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Article

The sharp bound of the Hankel determinant of the third kind for starlike functions with real coefficients

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Abstract: Let $\mathcal{SR}^*$ be the class of starlike functions with real coefficients, i.e., the class of analytic functions f which satisfy the condition $f(0) = 0 = f'(0) - 1$, $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, for $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $a_n := f^{(n)}(0)/n!$ is real for all $n \in \mathbb{N}$. In the present paper, the sharp estimates of the third Hankel determinant $H_{3,1}$ over the class $\mathcal{SR}^*$ are computed.

Keywords: starlike functions; Hankel determinant; Carathéodory functions; Schwarz functions

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. That is, for $z \in \mathbb{D}$, $f \in \mathcal{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of functions $f \in \mathcal{A}$ of the form (1) are defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

Computing the upper bound of $H_{q,n}$ over subfamilies of $\mathcal{A}$ is an interesting problem to study. Recently many authors have examined the Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ of order 2 (see e.g., [1–6]). Note that $H_{2,1}(f) = a_3 - a_2^2$ is the well-known functional which for the class of univalent functions was estimated by Bieberbach (see, e.g., [7] (Vol. I, p. 35)). Especially, the functional $H_{3,1}(f)$, Hankel determinant of order 3, is presented by

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \end{aligned}$$

The bounds of $H_{3,1}(f)$ over several subfamilies of $\mathcal{A}$ were studied in [8–16].

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Let $\mathcal{S}^*$ be the class of starlike functions in $\mathcal{A}$. That is, the class $\mathcal{S}^*$ consists of all functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{3}$$

The leading example of a function of class $\mathcal{S}^*$ is the Koebe function k, defined by

$$k(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + \cdots, \quad z \in \mathbb{D}.$$

In [4], Janteng *et. al.* obtained the sharp inequality $|H_{2,2}(f)| \le 1 = |H_{2,2}(k)|$ for $f \in \mathcal{S}^*$. For the estimates on the Hankel determinant $H_{3,1}(f)$ over the class $\mathcal{S}^*$, Babalola [17] obtained the inequality $|H_{3,1}(f)| \le 16$. And Zaprawa [18] improved the result by proving $|H_{3,1}(f)| \le 1$. Next, Kwon *et. al.* [13], recently, found the inequality $|H_{3,1}(f)| \le 8/9$ and we conjectured that

$$|H_{3,1}(f)| \le 4/9, \quad f \in \mathcal{S}^*. \tag{4}$$

The sharp bound of $|H_{3,1}(f)|$ over the class $\mathcal{S}^*$ is still open.

Let $\mathcal{SR}^*$ be the class of starlike functions in $\mathcal{A}$ with real coefficients. Hence, if $f \in \mathcal{A}$ belongs to the class $\mathcal{SR}^*$, then f has the form given by (1) with $a_n \in \mathbb{R}$, $n \in \mathbb{N} \setminus \{1\}$ and satisfies the condition (3).

In this paper, it will be derived that

$$-\frac{4}{9} \le H_{3,1}(f) \le \frac{1}{9}\sqrt{3}, \quad f \in \mathcal{SR}^*. \tag{5}$$

So, from (5), it is remarkable that the inequality (4) is true for $f \in \mathcal{SR}^*$.

2. Carathéodory and Schwarz functions

Let $\mathcal{P}$ be the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{6}$$

having a positive real part in $\mathbb{D}$, i.e., the Carathéodory class of functions. It is well known, e.g., [19] (p. 166), that for $p \in \mathcal{P}$ with the form given by (6),

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{7}$$

for some $\zeta \in \overline{\mathbb{D}}$. Moreover, the following lemma will be used for our investigation.

Lemma 1. [20] *The formula (7) with $c_1 \in [0,2)$ and $\zeta \in \mathbb{T}$ holds only for the function $p \in \mathcal{P}$ defined by*

$$p(z) = \frac{1 + \tau(1+\zeta)z + \zeta z^2}{1 - \tau(1-\zeta)z - \zeta z^2}, \quad z \in \mathbb{D},$$

where $\tau \in [0,1)$.

Let $\mathcal{B}_0$ be the subclass of $\mathcal{H}$ of all self-mappings ω of $\mathbb{D}$ of the form

$$\omega(z) = \sum_{n=1}^{\infty} \beta_n z^n, \quad z \in \mathbb{D}, \tag{8}$$

i.e., the class of Schwarz functions. It is well known that $\omega \in \mathcal{B}_0$ if and only if $p = (1+\omega)/(1-\omega) \in \mathcal{P}$. For coefficients of functions in $\mathcal{B}_0$, the following properties, which can be found in [7] (Vol. I, pp. 84–85 and Vol. II, p. 78) and [21] (p. 128), will be used for our proof.

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Lemma 2. *If $\omega \in \mathcal{B}_0$ is of the form given by (8), then*

(1) $|\beta_1| \leq 1$,
(2) $|\beta_2| \leq 1 - |\beta_1|^2$,
(3) $|\beta_3(1-|\beta_1|^2) + \overline{\beta_1}\beta_2^2| \leq (1-|\beta_1|^2)^2 - |\beta_2|^2$.

The following inequalities, which will be used, hold for the fourth coefficients for Schwarz functions with real coefficients.

Lemma 3. [22] *If $\omega \in \mathcal{B}_0$ is the form (8), $\beta_n \in \mathbb{R}$, $n \in \mathbb{N}$, and $\beta_2^2 \neq (1-\beta_1^2)^2$, then*

$$\Psi_L \leq \beta_4 \leq \Psi_U, \tag{9}$$

where

$$\Psi_L := \frac{1+\beta_1^4+\beta_2-\beta_2^2-\beta_2^3-2\beta_1^2-\beta_1^2\beta_2+2\beta_1\beta_2\beta_3-\beta_3^2}{-1+\beta_1^2-\beta_2} \tag{10}$$

and

$$\Psi_U := \frac{1+\beta_1^4-\beta_2-\beta_2^2+\beta_2^3-2\beta_1^2+\beta_1^2\beta_2-2\beta_1\beta_2\beta_3-\beta_3^2}{1-\beta_1^2-\beta_2}. \tag{11}$$

3. Propositions

For given a set A, let int$A$, cl$A$ and ∂A be the sets of interior, closure and boundary, respectively, points of A. And let $R = [0,1] \times [-1,1]$ be a rectangle in $\mathbb{R}^2$. In this section, we obtain several inequalities for functions, defined in subsets of R, which will be used for our main result.

Proposition 1. *Define a function F_1 by*

$$F_1(x,y) = \sum_{n=0}^{4} b_n(x)y^n, \tag{12}$$

where

$$\begin{aligned}
b_4(x) &= (1-x)^2(1+x)^4,\\
b_3(x) &= -x(1+x)^3(10-11x+x^2),\\
b_2(x) &= (1+x)^2(7-16x+14x^3-5x^4),\\
b_1(x) &= x(10+9x-2x^2-6x^3-8x^4-3x^5),\\
b_0(x) &= -8+16x^2+6x^3-8x^4-6x^5.
\end{aligned}$$

Then $F_1(x,y) < 2\sqrt{3}$ holds for all $(x,y) \in R$.

Proof. Let $(x,y) \in R$. Since $b_4(x) \geq 0$, we have $b_4(x)y^4 \leq b_4(x)y^2$ and

$$F_1(x,y) \leq G(x,y), \quad (x,y) \in R,$$

where

$$G(x,y) = b_3(x)y^3 + (b_4(x)+b_2(x))y^2 + b_1(x)y + b_0(x).$$

We will show that $G(x,y) < 2\sqrt{3}$ holds for $(x,y) \in R$.

When $x = 0$, we have $G(0,y) = -8(1-y^2) \leq 0$, for $y \in [-1,1]$. And, when $x = 1$, we have $G(1,y) \equiv 0$.

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Now, let $x \in (0,1)$ be fixed and put $b_i = b_i(x)$ ($i \in \{0,1,2,3,4\}$). Then $b_3 < 0$. Define a function g_x by $g_x(y) = G(x,y)$. Note that

$$g_x(-1) = 0 \quad \text{and} \quad g_x(1) = 4x^2(1-x^2)(5-2x^2) \leq 0. \tag{13}$$

Also,

$$g_x'(y) = 3b_3y^2 + 2(b_4+b_2)y + b_1 = 0 \tag{14}$$

occurs at $y = \zeta_1$ or ζ_2, where

$$\zeta_i = \frac{-(b_4+b_2) + (-1)^{i+1}\sqrt{(b_4+b_2)^2 - 3b_1b_3}}{3b_3}, \quad i \in \{1,2\}.$$

It is trivial that $\zeta_1 < 0 < \zeta_2$. Furthermore, since $b_3 < 0$, g_x has the local minimum at $y = \zeta_1$. Let $\alpha = 0.322818\cdots$ be a zero of polynomial q, where

$$q(y) = 8 - 10y - 42y^2 - 14y^3 + 7y^4.$$

Note that $\zeta_2 \geq 1$ holds for x satisfying

$$2(1-x^2)q(x) = b_1 + 2(b_4+b_2) + 3b_3 \geq 0.$$

Hence we obtain

$$\begin{cases} \zeta_2 \geq 1, & \text{when } x \in (0,\alpha], \\ \zeta_2 \leq 1, & \text{when } x \in [\alpha,1). \end{cases}$$

(a) When $x \in (0,\alpha]$, since $\zeta_2 \geq 1$, g_x is convex in $[-1,1]$. So, it holds that

$$g_x(y) \leq \max\{g_x(-1), g_x(1)\}, \quad y \in [-1,1].$$

32 Hence, by (13), we get $g_x(y) \leq 0 < 2\sqrt{3}$ for $y \in [-1,1]$.

(b) When $x \in [\alpha,1)$, g_x has its local maximum $g_x(\zeta_2)$. Using the fact that ζ_2 is a solution of the equation given by (14) leads us to get

$$g_x(\zeta_2) = \left(\frac{2}{3}b_1 - \frac{2(b_2+b_4)^2}{9b_3}\right)\zeta_2 + \left(b_0 - \frac{b_1(b_2+b_4)}{9b_3}\right).$$

We claim that $g_x(\zeta_2) - 3 < 0$ holds for all $x \in [\alpha,1)$. A compuation gives

$$g_x(\zeta_2) - 3 = \frac{1}{9b_3}(1-x)(1+x)^3[-2(1-x)(1+x)\kappa_1\zeta_2 + x\kappa_2],$$

where

$$\kappa_1 = 64 - 128x + 204x^2 + 464x^3 + 249x^4 - 14x^5 + 7x^6$$

and

$$\kappa_2 = 910 - 11x - 1340x^2 - 414x^3 + 752x^4 + 398x^5 - 64x^6 + 12x^7.$$

Since $b_3 < 0$, $g_x(\zeta_2) - 3 < 0$ is equivalent to

$$2(1-x^2)\kappa_1\sqrt{(b_4+b_2)^2 - 3b_1b_3} < -3x\kappa_2b_3 - 2(1-x^2)\kappa_1(b_4+b_2). \tag{15}$$

We can see that the right-side of the above equation is positive for all $x \in [\alpha,1)$. Thus, by squaring the both sides of (15), we have $g_x(\zeta_2) < 0$ is equivalent to $\Psi > 0$, where

$$\Psi = [3x\kappa_2b_3 + 2(1-x^2)\kappa_1(b_4+b_2)]^2 - 4(1-x^2)^2\kappa_1^2[(b_4+b_2)^2 - 3b_1b_3].$$

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By a simple calculation we have

$$\Psi = -27x^2(10-x)^2(1-x)^2(1+x)^6\Lambda_x, \tag{16}$$

where

$$\begin{aligned}\Lambda_x := {} & 22528 - 90112x - 143980x^2 + 177084x^3 + 333021x^4 - 21120x^5 - 258308x^6 \\ & - 143200x^7 + 452x^8 + 28728x^9 + 37512x^{10} + 24288x^{11} + 9748x^{12} + 2720x^{13} \\ & + 968x^{14} - 48x^{15} + 36x^{16}.\end{aligned}$$

Since $\Lambda_x < 0$ holds for all $x \in [\alpha, 1)$, from (16), $\Psi > 0$, which implies

$$g_x(\zeta_2) < 3. \tag{17}$$

Finally, since

$$g_x(y) \le \max\{g_x(-1), g_x(1), g_x(\zeta_2)\}, \quad y \in [-1,1],$$

it follows from (13) and (17) that $g_x(y) < 3 < 2\sqrt{3}$ holds for all $y \in [-1,1]$. Thus the proof of Proposition 1 is completed. □

Proposition 2. *Let*

$$\Omega = \left\{(x,y) \in [0,1/2) \times [0,1) : 0 \le x \le \frac{y}{1+y}\right\} \subset R.$$

Define a function $F_2 : \Omega \to \mathbb{R}$ by

$$F_2(x,y) = \frac{1-x}{8+y-x(17+y)} H_1(x,y), \tag{18}$$

where $H_1(x,y) = \sum_{n=0}^{3} d_n(y)x^n$ with

$$d_3(y) = (1+y)^2(1-6y+y^2), \quad d_2(y) = 17 + 24y + 10y^2 - 3y^4,$$

$$d_1(y) = -8 - 26y - y^2 + 12y^3 + 3y^4 \quad \text{and} \quad d_0(y) = y(8 + y - 8y^2 - y^3).$$

Then $F_2(x,y) \le (2/9)\sqrt{3}$ holds for all $(x,y) \in \Omega$.

Proof. First of all, we note that F_2 is well-defined, since $8 + y - x(17+y) > 0$ holds for all $(x,y) \in \Omega$. Differentiating F_2 with respect to x twice gives

$$\frac{1}{2}[8+y-x(17+y)]^3 \frac{\partial^2 F_2}{\partial x^2}(x,y) = \sum_{n=0}^{4} \tilde{d}_n(y)x^n, \tag{19}$$

where

$$\begin{aligned}\tilde{d}_4(y) &= -3(1-6y+y^2)(17+18y+y^2)^2, \\ \tilde{d}_3(y) &= -4(884 + 3197y + 4605y^2 + 2062y^3 - 302y^4 - 75y^5 - 3y^6), \\ \tilde{d}_2(y) &= 6(1024 + 2344y + 2421y^2 + 956y^3 - 202y^4 - 60y^5 - 3y^6), \\ \tilde{d}_1(y) &= -12(8+y)^2(4 + 7y + 5y^2 + y^3 - y^4), \\ \tilde{d}_0(y) &= 512 + 1088y + 960y^2 - 176y^3 - 83y^4 - 30y^5 - 3y^6.\end{aligned}$$

Fix now $y \in [0,1)$ and put $y_0 = y/(1+y) \in [0,1/2)$. Let us define a function $g_y : [0,y_0] \to \mathbb{R}$ by $g_y(x) = \sum_{n=0}^{4} \tilde{d}_n(y)x^n$. Then we have

$$g_y'(x) = -12(1+y)[8+y-x(17+y)]^2\varphi(x), \tag{20}$$

where

$$\varphi(x) = 4+3y+2y^2-y^3+(1+y)(1-6y+y^2)x.$$

Since $-4 \le 1-6y+y^2 \le 1$, we have

$$\varphi(x) \ge 4+3y+2y^2-y^3-4x(1+y) \ge 4-y+2y^2-y^3 > 0, \quad x \in [0,y_0].$$

Thus, by (20), we get $g_y'(x) < 0$, when $x \in [0,y_0]$. So g_y is decreasing on the interval $[0,y_0]$, which yields

$$g_y(x) \ge g_y(y_0) = \frac{64(1-y)(8-7y+2y^2+33y^3)}{(1+y)^2} \ge 0, \quad x \in [0,y_0].$$

Since $8+y-x(17+y) > 0$ holds for all $(x,y) \in \Omega$, by (19), $F_2(x,\cdot)$ is convex on $[0,y_0]$. This gives us that

$$F_2(x,y) \le \max\{F_2(0,y), F_2(y_0,y)\} = F_2(0,y) = y-y^3 \le \frac{2}{9}\sqrt{3}, \quad (x,y) \in \Omega,$$

as we asserted. □

Proposition 3. *Define a function F_3 by*

$$F_3(x,y) = \frac{9(1-x)(1+y)}{8-y+x(1+y)}H_2(x,y), \tag{21}$$

where $H_2(x,y) = \sum_{n=0}^{3} k_n(y)x^n$ with

$$k_3(y) = (1+y)^3, \quad k_2(y) = 1+7y+3y^2-3y^3,$$
$$k_1(y) = 8-2y-15y^2+3y^3 \quad \text{and} \quad k_0(y) = -y(8-9y+y^2).$$

Then $F_3(x,y) \le 2\sqrt{3}$ holds for all $(x,y) \in R$.

Proof. First of all, by simple calculations, the equation $(\partial F_3/\partial x)(x,y) = 0$ gives us

$$(1-x)(8-y+x(1+y))\frac{\partial H_2}{\partial x}(x,y) = 9H_2(x,y). \tag{22}$$

Also, the equation $(\partial F_3/\partial y)(x,y) = 0$ holds when

$$-(1+y)(8-y+x(1+y))\frac{\partial H_2}{\partial y}(x,y) = 9H_2(x,y). \tag{23}$$

Assume that the function F_3 has its critical point at $(x_0,y_0) \in \mathrm{int}R$. Since $8-y_0+x_0(1+y_0) \neq 0$, from (22) and (23), we have

$$(1-x_0)\frac{\partial H_2}{\partial x}(x_0,y_0) + (1+y_0)\frac{\partial H_2}{\partial y}(x_0,y_0) = 0,$$

or, equivalently, $y_0 = x_0/(1-x_0)$. However, it holds that

$$(1-x_0)(8-y_0+x_0(1+y_0))\frac{\partial H_2}{\partial x}(x_0,y_0) - 9H_2(x_0,y_0) = 64(1-x_0) \neq 0,$$

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since $x_0 \in (0,1)$. This contradicts to (22). Hence F_3 does not have any critical points in int$R$. Thus F_3 has its maximum on ∂R.

We now consider F_3 on ∂R.

(a) On the side $x = 1$, we have $F_3(1, y) \equiv 0$.

(b) On the side $y = -1$, we have $F_3(x, -1) \equiv 0$.

(c) On the side $y = 1$, we have

$$F_3(x,1) = \frac{-36x(3-7x+4x^3)}{7+2x} =: \varphi(x), \quad x \in [0,1]. \tag{24}$$

Since the inequality $2(7+56x-126x^2+72x^4) > 0$ holds for all $x \in [0,1]$, it follows that $\varphi(x) < 2$ ($x \in [0,1]$). This inequality with (24) implies $F_3(x,1) < 2 < 2\sqrt{3}$ holds for $x \in [0,1]$.

(d) On the side $x = 0$, we have

$$F_3(0,y) = -9y(1-y^2) =: \psi(y). \tag{25}$$

And the inequality $F_3(0,y) \le 2\sqrt{3}$ ($y \in [-1,1]$) comes directly from (25) and

$$\psi(y) \le \psi(-1/\sqrt{3}) = 2\sqrt{3}, \quad y \in [-1,1].$$

From (a)–(d), for all $(x,y) \in \partial R$, the inequality $F_3(x,y) \le 2\sqrt{3}$ holds. Thus the proof of Proposition 3 is completed. □

Proposition 4. *For F_1 defined by (12), the inequality*

$$F_1(x,y) \ge -8$$

holds for $(x,y) \in [0,1] \times [-1,0]$.

Proof. Define a function $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(x,y) = F(x,-y) - b_4(x)y^4 + 8 = l_3(x)y^3 + l_2(x)y^2 + l_1(x)y + l_0(x),$$

where $l_3(x) = -b_3(x)$, $l_2(x) = b_2(x)$, $l_1(x) = -b_1(x)$ and $l_0(x) = b_0(x) + 8$. Then we have

$$F(x,y) + 8 \ge G(x,-y), \quad (x,y) \in [0,1] \times [-1,0].$$

We note that, when $x = 0$, $G(0,y) = 7y^2 \ge 0$ holds for $y \in [-1,1]$. And, when $x = 1$, $G(1,y) \equiv 8 > 0$.

Let $x \in (0,1)$ be fixed and put $l_i = l_i(x)$ ($i \in \{0,1,2,3\}$). Define a function $g_x : [0,1] \to \mathbb{R}$ by $g_x(y) = G(x,y)$. We will show that the inequality $g_x(y) \ge 0$ holds for all $y \in [0,1]$.

Note that $l_3 > 0$ and $l_1 < 0$. Let

$$\zeta_i = \frac{-l_2 + (-1)^i \sqrt{l_2^2 - 3l_1 l_3}}{l_3}, \quad i = 1,2$$

be the roots of the equation

$$g_x'(y) = 3l_3y^2 + 2l_2y + l_1 = 0.$$

Then it is easily seen that $\zeta_1 < 0 < \zeta_2$. Moreover $\zeta_2 < 1$ holds. Indeed, $\zeta_2 < 1$ is equivalent to $l_1l_3 + 3l_3^2 + 2l_2l_3 > 0$. And a computation gives

$$l_1l_3 + 3l_3^2 + 2l_2l_3 = -2x(1-x)^2(1+x)^4\varphi(x), \tag{26}$$

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where

$$\varphi(x) = -70 - 73x - 52x^2 - 34x^3 - 16x^4 + 2x^5.$$

Since $\varphi(x) < 0$, by (26), we get $l_1 l_3 + 3l_3^2 + 2l_2 l_3 > 0$ and $\zeta_2 < 1$. Therefore, we have

$$g_x(y) \geq g_x(\zeta_2), \quad y \in [0,1]. \tag{27}$$

On the other hand, simple calculations give us that

$$\begin{aligned} g_x(\zeta_2) &= \frac{1}{9l_3}[(6l_1 l_3 - 2l_2^2)\zeta_2 + (9l_0 l_3 - l_1 l_2)] \\ &= \frac{-1}{9l_3}(1-x)(1+x)^3[2(1-x^2)\kappa_1\zeta_2 + x\kappa_2], \end{aligned}$$

where

$$\kappa_1 = 49 - 126x + 255x^2 + 472x^3 + 204x^4 - 24x^5 + 16x^6$$

and

$$\kappa_2 = -70 + 97x - 1352x^2 - 429x^3 + 746x^4 + 401x^5 - 56x^6 + 15x^7.$$

Since $l_3 > 0$, $g_x(\zeta_2) \geq 0$ holds, if

$$2(1-x^2)\kappa_1\zeta_2 + x\kappa_2 \leq 0. \tag{28}$$

Moreover (28) is equivalent to $\Psi \geq 0$, where

$$\Psi = [2(1-x^2)\kappa_1 l_2 - 3x\kappa_2 l_3]^2 - 4(1-x^2)^2\kappa_1^2(l_2^2 - 3l_1 l_3).$$

We represent Ψ by

$$\Psi = -27x^4(10-x)^2(1-x)^2(1+x)^6\tilde{\Lambda}_x, \tag{29}$$

where

$$\begin{aligned} \tilde{\Lambda}_x =& -17052 + 84812x - 222415x^2 - 10212x^3 + 78990x^4 - 226456x^5 \\ &- 152793x^6 + 198120x^7 + 169280x^8 - 11796x^9 - 33413x^{10} + 1068x^{11} \\ &+ 2790x^{12} - 1008x^{13} + 117x^{14}. \end{aligned} \tag{30}$$

Since $\tilde{\Lambda}_x < 0$ holds for all $x \in (0,1)$, from (29), $\Psi \geq 0$ is true. We thus have $g_x(\zeta_2) \geq 0$. Finally, it follows from (27) that $g_x(y) \geq 0$ holds for all $y \in [0,1]$. The proof of Proposition 4 is completed. □

Proposition 5. *For a function F_4 defined by*

$$F_4(x,y) = F_1(-x,y), \tag{31}$$

where F_1 is defined by (12), *we have*

$$F_4(x,y) \geq -8, \quad (x,y) \in [0,1] \times [-1/3,1].$$

Proof. It is easily checked that $F_4(x,y) \geq -8$ holds for $y \in [-1/3,1]$ when $x = 0$ or $x = 1$. Let $x \in (0,1)$ be fixed and put $m_i = b_i(-x)$ $(i \in \{0,1,2,3,4\})$. Define a function $g_x : [-1/3,1] \to \mathbb{R}$ by $g_x(y) = F_4(x,y)$.

First, we will show that $g_x(y) \geq -8$ holds for $y \in [-1/3,0]$. Since $m_3 > 0$ and $m_4 > 0$, we have $m_4 y^4 \geq 0$ and $m_3 y^3 \geq -m_3 y^2/3$ for $y \in [-1/3,0]$. Hence, we obtain

$$g_x(y) + 8 \geq \varphi_x(-y), \quad y \in [-1/3,0], \tag{32}$$

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where $\varphi_x : [0, 1/3] \to \mathbb{R}$ is the function defined by

$$\varphi_x(y) = \left(-\frac{1}{3}m_3 + m_2\right) y^2 - m_1 y + m_0 + 8, \quad y \in [0, 1/3].$$

Since $m_1 < 0$ and

$$-\frac{1}{3}m_3 + m_2 = \frac{1}{3}(1 - x^2)^2(21 - 4x + 14x^2) > 0, \quad x \in (0, 1),$$

we get

$$\varphi_x'(y) = 2\left(-\frac{1}{3}m_3 + m_2\right) y - m_1 > 0, \quad y \in [0, 1/3].$$

Therefore φ_x is increasing on $[0, 1/3]$ and we get

$$\varphi_x(y) \geq \varphi_x(0) = m_0 + 8 = x^2(16 - 6x - 8x^2 + 6x^3) \geq 0, \quad y \in [0, 1/3].$$

Thus, by (32), $g_x(y) \geq -8$ holds for $y \in [-1/3, 0]$.

Next, we will show that $g_x(y) \geq -8$ holds for $y \in [0, 1]$. For this, define a function $\psi_x : [0, 1] \to \mathbb{R}$ by

$$\psi_x(y) = g_x(y) - m_4 y^4 + 8 = m_3 y^3 + m_2 y^2 + m_1 y + m_0 + 8.$$

It is sufficient to show that $\psi_x(y) \geq 0$ holds for $y \in [0, 1]$, since

$$g_x(y) + 8 \geq \psi_x(y), \quad y \in [0, 1].$$

Let

$$\zeta_i = \frac{-m_2 + (-1)^i \sqrt{m_2^2 - 3m_1 m_3}}{3m_3}, \quad i \in \{1, 2\}$$

be the roots of the equation

$$\psi_x'(y) = 3m_3 y^2 + 2m_2 y + m_1 = 0.$$

Clearly, $\zeta_1 < 0$. Thus we have

$$\psi_x(y) \geq \min\{\psi_x(1), \psi_x(\zeta_2)\}, \quad y \in [0, 1]. \tag{33}$$

Since

$$\psi_x(1) = 7 + 2x - 19x^2 - 4x^3 + 29x^4 + 2x^5 - 9x^6 > 0, \quad x \in (0, 1),$$

it is enough to show that $\psi_x(\zeta_2) \geq 0$ holds. A similar argument with the proof of Proposition 4, for $x \in (0, 1)$, $\psi_x(\zeta_2) \geq 0$ holds if $\tilde{\Lambda}_{-x} < 0$, where $\tilde{\Lambda}_x$ is the quantity defined by (30). It can be checked that $\tilde{\Lambda}_x < 0$ holds for all $x \in (-1, 0)$. Consequently, $\psi_x(\zeta_2) \geq 0$, when $x \in (0, 1)$, follows. Hence, by (33), $\psi_x(y) \geq 0$ holds for $y \in [0, 1]$. It completes the proof of Proposition 5. □

4. Main result

By using all lemmas in Section 2 and propositions in Section 3, the sharp bound of Hankel determinant of the third kind for starlike functions with real coefficients can be derived as the following result.

Theorem 1. *If $f \in \mathcal{SR}^*$ is the form* (1), *then the following inequalities hold:*

$$-\frac{4}{9} \leq H_{3,1}(f) \leq \frac{1}{9}\sqrt{3}. \tag{34}$$

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The first inequality is sharp for the function $f = f_1 \in \mathcal{SR}^*$, *where*

$$f_1(z) := z(1-z^3)^{-2/3} = z + \frac{2}{3}z^4 + \frac{5}{9}z^7 + \cdots, \quad z \in \mathbb{D}.$$

The second inequality is sharp for the function $f = f_2 \in \mathcal{SR}^*$, *where*

$$\begin{aligned} f_2(z) &:= z\exp\left(-\int_0^z \frac{(2/\sqrt{3})\zeta + 2\zeta^3}{1+(2/\sqrt{3})\zeta^2+\zeta^4} d\zeta\right) \\ &= z - \frac{z^3}{\sqrt{3}} + \frac{2z^7}{3\sqrt{3}} - \frac{7z^9}{18} + \cdots, \quad z \in \mathbb{D}. \end{aligned}$$

Proof. Let $f \in \mathcal{SR}^*$ be of the form (1). Then by (3) there exists a $\omega \in \mathcal{B}_0$ of the form (8) such that

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)}. \tag{35}$$

Substituting the series (1) and (8) into (35), by equating the coefficients we get

$$18H_{3,1}(f) = 3\beta_1^4\beta_2 + 6\beta_1^3\beta_3 + 10\beta_1\beta_2\beta_3 - 8\beta_3^2 - 11\beta_1^2\beta_2^2 + 9(\beta_2 - \beta_1^2)\beta_4. \tag{36}$$

Since $H_{3,1}(f) = H_{3,1}(\tilde{f})$, where $\tilde{f}(z) = -f(-z) \in \mathcal{SR}^*$, we may assume that $\beta_1 \in [0,1]$.

I. When $\beta_1 = 1$, then by Schwarz's lemma, $\beta_n = 0$ for all $n \geq 2$. Thus, by (36), $H_{3,1}(f) = 0$.

II. When $\omega \in \mathcal{B}_0$ be such that $|\beta_2| = 1 - \beta_1^2$ and $\beta_1 \in [0,1)$. Let $p = (1+\omega)/(1-\omega) \in \mathcal{P}$ be of the form (6). From the relations

$$c_1 = 2\beta_1 \quad \text{and} \quad c_2 = 2(\beta_1^2 + \beta_2),$$

it follows from that $c_1 \in [0,2)$ and $2c_2 = c_1^2 + (4-c_1^2)\zeta$, where $\zeta = \pm 1 \in \mathbb{T}$.

II(a) Assume that $\zeta = 1$. Then, by Lemma 1, $p = p_1$, where

$$p_1(z) = \frac{1+2\tau z + z^2}{1-z^2} = 1 + 2\tau z + 2z^2 + 2\tau z^3 + \cdots, \quad z \in \mathbb{D}$$

with $\tau \in [0,1)$. And, from $p = (1+\omega)/(1-\omega)$, we have

$$\beta_1 = \tau, \quad \beta_2 = 1-\tau^2, \quad \beta_3 = -\tau + \tau^3 \quad \text{and} \quad \beta_4 = \tau^2 - \tau^4. \tag{37}$$

Substituting (37) into (36), we get

$$H_{3,1}(f) = -\frac{2}{9}\tau^2(5 - 7\tau^2 + 2\tau^4) =: g(\tau^2), \tag{38}$$

where

$$g(x) = -\frac{2}{9}x(1-x)(5-2x).$$

It can be easily checked that $g(x) \leq g(0) = 0$, for $x \in [0,1)$. Moreover, since $g'(x) = 0$ occurs only when $x = x_1 := (7-\sqrt{19})/6 = 0.440184\cdots \in [0,1)$ and $g''(x_1) = 4\sqrt{19}/9 > 0$, it holds that

$$g(x) \geq g(x_1) = \frac{1}{243}(28 - 19\sqrt{19}) \geq -\frac{4}{9}, \quad x \in [0,1).$$

So, from (38), the inequality (34) holds.

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II(b) Now assume that $\zeta = -1$. Then, by Lemma 1 again, we get $p = p_2$, where

$$\begin{aligned} p_2(z) &= \frac{1-z^2}{1-2\tau z+z^2} \\ &= 1+2\tau z+(-2+4\tau^2)z^2+(-6\tau+8\tau^3)z^3+(2-16\tau^2+16\tau^4)z^4+\cdots, \quad z\in\mathbb{D} \end{aligned}$$

with $\tau \in [0,1)$. Thus, we have

$$\beta_1 = \tau, \quad \beta_2 = \tau^2 - 1, \quad \beta_3 = \tau^3 - \tau \quad \text{and} \quad \beta_4 = \tau^4 - \tau^2. \tag{39}$$

Substituting (39) into (36), we get $H_{3,1}(f) = 0$ and the inequality (34) holds.

III. Let now $|\beta_2| \neq 1 - \beta_1^2$ and $\beta_1 \neq 1$.

At first, we will show that the second inequality in (34) holds. Since β_1, β_2 and β_3 are real, by Lemma 2 for $s \in [0,1]$ and $t, u \in [-1,1]$ we have

$$\beta_1 = s, \quad \beta_2 = (1-s^2)t, \quad \beta_3 = (1-s^2)(u(1-t^2) - st^2). \tag{40}$$

Substituting (40) into (10) and (11), we have

$$\Psi_U = (1-s^2)[1-u^2-u(u+2s)t-(1-u^2)t^2+(u+s)^2t^3] \tag{41}$$

and

$$\Psi_L = (1-s^2)[-1+u^2-u(u+2s)t+(1-u^2)t^2+(u+s)^2t^3]. \tag{42}$$

We also have $(s,t) \notin C$, where C is a curve defined by

$$C = \{(s,t) \in R : s = 1 \ \text{ or } \ t = \pm 1\} \subset \partial R.$$

III(a) Consider the case $\beta_2 \geq \beta_1^2$, i.e. $(s,t) \in \Omega_1$, where Ω_1 is the set defined by

$$\Omega_1 = \left\{ (s,t) \in [0, 1/\sqrt{2}) \times [0,1) : \frac{s^2}{1-s^2} \leq t < 1 \right\}$$

so that $\Omega_1 \cap C = \emptyset$. In this case, by (41), we have

$$\begin{aligned} 18H_{3,1}(f) &\leq 3\beta_1^4\beta_2 + 6\beta_1^3\beta_3 + 10\beta_1\beta_2\beta_3 - 8\beta_3^2 - 11\beta_1^2\beta_2^2 + 9(\beta_2 - \beta_1^2)\Psi_U \\ &= -(1-s^2)(1+t)\Phi(s,t,u), \quad (s,t,u) \in \Omega_1 \times [-1,1], \end{aligned} \tag{43}$$

where

$$\Phi(s,t,u) = \Phi_0 + \Phi_1 u + \Phi_2 u^2 \tag{44}$$

with

$$\begin{aligned} \Phi_0 &= \Phi_0(s,t) := -9(1-t)t - s^4 t(3+2t-t^2) + s^2(9+2t^2-t^3), \\ \Phi_1 &= \Phi_1(s,t) := -2s(1-t)[(5-t)t + s^2(3+4t+t^2)], \\ \Phi_2 &= \Phi_2(s,t) := (1-t^2)[8+t-s^2(17+t)]. \end{aligned}$$

We note that $\Phi_2 > 0$, since

$$8 + t - s^2(17+t) \geq \frac{8(1-t)}{1+t} > 0, \quad (s,t) \in \Omega_1.$$

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Let $u_1 = -\Phi_1/(2\Phi_2)$ be the root of the equation $(\partial\Phi/\partial u)(s,t,u) = 0$. Then it can be seen that $u_1 \geq -1$. Indeed, we note that $2\Phi_2 - \Phi_1 = 2(1-t)\Upsilon(s,t)$, where $\Upsilon(s,t) = \lambda_2(s)t^2 + \lambda_1(s)t + \lambda_0(s)$, where

$$\lambda_2(s) = (1-s)^2(1+s), \quad \lambda_1(s) = 9 + 5s - 18s^2 + 4s^3$$

and

$$\lambda_0(s) = 8 - 17s^2 + 3s^3.$$

Since $\lambda_i(s) \geq 0$ when $s \in [0, 1/\sqrt{2})$ for $i \in \{1,2\}$, we have

$$\Upsilon(s,t) \geq \Upsilon\left(s, \frac{s^2}{1-s^2}\right) = \frac{8(1+s-s^2)}{1+s} \geq 0, \quad (s,t) \in \Omega_1.$$

Hence, we get $2\Phi_2 - \Phi_1 \geq 0$ and it follows from $\Phi_2 > 0$ that $u_1 \geq -1$.

(i) Assume that $u_1 \geq 1$. Then we have

$$\Phi(s,t,u) \geq \Phi(s,t,1) = \Phi_0 + \Phi_1 + \Phi_2, \quad (s,t,u) \in \Omega_1 \times [-1,1].$$

Therefore, by (43), it holds that

$$18H_{3,1}(f) \leq -(1-s^2)(1+t)(\Phi_0 + \Phi_1 + \Phi_2) = F_1(s,t), \quad (s,t) \in \Omega_1, \tag{45}$$

where F_1 is the function defined by (12). From Proposition 1 and (45), we thus have $H_{3,1}(f) \leq \sqrt{3}/9$.

(ii) Assume that $-1 \leq u_1 \leq 1$. Then we have

$$\Phi(s,t,u) \geq \Phi(s,t,u_1) = \Phi_0 - \frac{\Phi_1^2}{4\Phi_2}, \quad (s,t,u) \in \Omega_1 \times [-1,1].$$

Therefore, by (43), it holds that

$$18H_{3,1}(f) \leq -(1-s^2)(1+t)\left(\Phi_0 - \frac{\Phi_1^2}{4\Phi_2}\right) = 9F_2(s^2,t), \quad (s,t) \in \Omega_1,$$

where F_2 is the function defined by (18). Therefore, by Proposition 2, $H_{3,1}(f) \leq \sqrt{3}/9$ holds.

III(b) Consider the case $\beta_2 \leq \beta_1^2$, i.e. $(s,t) \in \Omega_2$, where Ω_2 is the set defined by $\Omega_2 = \mathrm{cl}(R \setminus \Omega_1) \setminus C$. Then, from (42), we have

$$\begin{aligned} 18H_{3,1}(f) &\leq 3\beta_1^4\beta_2 + 6\beta_1^3\beta_3 + 10\beta_1\beta_2\beta_3 - 8\beta_3^2 - 11\beta_1^2\beta_2^2 + 9(\beta_2 - \beta_1^2)\Psi_L \\ &= -(1-s^2)(1+t)\hat{\Phi}(s,t,u), \quad (s,t,u) \in \Omega_2 \times [-1,1], \end{aligned} \tag{46}$$

where

$$\hat{\Phi}(s,t,u) = \hat{\Phi}_0 + \hat{\Phi}_1 u + \hat{\Phi}_2 u^2 \tag{47}$$

with

$$\begin{aligned} \hat{\Phi}_0 &= \hat{\Phi}_0(s,t) := 9(1-t)t - s^4 t(3+2t-t^2) - s^2(9 - 20t^2 + t^3), \\ \hat{\Phi}_1 &= \hat{\Phi}_1(s,t) := -2s(1-t)[(5-t)t + s^2(3+4t+t^2)], \\ \hat{\Phi}_2 &= \hat{\Phi}_2(s,t) := (1-t)^2[8 - t + s^2(1+t)]. \end{aligned}$$

Using the inequality $s^2 \geq t/(1+t)$, we have $\hat{\Phi}_2 \geq 8(1-t)^2 > 0$ for $(s,t) \in \Omega_2$. Let $u_2 = -\hat{\Phi}_1/(2\hat{\Phi}_2)$ be the root of the equation $(\partial\hat{\Phi}/\partial u)(s,t,u) = 0$. Then, by a similar procedure with Part III(a), it can be seen that $u_2 \geq -1$.

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(i) Assume that $u_2 \geq 1$. Then we have

$$\hat{\Phi}(s,t,u) \geq \hat{\Phi}(s,t,1) = \hat{\Phi}_0 + \hat{\Phi}_1 + \hat{\Phi}_2, \quad (s,t,u) \in \Omega_2 \times [-1,1].$$

Therefore, by (46), it holds that

$$18H_{3,1}(f) \leq -(1-s^2)(1+t)(\hat{\Phi}_0 + \hat{\Phi}_1 + \hat{\Phi}_2) = F_1(s,t), \quad (s,t) \in \Omega_2,$$

where F_1 is the function defined by (12). Thus, by Proposition 1, $H_{3,1}(f) \leq \sqrt{3}/9$ holds.

(ii) Assume that $-1 \leq u_2 \leq 1$. Then we have

$$\hat{\Phi}(s,t,u) \geq \hat{\Phi}(s,t,u_2) = \hat{\Phi}_0 - \frac{\hat{\Phi}_1^2}{4\hat{\Phi}_2}, \quad (s,t,u) \in \Omega_2 \times [-1,1].$$

Therefore, by (46), it holds that

$$18H_{3,1}(f) \leq -(1-s^2)(1+t)\left(\hat{\Phi}_0 - \frac{\hat{\Phi}_1^2}{4\hat{\Phi}_2}\right) = F_3(s^2,t), \quad (s,t) \in \Omega_2,$$

where F_3 is the function defined by (21). Therefore, by Proposition 3, we obtain $H_{3,1}(f) \leq \sqrt{3}/9$.

Next, we will show that the first inequality in (34) holds.

IV(a) Consider the case $\beta_2 \geq \beta_1^2$. Then we have

$$18H_{3,1}(f) \geq -(1-s^2)(1+t)\hat{\Phi}(s,t,u), \quad (s,t,u) \in \Omega_1 \times [-1,1], \tag{48}$$

where $\hat{\Phi}$ is the function defined by (47). Since $\hat{\Phi}_1 \leq 0$ and $\hat{\Phi}_2 > 0$, it holds that

$$\begin{aligned}\hat{\Phi}(s,t,u) &\leq \max\{\hat{\Phi}(s,t,-1), \hat{\Phi}(s,t,1)\} \\ &= \hat{\Phi}(s,t,-1) = \hat{\Phi}_2 - \hat{\Phi}_1 + \hat{\Phi}_0, \quad (s,t,u) \in \Omega_1 \times [-1,1].\end{aligned}$$

Hence, from (48), we obtain

$$H_{3,1}(f) \geq -(1-s^2)(1+t)(\hat{\Phi}_2 - \hat{\Phi}_1 + \hat{\Phi}_0) = F_4(s,t), \quad (s,t) \in \Omega_1, \tag{49}$$

where F_4 is the function defined by (31). Thus, by Proposition 5 and (49), we get $H_{3,1}(f) \geq -4/9$.

IV(b) We consider the case $\beta_2 \leq \beta_1^2$. Then we have

$$18H_{3,1}(f) \geq -(1-s^2)(1+t)\Phi(s,t,u), \quad (s,t,u) \in \Omega_2 \times [-1,1],$$

where Φ is the function defined by (44).

For $t \in [-1/3, 0]$, let

$$s_t = \frac{t^2 - 5t}{t^2 + 4t + 3}$$

so that $0 = s_0 \leq s_t \leq s_{-1/3} = 1$ holds for $t \in [-1/3, 0]$. And let

$$\Omega_3 = \{(s,t) \in \Omega_2 : s \leq s_t\} \quad \text{and} \quad \Omega_4 = \{(s,t) \in \Omega_2 : s \geq s_t\}.$$

We note that $\Omega_3 \subset [0,1] \times [-1,0]$ and $\Omega_4 \subset [0,1] \times [-1/3,1]$. Then $\Phi_1 \geq 0$ when $(s,t) \in \Omega_3$, and $\Phi_1 \leq 0$ when $(s,t) \in \Omega_4$.

(i) For the case $(s,t) \in \Omega_3$, since $\Phi_1 \geq 0$ and $\Phi_2 \geq 0$, we have

$$\Phi(s,t,u) \leq \Phi(s,t,1) = \Phi_2 + \Phi_1 + \Phi_0, \quad (s,t,u) \in \Omega_3 \times [-1,1]$$

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and, therefore, we get

$$18H_{3,1}(f) \geq -(1-s^2)(1+t)(\Phi_2+\Phi_1+\Phi_0) = F_1(s,t), \quad (s,t) \in \Omega_3,$$

where F_1 is the function defined by (12). Since $\Omega_3 \subset [0,1] \times [-1,0]$, Proposition 4 gives us that $H_{3,1}(f) \geq -4/9$ holds.

(ii) For the case $(s,t) \in \Omega_4$, we have

$$18H_{3,1}(f) \geq -(1-s^2)(1+t)(\Phi_2-\Phi_1+\Phi_0) = F_4(s,t), \quad (s,t) \in \Omega_4,$$

where F_4 is the funciton defined by (31). Since $\Omega_4 \subset [0,1] \times [-1/3,1]$, Proposition 5 gives us that $H_{3,1}(f) \geq -4/9$ holds. □

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