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Estimating Parameters in Mathematical Model for Societal Booms through Bayesian Inference Approach

by

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Abstract

In this study, based on our previous study, we examined the mathematical properties, especially the stability of the equilibrium for our proposed mathematical model. By means of the results of the stability in this study, we also used actual data representing transient booms and resurgent booms, and conducted parameter estimation for our proposed model using Bayesian inference. In addition, we conducted a model fitting to five actual data. By this study, we reconfirmed that we can express the resurgences or minute vibrations of actual data by means of our proposed model.

Keywords: societal booms model; time delay; Bayesian inference approach

1 Introduction

Booms emerge in many fields and are closely tied to our everyday life. For example, a fashion in clothing, makeup, sports, a movie and food(we call "societal booms"). These examples show that "interesting information" about the individual boom passed at a rapid rate to a large number of people in a short period of time. Furthermore, in a sense, we can regard an infectious disease as a boom, like influenza or SARS(we call "epidemiological booms"), which is infected by viruses that are transmitted from person to person. In the above examples, it is the most important things that both "interesting information" and "viruses" are transmitted by some form of contact. Hence, we considered that a spread with an interest in products, movie, food, etc resemble the transmission dynamics of viruses in ways.

Studies on epidemiological booms that employed mathematical models were founded on differential equations in the late 19th century, and mathematical modeling using differential equations was developed by Kermack and McKendrick in

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[0]**Keywords and Phrases:** Societal booms model; Time delay; Bayesian inference approach.

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the 1920s. This field of study gained attention among researchers owing to the spread of emerging infectious diseases, such as AIDS in the 1980s, that posed risks in developed countries. This research continues to progress ([2], [4], [10]). On the other hand, there are many studies that researched societal booms from a sociological or psychological perspective, but few were conducted from a mathematical point of view ([1], [9], [13], [17], [19]). However, in recent years, companies have concentrated their marketing efforts on the development of hit products that emphasize customer taste and trend analysis by using social networking services (SNS) such as Twitter and Facebook.

In this study, we focus on societal booms and we develop research for it. Ishi et al. [7] have derived a mathematical model for the 'hit' phenomenon in entertainment within a society, which is presented as a stochastic process of human dynamics interactions. Ishi et al. have performed calculations using their proposed equation for many movies in the Japanese market. Moreover, Nakagiri and Kurita [14] conducted one study that focused on societal booms. Nakagiri et al. used a system of simultaneous linear differential equations to develop a mathematical model to describe problems in societal booms, and performed a model fitting to actual data. The mathematical model proposed in this study is simple but extremely versatile. Additionally, Ueda and Asahi [21] expanded on the model developed by Nakagiri et al. to conduct an analysis using actual data by constructing and verifying a model of the changing interests among Twitter users. In [16], we proposed the mathematical boom model developed by Nakagiri et al. in consideration of the SIR model [9], which is a leading idea to describe biological booms such as viral infections, and the Diffusion of Innovation theory [19] proposed by sociologist E.M. Rogers. In this study, we examine the stability of the equilibrium of our proposed model in [16]. Moreover, using actual booms data we evaluate the parameters and examine the fit of our proposed model.

This study is divided into six parts. In Section 2, we explain the ideas at the core of our proposed mathematical model for Societal Booms, which was proposed in our previous study. In Section 3, we investigate the stability of the equilibrium point to our proposed model and derive the sufficient condition for parameters of our proposed model. In Section 4, we explain the Bayesian inference method, which was used to estimate the parameters of our proposed model, and discusses the numerical exploration of the posterior state space by the MCMC method. Moreover, we introduce the coefficient of determination that forms the standard for the fit. In Section 5, we evaluate the parameters of our proposed model and examine fitting our proposed model to actual data, using five actual data for societal booms.

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2 Mathematical Model

In this section, we explain a mathematical model for societal booms which was derived in [16].

2.1 Three key points to derive our proposed model

Here, we explain the three key points which were discussed to derive our mathematical model.

The first point is the contact. Infectious disease epidemics such as influenza are thought to occur when a virus invades and infects a healthy person's body from contact with an infected person. In our proposed model, we define "interesting information" to be a "virus" , which is transmitted from people in an on-boom state to those whom the boom has not reached yet (pre-boom). Thus, our proposed model incorporates the perspective of the contact, which was not considered in [14].

The second point is the time delay. The Diffusion of Innovation theory, developed by E.M. Rogers, separates consumers into five categories based on the speed at which people are likely to adopt innovation (innovators, early adopters, early majority, late majority, and laggards). Based on this theory, we think that time lags exist in the adoption of booms by people in a social system, and thus developed a model that considers the effects of a time delay. Hutchinson [6] suggested the following logistic equation with time delay τ. This equation shows that the solution does not fluctuate monotonously but exhibits complex behavior such as oscillatory behavior depending on the magnitude of the time delay:

$$\frac{dx(t)}{dt} = \alpha x(t)\left(1 - \frac{x(t-\tau)}{K}\right) \qquad (\alpha, K > 0) \tag{2.1}$$

Furthermore, the biologist R.M. May [11] regarded (2.1) as a mathematical model that expresses the temporal changes of the herbivorous animal population $x(t)$, and asserted that the biological definition of a time delay was "the time required for the regeneration of plants that is suitable for animals to eat". Additionally, May received acclaim for fitting results from an experiment on the Australian sheep blowfly (Lucilia cuprina) conducted by Nicholson [15]. Based on these experimental results, we regarded the definition of a time delay for societal booms as "the time required for a boom adopter associated with contact and resurgence to pick up a boom and take action", and incorporated the concept of the time delay into the derivation of a mathematical model.

The last point is the existence of influencer and "Sakura[1]". In the current landscape, companies actively employ influencer and "Sakura" in their marketing

[1] "Sakura" are people who were compelled to boom state.

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strategies. Our proposed model expresses their presence by depicting a resurgence caused by opinion leaders and forced changes in the number of people.

As described above, our proposed model is a natural extension of the boom model developed in [14] and is derived from the above three perspectives. We expect that the model will be able to capture various types of boom data.

2.2 Mathematical model for societal booms

In [16], we proposed a new mathematical model to explain societal booms based on the background described above. Here, according to [16] we derive the mathematical model for societal booms.

First, we assume the state of the boom participants at any given time to be one of the following:

- **State 1【pre-boom】**: Condition where there is a potential to adopt a boom
- **State 2 【on-boom】** : Condition where the boom is captured
- **State 3 【rooted boom】** : Condition where the boom is retained
- **State 4 【unrooted boom】** : Condition where the boom did not take off

Furthermore, at a given time t, we assume the number of boom participants in each state to be $y_1(t), y_2(t), y_3(t)$, and $y_4(t)$ respectively, for States $1-4$. Then we represent the changes of a customer's state by using the following equations:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -\alpha y_1(t) y_2(t-\tau_1) - \delta y_1(t) + \varepsilon y_2(t-\tau_2) - \zeta \\ \dfrac{dy_2(t)}{dt} = \alpha y_1(t) y_2(t-\tau_1) - (\beta+\gamma) y_2(t) + \delta y_1(t) \\ \dfrac{dy_3(t)}{dt} = \beta y_2(t) - \varepsilon y_2(t-\tau_2) + \zeta \\ \dfrac{dy_4(t)}{dt} = \gamma y_2(t) \end{cases} \tag{2.2}$$

Here, the variables $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and τ_1, τ_2 in (2.2) respectively represent the rate of transmission ("infection") of the boom among people in a pre-boom state per unit time, rate of retention among people in an on-boom state, rate of people who quit the boom, adoption rate of the boom by people in a pre-boom state, rate of resurgence from rooted boom to pre-rooted state, and the production rate of people in an rooted boom state. τ_1 and τ_2 are parameters that show the time delay, and $\tau_1 < \tau_2$. Here, $\alpha > 0, \beta + \gamma > 0, \delta > 0$. In particular, $\alpha y_2(t-\tau_1)$ is called as infectivity and is an important indicator that characterizes the boom model.

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From (3.5), (3.6) or (3.7), (3.8), we have

$$\frac{(\beta+\gamma)\delta A^2 + (1-\varepsilon\tau_2)B^2}{AB} > 0.$$

This is contradiction shows that every λ has negative real part. Hence every solution of Esq. (2.2) tends to zero exponentially as $t \to \infty$.

4 Bayesian approach for estimating parameters

When using the mathematical model based on a differential equation for various phenomena, it creates a problem of how to estimate parameters of its model. In particular, when it is difficult to analytically obtain the solution of a differential equation which becomes the maximum likelihood estimated value, we must provide a maximum likelihood estimation method which allows processing for digitally obtaining a maximum likelihood estimated value from a likelihood function. In this section, we describe a Bayesian inference approach for estimating parameters in a mathematical ODE model.

Recently, the Bayesian inference approach has been greatly extended through the development of analytical techniques such as MCMC (see [8]). The underlying concept of Bayesian inference approach is Bayes' theorem, which relates the parameters θ and the observed data Y as follows:

$$f(\theta|Y) = \frac{f(Y|\theta)f(\theta)}{f(Y)}. \tag{4.1}$$

(4.1) states that the posterior probability density function(PPDF) $f(\theta|Y)$ for parameter θ is proportional to the product of the likelihood function $f(Y|\theta)$ and the prior density function $f(\theta)$.

Here, let us define $m-$dimensional vectors $\mathbf{Y}$, $\mathbf{F}^\theta$ and $\mathbf{E}$ as follows :

$$\begin{aligned}
\mathbf{Y} &= (Y_1, Y_2, \cdots, Y_m) = (Y(t_1), Y(t_2), \cdots, Y(t_m)) \\
\mathbf{F}^\theta &= (F_1^\theta, F_2^\theta, \cdots, F_m^\theta) \\
&= (y_2^\theta(t_1) + y_3^\theta(t_1), y_2^\theta(t_2) + y_3^\theta(t_2), \cdots, y_2^\theta(t_m) + y_3^\theta(t_m)) \\
\mathbf{E} &= (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)
\end{aligned}$$

where $t_j (j = 1, \cdots, m)$ are the measurement points, $F_j^\theta = y_2^\theta(t_j) + y_3^\theta(t_j)$ are the solution of proposed model (2.2) for the unknown parameters θ. Moreover, ε_j is the noise, assumed as white Gaussian noise as follows:

$$\varepsilon_j \sim \text{i.i.d. } N(0, \Sigma_\theta^2)$$

where the $i.i.d.$ means every residual is independent and identically distributed. They all have the same distribution, which is defined right afterward.

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Then we seek the parameters $\bar{\theta}$, which assumedly represent the true value of θ, such that

$$\mathbf{Y} = \mathbf{F}^{\theta} + \mathbf{E}. \tag{4.2}$$

Then the likelihood function $f(\mathbf{Y}|\theta)$ is then given as

$$f(\mathbf{Y}|\theta) = \exp\left\{-\frac{(\mathbf{Y}-\mathbf{F}^{\theta})^T(\mathbf{Y}-\mathbf{F}^{\theta})}{2\Sigma_{\varepsilon}^2}\right\}. \tag{4.3}$$

The prior density function is simply assumed as $f(\theta) = U_{[-\theta_0,\theta_0]}$, where $U_{[-\theta_0,\theta_0]}$is a uniform distribution with the parameter θ_0 which is a sufficiently large positive constant. Eventually the PPDF of the parameters θ can be written as follows:

$$f(\theta|\mathbf{Y}) \propto \exp\left\{-\frac{(\mathbf{Y}-\mathbf{F}^{\theta})^T(\mathbf{Y}-\mathbf{F}^{\theta})}{2\Sigma_{\varepsilon}^2}\right\}. \tag{4.4}$$

Here, the standard derivation Σ_{ε} is known and can be regarded as a regularization parameter.

4.1 MCMC methods

In Bayesian inference approach, the complicated and intractable probabilistic models can be estimated by numerical sampling methods such as MCMC, which has been widely applied in recent years. The details of MCMC methods are given in Robert and Casella [18].

Monte Carlo simulation generates pseudo–random numbers for exploring posterior distributions. In the MCMC algorithm, the pseudo–random number is a Markov chain. The MCMC algorithm exploits the property of a Markov chain to generate pseudo–random numbers from a posterior distribution, even for a complicated model. It first constructs an ergodic Markov chain with a stationary distribution equaling the target distribution. By iterating the Markov chain transitions from suitable initial value, it eventually obtains the target distribution.

In this paper, we employs a typical MCMC algorithm called the Metropolis–Hastings (M–H) algorithm (see Metropolis et al. [13]; Hastings [5]). The M–H (Algorithm **??**) given below builds its Markov chain by accepting or rejecting samples extracted from a proposed distribution. This algorithm is generally used in Bayesian inference approach (cf. [8]).

M–H Algorithm

- **Step1** : Generate $\theta' \sim q(\cdot|\theta_k) = N(\theta_k, \sigma_{\theta}^2)$ (the normal distribution) with a given stander derivation $\sigma > 0$ for given θ_k.
- **Step2** : Calculate the choice $\alpha(\theta', \theta_k) = \min\{1, f(\theta'|Y)/f(\theta_k|Y)\}$.

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we observe that it is able to largely replicate the subsequent curves including the leveling-off and decrease rapidly towards the end of the period after two small peaks. In addition, the coefficient of determination, which is the measure of how well a model explains the data, shows a high value at $R^2 = 0.9688$; this reveals that the proposed model has a high degree of fitness. On the other hand, we observe that fitting to two small peaks in the middle of the period is not good.

5.3.2 "Ten-nensui (Pure Water)"

Suntory "Ten-nensui" is best-selling mineral water which made with water from renowned water resources in Japan, including the Minami-Alps. All Suntory Ten-nensui products are made from "soft water", clear in color, and beloved for their refreshing taste[2].

In this study, we used Twitter data from before and after the product launch on 4/17/2019 to test the effectiveness of the "GREEN TEA CAMPAIGN" which is the new product of "Ten-nensui (Pure water)" with respect to the fit proposed model to actual data. The actual data in Figure 4 show a decline after the first

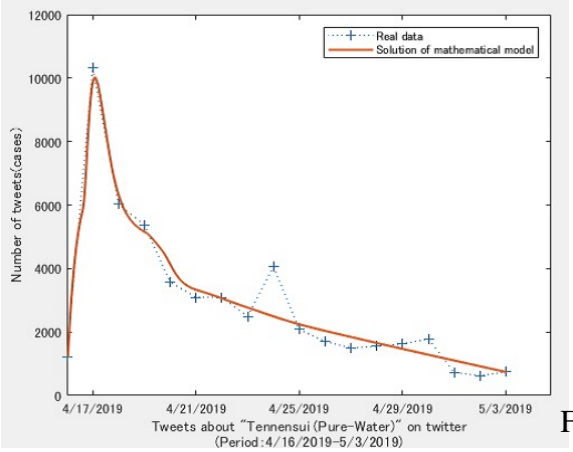

Figure 4: Fitting to "Tennensui" Pure-Water data

Pure_Water			
α	0.000041	$y_1(0)$	19000
β	0.021198	$y_2(0)$	0
γ	0.072868	$y_3(0)$	1207
δ	0.035978	$y_4(0)$	0
ε	0.010408	τ_1	14
ζ	8.579725	τ_2	72

Figure 5: Various parameter values

peak. After that, it has a small second peak once more, but it decreases again. Similar to the other examples, the proposed model shows a high degree of fit to the first peak. In addition, we observe that the graph from the proposed model is able to well reconstruct the processes of declining and even expresses the small decline that occurs from 4/18/2019 to 4/21/2019. The high degree of accuracy in the fitness is evident from the large coefficient of determination, $R^2 = 0.9612$. On

[2]https://www.suntory.com/brands/suntorytennensui/

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the other hand, we observe that the graph from the proposed model can't express well the second peak in the middle of the period.

5.3.3 Honkirin Beer

Honkirin Beer is a Happoshu (low-malt) beer that was introduced on March 13, 2018. In a cost-conscious environment, Honkirin Beer became the biggest hit among new beer releases in FY 2018 owing to its high quality and low price. According to the brewer Kirin, Honkirin Beer underwent a product renewal in mid-January 2019 for an even more refined authentic taste. As a result, the product logged a record sales volume in February (1.12 million cases), second only to its release in March 2018 (1.17 million cases).

Here, too, we used Twitter data from before and after the product launch on 4/22/2019 to test the effectiveness of the Honkirin Beer campaign with respect to the fit of the proposed model to actual data. The actual data in Figure 6 show

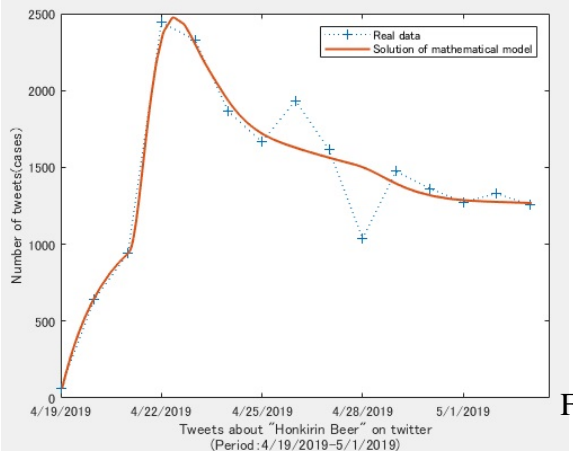

Figure 6: Fitting to Honkirin Beer data

Honkirin Beer			
α	0.000128	$y_1(0)$	4700
β	0.014264	$y_2(0)$	0
γ	0.027706	$y_3(0)$	63
δ	0.008074	$y_4(0)$	0
ε	0.002992	τ_1	48
ζ	0.714912	τ_2	158

Figure 7: Various parameter values

a decline after the first peak slowly with one peak and one bottom on April 26 and 28 2019. Similar to the other examples, the graph from the proposed model shows a high degree of fit to the first peak. In addition, we observe that it is able to fit the subsequent curves to the actual data except for two specific points. The high degree of accuracy in the fitness is evident from the large coefficient of determination, $R^2 = 0.9365$. On the other hand, we observe that the graph from the proposed model isn't able to express well two specific points. Future challenges include improvements to these points.

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