Logarithmic Space Verifiers on NP-complete

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Abstract

P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? A precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. NP is the complexity class of languages defined by polynomial time verifiers M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in P. Another major complexity classes are L and NL. The certificate-based definition of NL is based on logarithmic space Turing machine with an additional special read-once input tape: This is called a logarithmic space verifier. NL is the complexity class of languages defined by logarithmic space verifiers M such that when the input is an element of the language with its certificate, then M outputs 1. To attack the P versus NP problem, the NP-completeness is a useful concept. We demonstrate there is an NP-complete language defined by a logarithmic space verifier M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in L.

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1 Introduction

The P versus NP problem is a major unsolved problem in computer science [4]. This is considered by many to be the most important open problem in the field [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US$1,000,000 prize for the first correct solution [4]. It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency [1]. However, the precise statement of the $P = NP$ problem was introduced in 1971 by Stephen Cook in a seminal paper [4]. In 2012, a poll of 151 researchers showed that 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore impossible to prove or disprove, 8 (5%) said either do not know or do not care or don’t want the answer to be yes nor the problem to be resolved [7]. It is fully expected that $P \neq NP$ [11]. Indeed, if $P = NP$ then there are stunning practical consequences [11]. For that reason, $P = NP$ is considered as a very unlikely event [11]. Certainly, $P$ versus $NP$ is one of the greatest open problems in science and a correct solution for this incognita will have a great impact not only in computer science, but for many other fields as well [1]. Whether $P = NP$ or not is still a controversial and unsolved problem [1]. In this work, we show some results that might be helpful in facing one of the most important open problems in computer science.

2 Preliminaries

In 1936, Turing developed his theoretical computational model [13]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions
2 Logarithmic Space Verifiers on NP-complete

related to this theoretical model for computation [13]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [13]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [13].

Let $\Sigma$ be a finite alphabet with at least two elements, and let $\Sigma^*$ be the set of finite strings over $\Sigma$ [3]. A Turing machine $M$ has an associated input alphabet $\Sigma$ [3]. For each string $w$ in $\Sigma^*$ there is a computation associated with $M$ on input $w$ [3]. We say that $M$ accepts $w$ if this computation terminates in the accepting state, that is $M(w) = 1$ (when $M$ outputs 1 on the input $w$) [3]. Note that $M$ fails to accept $w$ either if this computation ends in the rejecting state, that is $M(w) = 0$, or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when $M$ outputs the string $y$ on the input $w$) [3].

Another relevant advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet $\Sigma$. We denote by $\text{L}(\Sigma)$ the set of all strings over $\Sigma$ [3].

A complexity class is a set of problems, which are represented as a language, grouped by a complexity measure such as the running time, memory, etc [5]. The language accepted by a Turing machine $M$, denoted $L(M)$, has an associated alphabet $\Sigma$ and is defined by:

$$L(M) = \{ w \in \Sigma^n : M(w) = 1 \}.$$

We denote by $t_M(w)$ the number of steps in the computation of $M$ on input $w$ [3]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of $M$; that is:

$$T_M(n) = \max\{ t_M(w) : w \in \Sigma^n \}$$

where $\Sigma^n$ is the set of all strings over $\Sigma$ of length $n$ [3]. We say that $M$ runs in polynomial time if there is a constant $k$ such that for all $n$, $T_M(n) \leq n^k + k$ [3]. In other words, this means the language $L(M)$ can be decided by the Turing machine $M$ in polynomial time. Therefore, $P$ is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [5]. A verifier for a language $L_1$ is a deterministic Turing machine $M$, where:

$$L_1 = \{ w : M(w,c) = 1 \text{ for some string } c \}.$$

We measure the time of a verifier only in terms of the length of $w$, so a polynomial time verifier runs in polynomial time in the length of $w$ [3]. A verifier uses additional information, represented by the symbol $c$, to verify that a string $w$ is a member of $L_1$. This information is called certificate. $NP$ is the complexity class of languages defined by polynomial time verifiers [11].

**Lemma 1.** Given a language $L_1 \in P$, a language $L_2$ is in $NP$ if there is a deterministic Turing machine $M$, where:

$$L_2 = \{ w : M(w,c) = y \text{ for some string } c \text{ such that } y \in L_1 \}$$

and $M$ runs in polynomial time in the length of $w$. In this way, $NP$ is the complexity class of languages defined by polynomial time verifiers $M$ such that when the input is an element of the language with its certificate, then $M$ outputs a string which belongs to a single language in $P$.

**Proof.** If $L_1$ can be decided by the Turing machine $M'$ in polynomial time, then the deterministic Turing machine $M''(w,c) = M'(M(w,c))$ will output 1 when $w \in L_2$. Consequently, $M''$ is a polynomial time verifier of $L_2$ and thus, $L_2$ is in $NP$. □
3 Hypothesis

A function \( f : \Sigma^* \rightarrow \Sigma^* \) is a polynomial time computable function if some deterministic Turing machine \( M \), on every input \( w \), halts in polynomial time with just \( f(w) \) on its tape [13]. Let \( \{0,1\}^* \) be the infinite set of binary strings, we say that a language \( L_1 \subseteq \{0,1\}^* \) is polynomial time reducible to a language \( L_2 \subseteq \{0,1\}^* \), written \( L_1 \leq_p L_2 \), if there is a polynomial time computable function \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) such that for all \( x \in \{0,1\}^* \):

\[
x \in L_1 \text{ if and only if } f(x) \in L_2.
\]

An important complexity class is \( NP\text{-complete} \) [8]. A language \( L_1 \subseteq \{0,1\}^* \) is \( NP\text{-complete} \) if:

1. \( L_1 \in NP \), and
2. \( L' \leq_p L_1 \) for every \( L' \in NP \).

If \( L_1 \) is a language such that \( L' \leq_p L_1 \) for some \( L' \in NP\text{-complete} \), then \( L_1 \) is \( NP\text{-hard} \) [5]. Moreover, if \( L_1 \in NP \), then \( L_1 \in NP\text{-complete} \) [5]. A principal \( NP\text{-complete} \) problem is \( SAT \) [6]. An instance of \( SAT \) is a Boolean formula \( \phi \) which is composed of:

1. Boolean variables: \( x_1, x_2, \ldots, x_n \);
2. Boolean connectives: Any Boolean function with one or two inputs and one output, such as \( \land \) (AND), \( \lor \) (OR), \( \neg \) (NOT), \( \Rightarrow \) (implication), \( \Leftrightarrow \) (if and only if);
3. and parentheses.

A truth assignment for a Boolean formula \( \phi \) is a set of values for the variables in \( \phi \). A satisfying truth assignment is a truth assignment that causes \( \phi \) to be evaluated as true. A formula with a satisfying truth assignment is a satisfiable formula. The problem \( SAT \) asks whether a given Boolean formula is satisfiable [6]. We define a \( CNF \) Boolean formula using the following terms:

A literal in a Boolean formula is an occurrence of a variable or its negation [5]. A Boolean formula is in conjunctive normal form, or \( CNF \), if it is expressed as an AND of clauses, each of which is the OR of one or more literals [5]. A Boolean formula is in 3-conjunctive normal form or \( 3CNF \), if each clause has exactly three distinct literals [5].

For example, the Boolean formula:

\[
(x_1 \lor \neg x_1 \lor x_2) \land (x_3 \lor x_2 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor \neg x_4)
\]

is in \( 3CNF \). The first of its three clauses is \((x_1 \lor \neg x_1 \lor x_2)\), which contains the three literals \( x_1, \neg x_1 \), and \( x_2 \). Another relevant \( NP\text{-complete} \) language is \( 3CNF \) satisfiability, or \( 3SAT \) [5]. In \( 3SAT \), it is asked whether a given Boolean formula \( \phi \) in \( 3CNF \) is satisfiable.

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [13]. The work tapes may contain at most \( O(\log n) \) symbols [13]. In computational complexity theory, \( L \) is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [11]. \( NL \) is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [11].

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and read/write work tapes [13]. The work tapes must contain at most \( O(\log n) \) symbols [13]. A logarithmic space transducer \( M \) computes a function \( f : \Sigma^* \rightarrow \Sigma^* \), where \( f(w) \) is the string remaining on the output tape after \( M \) halts when it is started with...
4 Logarithmic Space Verifiers on NP-complete

We call \( f \) a logarithmic space computable function [13]. We say that a language \( L_1 \subseteq \{0,1\}^* \) is logarithmic space reducible to a language \( L_2 \subseteq \{0,1\}^* \), written \( L_1 \leq_l L_2 \), if there exists a logarithmic space computable function \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) such that for all \( x \in \{0,1\}^* \),

\[
x \in L_1 \text{ if and only if } f(x) \in L_2.
\]

The logarithmic space reduction is frequently used for \( L \) and \( NL \) [11]. A Boolean formula is in 2-conjunctive normal form, or 2CNF, if it is in CNF and each clause has exactly two distinct literals. There is a problem called 2SAT, where we asked whether a given Boolean formula \( \phi \) in 2CNF is satisfiable. 2SAT is complete for \( NL \) [11]. Another special case is the class of problems where each clause contains XOR (i.e. exclusive or) rather than (plain) OR operators. This is in \( P \), since an XOR SAT formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [9]. We denote the XOR function as \( \oplus \). The XOR 2SAT problem will be equivalent to XOR SAT, but the clauses in the formula have exactly two distinct literals. XOR 2SAT is in \( L \) [2], [12].

We can give a certificate-based definition for \( NL \) [3]. The certificate-based definition of \( NL \) assumes that a logarithmic space Turing machine has another separated read-only tape [3]. On each step of the machine the machine’s head on that tape can either stay in place or move to the right [3]. In particular, it cannot reread any bit to the left of where the head currently is [3]. For that reason this kind of special tape is called “read once’ [3].

Definition 2. A language \( L_1 \) is in \( NL \) if there exists a deterministic logarithmic space Turing machine \( M \) with an additional special read-once input tape polynomial \( p : \mathbb{N} \rightarrow \mathbb{N} \) such that for every \( x \in \{0,1\}^* \),

\[
x \in L_1 \iff \exists u \in \{0,1\}^{p(|x|)} \text{ such that } M(x,u) = 1
\]

where by \( M(x,u) \) we denote the computation of \( M \) where \( x \) is placed on its input tape and \( u \) is placed on its special read-once tape, and \( M \) uses at most \( O(\log |x|) \) space on its read/write tapes for every input \( x \) where \( |\ldots| \) is the bit-length function [3]. \( M \) is called a logarithmic space verifier [3].

We state the following Hypothesis:

Hypothesis 3. Given a language \( L_1 \in L \), there is a language \( L_2 \) in \( NP-complete \) with a deterministic Turing machine \( M \), where:

\[
L_2 = \{w : M(w,u) = y \text{ for some string } u \text{ such that } y \in L_1\}
\]

when \( M \) runs in logarithmic space in the length of \( w \), \( u \) is placed on the special read-once tape of \( M \), and \( u \) is polynomially bounded by \( w \). In this way, there is an \( NP-complete \) language defined by a logarithmic space verifier \( M \) such that when the input is an element of the language with its certificate, then \( M \) outputs a string which belongs to a single language in \( L \).

4 Results

We show a previous known \( NP-complete \) problem:

Definition 4. MONOTONE NAE 3SAT
INSTANCE: A Boolean formula $\phi$ in 3CNF such that each clause has no negation variables.

QUESTION: Is there a truth assignment for $\phi$ such that each clause has at least one true literal and at least one false literal?

REMARKS: This is equivalent to the special case of the NP-complete problem known as SET SPLITTING when the sets in the input have exactly three elements and therefore, MONOTONE NAE 3SAT $\in$ NP-complete [6].

We define a new problem:

- **Definition 5. MINIMUM EXCLUSIVE-OR 2-SATISFIABILITY**
  INSTANCE: A positive integer $K$ and a Boolean formula $\phi$ that is an instance of XOR 2SAT such that each clause has no negation variables.
  QUESTION: Is there a truth assignment in $\phi$ such that at most $K$ clauses are unsatisfiable?
  REMARKS: We denote this problem as $MIN \oplus 2SAT$.

- **Theorem 6. $MIN \oplus 2SAT \in NP$-complete.**

  **Proof.** It is trivial to see $MIN \oplus 2SAT \in NP$ [11]. Given a Boolean formula $\phi$ in 3CNF with $n$ variables and $m$ clauses such that each clause has no negation variables, we create three new variables $a_{c_i}$, $b_{c_i}$ and $d_{c_i}$ for each clause $c_i = (x \lor y \lor z)$ in $\phi$, where $x$, $y$ and $z$ are positive literals, in the following formula:

  $$P_i = (a_{c_i} \oplus b_{c_i}) \land (b_{c_i} \oplus d_{c_i}) \land (a_{c_i} \oplus d_{c_i}) \land (x \oplus a_{c_i}) \land (y \oplus b_{c_i}) \land (z \oplus d_{c_i}).$$

  We can see $P_i$ has at most one unsatisfiable clause for some truth assignment if and only if at least one member of $\{x, y, z\}$ is true and at least one member of $\{x, y, z\}$ is false for the same truth assignment. Hence, we can create the Boolean formula $\psi$ as the conjunction of the $P_i$ formulas for every clause $c_i$ in $\phi$, such that $\psi = P_1 \land \ldots \land P_m$. Finally, we obtain that $\phi \in MONOTONE NAE 3SAT$ if and only if $(\psi, m) \in MIN \oplus 2SAT$.

  Consequently, we prove $MONOTONE NAE 3SAT \leq_p MIN \oplus 2SAT$ where we already know the language $MONOTONE NAE 3SAT \in NP$-complete [6]. To sum up, we show $MIN \oplus 2SAT \in NP$-hard and $MIN \oplus 2SAT \in NP$ and thus, $MIN \oplus 2SAT \in NP$-complete. ▶

- **Theorem 7. There is a deterministic Turing machine $M$, where:**

  $$MIN \oplus 2SAT = \{w : M(w, u) = y \text{ for some string } u \text{ such that } y \in XOR 2SAT\}$$

  when $M$ runs in logarithmic space in the length of $w$, $u$ is placed on the special read-once tape of $M$, and $u$ is polynomially bounded by $w$.

  **Proof.** Given a valid instance $(\psi, K)$ for $MIN \oplus 2SAT$ when $\psi$ has $m$ clauses, we can create a certificate array $A$ which contains $K$ different natural numbers in ascending order which represents the indexes of the clauses in $\psi$ that we are going to remove from the instance. We read at once the elements of the array $A$ and we reject whether this is not a valid certificate: That is when the numbers are not sorted in ascending order, or the array $A$ does not contain exactly $K$ elements, or the array $A$ contains a number that is not between 1 and $m$. While we read the elements of the array $A$, we remove the clauses from the instance $(\psi, K)$ for $MIN \oplus 2SAT$ just creating another instance $\phi$ for XOR 2SAT where the Boolean formula
6 Logarithmic Space Verifiers on NP-complete

φ does not contain the K different indexed clauses ψ represented by the numbers in A. Therefore, we obtain that:

\[(\psi, K) \in MIN \oplus 2SAT \text{ if and only if } \phi \in XOR 2SAT.\]

Furthermore, we can make this verification in logarithmic space such that the array A is placed on the special read-once tape, because we read at once the elements in the array A and we assume the clauses in the input ψ are indexed from left to right. Hence, we only need to iterate from the elements of the array A to verify whether the array is a valid certificate and also remove the K different clauses from the Boolean formula ψ when we write the final clauses to the output. This logarithmic space verification will be the Algorithm 1. We assume whether a value does not exist in the array A into the cell of some position i when \(A[i] = undefined\). In addition, we reject immediately when the following comparisons hold at least into one single binary digit. Note, in the loop \(j\) from \(min\) to \(max - 1\), we do not output any clause when \(max - 1 < min\).

**Algorithm 1 Logarithmic space verifier**

1: /*A valid instance for \(MIN \oplus 2SAT\) with its certificate*/
2: procedure VERIFIER((ψ, K), A)
3: /*Initialize minimum and maximum values*/
4: \(min \leftarrow 1\)
5: \(max \leftarrow 0\)
6: /*Iterate for the elements of the certificate array A*/
7: for \(i \leftarrow 1\) to \(K + 1\) do
8:     if \(i = K + 1\) then
9:         /*There exists a \(K + 1\) element in the array*/
10:            if \(A[i] \neq undefined\) then
11:                /*Reject the certificate*/
12:                    return 0
13:            end if
14:            /*m is the number of clauses in ψ*/
15:            \(max \leftarrow m + 1\)
16:        else if \(A[i] = undefined \lor A[i] \leq max \lor A[i] < 1 \lor A[i] > m\) then
17:            /*Reject the certificate*/
18:                return 0
19:        else
20:            \(max \leftarrow A[i]\)
21:        end if
22:     end if
23:     /*Iterate for the clauses of the Boolean formula ψ*/
24:     for \(j \leftarrow min\) to \(max - 1\) do
25:         /*Output the indexed \(j\) clause in ψ*/
26:             output “\(\land c_j\)”
27:     end for
28:     \(min \leftarrow max + 1\)
29: end procedure
Theorem 8. The Hypothesis 3 is true.

Proof. This is a consequence of Theorems 6 and 7.

5 Materials and Methods

This work is implemented into a GitHub Project programmed in Scala [14]. In this GitHub Project, we use the Assertion on the properties of the instances of each problem and the Unit Test for checking the correctness of every reduction [14]. We need to install JDK 8 in order to test the Scala Project [10]. In addition, we need to install SBT to run the unit test (we could run the unit test with the sbt test command) [10].

References