\boldsymbol{u}} + \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} (\boldsymbol{u}' \cdot \boldsymbol{\nabla}\boldsymbol{u}')dt$$
$$= -\frac{1}{\rho}\boldsymbol{\nabla}\bar{p} + \nu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} \tag{10}$$
$$\boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} = 0. \tag{11}$$

These integro-differential equations are called Reynolds-averaged Navier-Stokes equations (RANS), which were the equations (15) in [19] that Reynolds formulated, calling them equations of mean-motion at every point. It is clear that the RANS are deterministic equations rather than statistical ones [19, 20].

It is worth pointing out that for the deterministic velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, the Reynolds decomposition in Eqs. 1 and 2 can always be well-defined by Eqs. 3 and 4. In other words, Reynolds decomposition can always be constructed for any flow motion, regardless of turbulent flow or not.

The governing equations of flows are deterministic, not random. So fundamentally turbulent fields are not random because their generating dynamics is not random [29]. Random is not intrinsic properties of turbulent field while extrinsic ones. It is misleading to consider the essence of turbulence as a random motion, and the flow velocity field $\boldsymbol{u}$ as a statistical function [30]. Because there is consensus on turbulence, namely that the deterministic Navier-Stokes equations contain all the required information about turbulence [6], means that the essence of the flow velocity and its Navier-Stokes equations is deterministic, rather than random. Taylor [30] introduced a statistical approach and attempted to study turbulence from a statistical point of view; however, the approach is merely a mathematical method for solutions, which cannot change the deterministic nature of flow velocity $\boldsymbol{u}$ and Navier-Stokes equations. In physics, turbulence cannot be considered as a random motion simply because it is difficulty to solve. Generally speaking, any mathematical method used for solving a problem should not change the physics of the problem.

III. HOW MANY UNKNOWNS ARE THERE IN THE REYNOLDS STRESS TENSOR AND/OR REYNOLDS-AVERAGED NAVIER-STOKES EQUATIONS ?

All current literature including textbooks, report that the Reynolds stresses $\boldsymbol{\tau}$ have six unknowns (Later we will show that the Reynolds stresses only have three unknowns instead of six). This traditional understanding has resulted in consensus that there are 10 unknowns as shown in Eqs.5, 6 and/or Eqs.10, 11.

In his 1895 paper [19], Reynolds did not discuss the number of unknowns in the RANS. However, from his presentation we note that he never considered the term $\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}$ as independent unknowns, while for some cases he proposed explicit expressions for the velocity fluctuation $\boldsymbol{u}'$ on page 149, and equation (50) on page 158 of his paper [19]. For instance, one expression he proposed on page 149 of [19] is:

$$\sum \left[A_r \cos\left\{ r\left(nt + \frac{2\pi}{a}x\right)\right\} \right]$$

and in the section, *Expressions for the components of possible relative-mean-motion*, he proposed a similar expression on page 15 of [19], as follows:

$$u' = \sum_0^\infty \left\{ \left(\frac{d\alpha_n}{dy} + \frac{d\gamma_n}{dz}\right)\cos(nlx) + \left(\frac{d\beta_n}{dy} + \frac{d\delta_n}{dz}\right)\sin(nlx)\right\},$$

$$v' = \sum_0^\infty \left\{ n\alpha_n \sin(nlx) - nl\beta_n \cos(nlx)\right\},$$

$$w' = \sum_0^\infty \left\{ n\gamma_n \sin(nlx) - nl\delta_n \cos(nlx)\right\}.$$

Reynolds also compiled integrations by using the above expressions, for example, equation (58) on page 159 of Reynolds's research [19], which is shown below:

$$\iint \rho\overline{u'v'}\frac{d\bar{u}}{dy}dydz$$
$$= \frac{1}{2} \iint \sum \left[nl\left(\alpha_n\frac{d\beta_n}{dy} - \beta_n\frac{d\alpha_n}{dy}\right)\frac{d\bar{u}}{dy}\right]dydz.$$

Giving the above expressions means that Reynolds regarded the velocity fluctuation $\boldsymbol{u}'$ not only as independent unknowns but also deterministic quantity (not a random one !). Please see Reynolds research paper [19] for the meaning of the notations in the above expressions..

Due to limited references, the author does not know who was the first person who proposed the 10-unknowns perception in the RANS. This mistake in classical physics has hampered the development of turbulence.

The RANS in equation 5 and 6 have four independent equations that governs the mean velocity field, namely the three components of the Reynolds equation; equation 5 together with the mean continuity equation 6. As

a matter of fact, these four equations contain more than four independent unknowns. In addition to $\bar{\boldsymbol{u}}$ and $\bar{p}$, there are also the Reynolds stresses $\boldsymbol{\tau}$, which results in the Reynolds-averaged Navier-Stokes equations being unclosed. The closure problem has been considered as the number one topic in turbulence during the past several decades, while scientists and engineers have made many attempts to solve this closure problem, however, no universal modelling has been proposed.

Regarding the RANS closure problem, the current consensus is that there are six unknown components in the symmetric Reynolds stress tensor $\boldsymbol{\tau}$, namely $\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}$. However, this research study presents a complete different perspectives. The Reynolds stress tensor $\boldsymbol{\tau}$ only has three independent unknowns, which are fully determined by the velocity fluctuation components $u_i'\,(i = 1, 2, 3)$ owing to the fact that the Reynolds stress tensor is simply an integration of a second order dyadic tensor of flow velocity fluctuations rather than a general symmetric tensor.

IV. LEMMA

Following lemmas support the above statement.

Lemma 1 *Given two vectors,* $\boldsymbol{v}(\boldsymbol{x}, t) = v_i\boldsymbol{e}_i = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$ *and* $\boldsymbol{w}(\boldsymbol{x}, t) = w_j\boldsymbol{e}_j = w_1\boldsymbol{e}_1 + w_2\boldsymbol{e}_2 + w_3\boldsymbol{e}_3$, *we can define a dyadic tensor* $\boldsymbol{v} \otimes \boldsymbol{w}$ *as follows [66]*

$$
\begin{aligned}
\boldsymbol{v} \otimes \boldsymbol{w} &= v_i\boldsymbol{e}_i \otimes w_j\boldsymbol{e}_j = v_iw_j\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= v_1w_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + v_1w_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + v_1w_3\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\
&+ v_2w_1\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + v_2w_2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + v_2w_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\
&+ v_3w_1\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + v_3w_2\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + v_3w_3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3.
\end{aligned} \quad (12)
$$

In general, in the case of $\boldsymbol{v} \neq \boldsymbol{w}$, *the dyadic tensor* $\boldsymbol{v} \otimes \boldsymbol{w}$ *is a general tensor with 9 components (elements) and has 6 independent unknowns, namely* v_1, v_2 *and* v_3 *and* w_1, w_2 *and* w_3, *since the 9 components can be fully determined by the 6 independent unknowns.*

If $\boldsymbol{v} = \boldsymbol{w}$, *the tensor* $\boldsymbol{v} \otimes \boldsymbol{w}$ *is a 2rd order tensor with 9 components (elements), and have 3 independent unknowns, namely* v_1, v_2 *and* v_3 *and/or* w_1, w_2 *and* w_3, *since the 9 components can be fully determined by the 3 independent unknowns.*

Lemma 2 *Given three vectors,* $\boldsymbol{u}(\boldsymbol{x}, t) = u_i\boldsymbol{e}_i = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$, $\boldsymbol{v}(\boldsymbol{x}, t) = v_i\boldsymbol{e}_i = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$ *and* $\boldsymbol{w}(\boldsymbol{x}, t) = w_j\boldsymbol{e}_j = w_1\boldsymbol{e}_1 + w_2\boldsymbol{e}_2 + w_3\boldsymbol{e}_3$, *we can define a 3rd order tensor* $\boldsymbol{v} \otimes \boldsymbol{w}$ *as follows*

$$
\begin{aligned}
\boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{w} &= u_i\boldsymbol{e}_k \otimes v_k\boldsymbol{e}_i \otimes w_j\boldsymbol{e}_j \\
&= u_kv_iw_j\boldsymbol{e}_k \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j.
\end{aligned} \quad (13)
$$

In general, in the case of $\boldsymbol{u} \neq \boldsymbol{v} \neq \boldsymbol{w}$, *the tensor* $\boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{w}$ *is a general tensor with 27 components (elements) and has 9 unknowns, namely* u_1, u_2 *and* u_3, v_1, v_2 *and* v_3 *and* w_1, w_2 *and* w_3, *since the 27 components can be fully determined by the 9 unknowns.*

If $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$, *the tensor* $\boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{w}$ *is a 3rd order tensor with 27 components (elements) and has 3 unknowns, namely* u_1, u_2 *and* u_3, *or* v_1, v_2 *and* v_3 *and/or* w_1, w_2 *and* w_3, *since the 27 components can be fully determined by the 3 unknowns.*

Lemma 3 *Given a vector,* $\boldsymbol{v}(\boldsymbol{x}, t) = v_i\boldsymbol{e}_i = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$, *we can define a 2nd order symmetric dyadic tensor* $\boldsymbol{v} \otimes \boldsymbol{v}$ *and its mean value* $\boldsymbol{A}(\boldsymbol{x})$ *as follows*

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \boldsymbol{v} \otimes \boldsymbol{v}\, dt \\
&= \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} v_iv_j\boldsymbol{e}_i \otimes \boldsymbol{e}_j\, dt \\
&= \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} [v_1v_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + v_1v_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \\
&+ v_1v_3\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + v_2v_1\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + v_2v_2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\
&+ v_2v_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + v_3v_1\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 \\
&+ v_3v_2\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + v_3v_3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3]\, dt,
\end{aligned} \quad (14)
$$

where $v_iv_j = v_jv_i$.

Although $\boldsymbol{A}(\boldsymbol{x})$ *has six independent components, namely* $v_1v_1, v_1v_2, v_1v_3, v_2v_2, v_2v_3, v_3v_3$, *it is clear that there are only three independent quantities, namely* v_1, v_2 *and* v_3, *in the* $\boldsymbol{A}(\boldsymbol{x})$. *It is because the quantities* $v_1v_1, v_1v_2, v_1v_3, v_2v_2, v_2v_3, v_3v_3$ *can be fully determined by* v_1, v_2 *and* v_3.

Lemma 3 actually states that any (time) averaging operation is merely a method of data processing and will not change the number of unknowns within the problem.

It is clear that $A(\boldsymbol{x})$ will be the Reynolds stress tensor $\boldsymbol{\tau}$ if replacing v_1, v_2 and v_3 by the components of velocity fluctuations $u_i'\,(i = 1, 2, 3)$, respectively, namely $v_1 = u_1', v_2 = u_2', v_3 = u_3'$.

V. TWO PROOFS

The number of unknowns in the Reynolds stress tensor and/or in the RANS should not been a issue at all, since you can simply find it from Reynolds' papers [19]. However, due to the long-time misconception in turbulence research community, it has affected people's mind. To make it clear, it is necessary to explain the issue from the following aspects.

A. Direct proof by the definition of Reynolds stress tensor

The Lemma shows that the Reynolds stress tensor has three independent unknown components. This is proved when the Reynolds stress tensor is defined in the follow-

ing

$$
\begin{aligned}
\boldsymbol{\tau}(\boldsymbol{x}) &= -\rho \overline{\boldsymbol{u}' \otimes \boldsymbol{u}'} = -\rho \overline{u_i' \boldsymbol{e}_i \otimes u_j' \boldsymbol{e}_j} \\
&= -\rho \overline{u_i' u_j'} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= -\rho \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (u_i' u_j' \boldsymbol{e}_i \otimes \boldsymbol{e}_j) dt \\
&= \left[-\rho \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u_i' u_j' dt \right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= \tau_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j,
\end{aligned} \tag{15}
$$

and the fluctuation velocity convective terms are:

$$
\begin{aligned}
\overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}'} &= \overline{u_i' \boldsymbol{e}_i \cdot [\boldsymbol{e}_k \partial_k \otimes (u_j' \boldsymbol{e}_j)]} \\
&= \overline{u_i' u_{j,k}'} \boldsymbol{e}_i \cdot (\boldsymbol{e}_k \otimes \boldsymbol{e}_j) = \overline{u_i' u_{j,k}'} (\boldsymbol{e}_i \cdot \boldsymbol{e}_k) \boldsymbol{e}_j \\
&= \overline{u_i' u_{j,k}'} \delta_{ik} \boldsymbol{e}_j = \overline{u_i' u_{j,i}'} \boldsymbol{e}_j \\
&= \left(\lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} u_i' u_{j,i}' dt \right) \boldsymbol{e}_j \\
&= \tau_{ij,i} \boldsymbol{e}_j = \boldsymbol{\nabla} \cdot \tau,
\end{aligned} \tag{16}
$$

where the Reynolds stress tensor in index form τ_{ij} is defined by the following

$$
\tau_{ij} = \tau_{ji} = -\rho \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u_i' u_j' dt. \tag{17}
$$

The above indicates that any Reynolds stress tensor component τ_{ij} can be calculated by fluctuation velocity components u_1', u_2' and u_3', which means that the τ_{ij} is dependent on components u_1', u_2' and u_3'. In other words, components u_1', u_2' and u_3' are real independent unknowns. This is the main reason why stating that that the Reynolds stress tensor $\boldsymbol{\tau}$ has 6 independent unknowns.

Therefore, the formulation in equation 15 reveals that the Reynolds stress tensor $\boldsymbol{\tau} = -\rho \overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}$ can be fully calculated by three independent components of fluctuation velocity, namely u_1', u_2' and u_3'. In other words, the Reynolds stress tensor only has three unknowns rather than six. It means that the averaging technique is merely a mathematical process, which can provide a mean value, but cannot change the number of unknowns within the problem.

The misinterpretation of the number of independent unknown components in the literature may stem from considering the Reynolds stress tensor as a general 2nd order symmetric tensor with six independent components. However, the Reynolds stress tensor is not an arbitrary 2nd order tensor. In fact, its components are made by the bi-product of the fluctuation velocity components, which means that the Reynolds stress tensor is a dyadic tensor of the velocity fluctuation. The unknown components to construct the dyadic tensor are the three components of fluctuation velocity $\boldsymbol{u}'$. Therefore, the Reynolds stress tensor only has three independent unknowns, namely u_1', u_2' and u_3'. For two dimensional flow, of course, the 2D Reynolds stress tensor only has two independent unknowns, namely u_1', u_2'.

B. Proof of a particular case

One can use two scalar functions, $u' = \sqrt{2}U \cos(\omega t)$ and $v' = \sqrt{2}V \cos(\omega t + \theta)$, as independent unknowns, to construct a 2nd order tensor or matrix with 4 components, as follows: $(u')^2$, $u'v'$, $v'u'$, $(v')^2$. Thus, the time averages are:

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (u'u') dt = U^2, \tag{18}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (u'v') dt = UV \cos\theta, \tag{19}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (v'u') d = UV \cos\theta, \tag{20}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (v'v') dt = V^2. \tag{21}
$$

In the definition of the Reynolds stress tensor, $\theta = 0$, we have $u' = \sqrt{2}U \cos(\omega t)$ and $v' = \sqrt{2}V \cos(\omega t)$ and, the averaging are as follows

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (u'u') dt = U^2, \tag{22}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (u'v') dt = UV, \tag{23}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (v'u') dt = UV, \tag{24}
$$

$$
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (v'v') dt = V^2. \tag{25}
$$

Therefore, the Reynolds stress tensor is

$$
\begin{aligned}
\boldsymbol{\tau} = -\rho(U^2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + UV \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \\
+ VU \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + V^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2).
\end{aligned} \tag{26}
$$

The above process shows that, u' or U, and v' or V, are the independent unknowns, which culminate in the Reynolds stress tensor $\boldsymbol{\tau}$.

If the θ remains, then we can refer to the time correlation (autocorrelations) at the same point. The correlation between the same (Greek autos = self or same) fluctuating quantity measured at two different times (at the same point in space) is not relevant to the behaviour of turbulence, while its measurement requires a time delay mechanism (usually a tape recorder with movable heads or a digital sample-and-delay system).

In the same way, one can define the space correlation; however, all the literature, including textbooks, mention that the Reynolds stress tensor is the one-point or single-point velocity fluctuation correlation. Therefore, $\theta = 0$.

Even without $\theta = 0$, the Reynolds stress still has three independent unknowns, namely the three independent components of the velocity fluctuations.

It is proven once again that both the tensor $\boldsymbol{\sigma}$ and the Reynolds stress tensor $\boldsymbol{\tau}$ are the only function of

u'_1, u'_2, u'_3. Summarily, the number of independent unknowns in the Reynolds stress tensor is shown in Table I below:

TABLE I: Number of unknowns in the Reynolds stress tensor

	Current literature	This paper
Number	6	3
Unknowns	$\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}$	u'_1, u'_2 and u'_3

Although the Reynolds-averaged Navier-Stokes (RANS) equations are unclosed, the four-equations RANS in Eqs.5 and 6 contain only 7 independent unknowns instead of 10 as stated in current literatures. The list of unknowns in the RANS are summarized in the below Table II:

TABLE II: Number of independent unknowns in the RANS

	Current literature	This paper
Number	10	7
Unknowns	$\bar{u}_1, \bar{u}_2, \bar{u}_3$	$\bar{u}_1, \bar{u}_2, \bar{u}_3$
	$\bar{p}$	$\bar{p}$
	$\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}$	u'_1, u'_2 and u'_3

VI. TRANSPORT EQUATION OF REYNOLDS STRESS TENSOR

With the above new understanding of the number of independent unknowns, he following presents completely new perceptions on the higher-order correction of the Reynolds' stress tensor.

Certain tensor algebra from Eqs.Eqs.10 and 11 resulted in an important equation for the Reynolds' stress tensor, as follows

$$\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \bar{\boldsymbol{u}} \cdot \boldsymbol{\tau}$$
$$= 2\mu \boldsymbol{I} : \overline{\boldsymbol{\nabla} \boldsymbol{u}' \otimes \boldsymbol{\nabla} \boldsymbol{u}'}$$
$$+ \overline{\boldsymbol{u}' \otimes (\boldsymbol{\nabla} p')} + \overline{(\boldsymbol{\nabla} p') \otimes \boldsymbol{u}'}$$
$$+ \mu \boldsymbol{\nabla}^2 \boldsymbol{\tau} + \rho \boldsymbol{\nabla} \cdot \overline{(\boldsymbol{u}' \otimes \boldsymbol{u}' \otimes \boldsymbol{u}')}. \tag{27}$$

Current literature claims that Eq.27 has 31 independent unknowns. However, this research has a different opinion, and has proven that the Reynolds stress equation Eq. 27 only has 7 independent unknowns. Accoutring for all symmetries, these are listed in Table III below.

The above statement is proven by the following:

$$\boldsymbol{u}' \otimes \boldsymbol{u}' \otimes \boldsymbol{u}' = \overline{u'_i u'_j u'_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$
$$= \left(\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u'_i u'_j u'_k dt \right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k, \tag{28}$$

TABLE III: Number of unknowns in the Reynolds stress equation

	Current literature	This paper
Number	31	7
Unknowns	$\bar{\boldsymbol{u}}$ (3)	$\bar{u}_1, \bar{u}_2, \bar{u}_3$
	$\boldsymbol{u}' \otimes \boldsymbol{u}' \otimes \boldsymbol{u}'$ (10)	p'
	$\boldsymbol{I} : \overline{\boldsymbol{\nabla} \boldsymbol{u}' \otimes \boldsymbol{\nabla} \boldsymbol{u}'}$ (6)	u'_1, u'_2 and u'_3
	$\overline{\boldsymbol{u}' \otimes (\boldsymbol{\nabla} p')}$ (6)	
	$\boldsymbol{\tau}$ (6)	

and

$$\boldsymbol{I} : \overline{\boldsymbol{\nabla} \boldsymbol{u}' \otimes \boldsymbol{\nabla} \boldsymbol{u}'} = (\boldsymbol{e}_k \otimes \boldsymbol{e}_k) : (\overline{\boldsymbol{\nabla} \boldsymbol{u}' \otimes \boldsymbol{\nabla} \boldsymbol{u}'})$$
$$= \overline{(\boldsymbol{e}_k \otimes \boldsymbol{e}_k) : (\boldsymbol{\nabla} \boldsymbol{u}' \otimes \boldsymbol{\nabla} \boldsymbol{u}')}$$
$$= \overline{(\boldsymbol{e}_k \cdot \boldsymbol{\nabla} \boldsymbol{u}') \otimes (\boldsymbol{e}_k \cdot \boldsymbol{\nabla} \boldsymbol{u}')}$$
$$= \overline{(\boldsymbol{\nabla}_k \boldsymbol{u}') \otimes (\boldsymbol{\nabla}_k \boldsymbol{u}')} = \overline{(\boldsymbol{\nabla}_k \tilde{u}_i \boldsymbol{e}_i) \otimes (\boldsymbol{\nabla}_k \tilde{u}_j \boldsymbol{e}_j)}$$
$$= \overline{(\boldsymbol{\nabla}_k \tilde{u}_i)(\boldsymbol{\nabla}_k \tilde{u}_j)} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$
$$= \left(\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} dt \right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{29}$$

It is clear that the mean value of $\overline{u'_i u'_j u'_k}$ and $\overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}$ can be calculated by the velocity fluctuations u'_1, u'_2 and u'_3. It means that u'_1, u'_2 and u'_3 are unknowns. Similarly,

$$\overline{\boldsymbol{u}' \otimes (\boldsymbol{\nabla} p')} = \overline{u'_i \frac{\partial p'}{\partial x_j}} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$
$$= \left(\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u'_i \frac{\partial p'}{\partial x_j} dt \right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{30}$$

The mean value of $\overline{u'_i \frac{\partial p'}{\partial x_j}}$ can be calculated by p' and u'_1, u'_2 and u'_3.

The same can also be done for the fourth order and for higher orders as in [32] and in textbooks. However, no other unknowns can be created by any order equation in respect of the Reynolds' stress tensor.

VII. TRANSPORT EQUATION OF TURBULENCE KINETIC ENERGY

The contraction operation for index i and j in Eq.27 resulted in the following transport equation for turbulence kinetic energy k

$$\rho \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} k = \boldsymbol{\tau} : \boldsymbol{\nabla} \bar{\boldsymbol{u}}$$
$$- \mu \overline{\boldsymbol{\nabla} \boldsymbol{u}' : \boldsymbol{\nabla} \boldsymbol{u}'} + \mu \boldsymbol{\nabla}^2 k$$
$$- \boldsymbol{\nabla} \cdot (\overline{p' \boldsymbol{u}'}) - \frac{1}{2} \rho \boldsymbol{\nabla} \cdot \overline{[(\boldsymbol{u}' \cdot \boldsymbol{u}') \boldsymbol{u}']}, \tag{31}$$

where the kinetic energy $k(\boldsymbol{x}) = -\frac{1}{2} \tau_{kk} = \frac{1}{2} \overline{u'_k u'_k} = \frac{1}{2} \overline{\boldsymbol{u}' \cdot \boldsymbol{u}'} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \boldsymbol{u}' \cdot \boldsymbol{u}' dt$. The number of independent unknowns in the kinetic energy equation Eq.31 is listed in Table IV below.

TABLE IV: Number of unknowns in the kinetic energy equation

	Current literature	This paper
Number	21	7
Unknowns	k (3)	u_1', u_2' and u_3'
	$\bar{\boldsymbol{u}}$ (3)	$\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$
	$\boldsymbol{\tau}$ (6)	p'
	$\overline{\boldsymbol{\nabla u'} : \boldsymbol{\nabla u'}}$ (3)	
	$\overline{p'\boldsymbol{u'}}$ (3)	
	$\overline{[(\boldsymbol{u'} \cdot \boldsymbol{u'})\boldsymbol{u'}]}$ (3)	

VIII.   CONCLUSIONS

This article re-visited a fundamental problem in turbulence analysis, namely the number of independent unknowns in the Reynolds' stress tensor. The research study found that there are 3 independent unknowns in the Reynolds' stress tensor, namely 3 components of velocity fluctuations u_1', u_2' and u_3'. This study has not only clarified the number of independent unknowns in the formulations of the Reynolds-averaged Navier-Stokes equation, but has also discovered that the number of independent unknowns is much less than traditionally found and thought.

In future, turbulence modelling should focus on the modelling of velocity fluctuations u_1', u_2', u_3', instead of on the Reynolds stress τ_{ij}. An advantage of modelling the velocity fluctuations u_1', u_2', u_3' is that it reduces the six components of τ_{ij} into three ones, and from an experimental perspective, the components of velocity fluctuations are easiest to measure than the Reynolds' stress tensor.

It is important to point out that this study's ideas and methodologies are also applicable to compressible turbulence Navier-Stokes equations, where the mass density ρ and the temperature T must be taken into account, and their Reynolds decompositions, $\rho = \bar{\rho} + \tilde{\rho}$ and $T = \bar{T} + \hat{T}$, should be introduced.

The present investigation can be considered as a renaissance of Reynolds' study in 1895, which might assist with understanding the well-known closure problem of turbulence, and the puzzle of the turbulence closure problem that has eluded scientists and mathematicians for centuries [63].

IX.   ACKNOWLEDGEMENT

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