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Article

Optimality conditions for Vector Equilibrium Problems on Hadamard manifolds

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Abstract: The aim of this paper is to obtain Karush-Kuhn-Tucker optimality conditions for weakly efficient solutions to vector equilibrium problems with the addition of constraints in the novel context of Hadamard manifolds as opposed to the classical examples of Banach, normed or Hausdorff spaces. More specifically, classical necessary and sufficient conditions for weakly efficient solutions to the constrained vector optimization problem are presented. As well as some examples. The results presented in this paper generalize results obtained by Gong (2008) and Wei and Gong (2010) and Feng and Qiu (2014) from Hausdorff topological vector spaces, real normed spaces and real Banach spaces to Hadamard manifolds, respectively.

Keywords: Vector Equilibrium Problem; Generalized convexity; Hadamard manifolds; Weakly efficient solutions

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1. Introduction

The pursuit of equilibrium is a ubiquitous goal in most of the different areas of human activity. For example, in economics, the dynamics of offer and demand are typically described as equilibrium problems. In the same way, physical phenomena or other more human situations such as the distribution of traffic and telecommunication networks require us to think in terms of equilibriums.

In Fan [7] equilibrium theory in Euclidean spaces was firstly introduced. Mathematically, the simplest definition of a equilibrium problem consists in finding $x \in S$ such that

$$F(x,y) \geq 0, \ \forall y \in S$$

where $S \subseteq \mathbb{R}^p$ is a nonempty closed set and $F : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is an equilibrium bifunction, i.e. $F(x,x) = 0$ for all $x \in S$.

Some of principal mathematical problems that can be formulated as Equilibrium Problems are:

- A weak Pareto global minimum of the vector function $f = (f_1, \ldots, f_p)$ over a closed set $S \subseteq \mathbb{R}^p$ is any $\bar{x} \in S$ such that for any $y \in S$ exists an index i such that $f_i(y) - f_i(\bar{x}) \geq 0$. Finding a weak Pareto global minimum amounts to solving an Equilibrium Problem with

$$F(x,y) = \max_{i=1,\ldots,p} [f_i(y) - f_i(x)]$$

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- Complementarity Problems find a point $\bar{x} \in S$ such that $< G(\bar{x}), v > \geq 0$ for any $v \in C$ where $G : \mathbb{R}^p \to \mathbb{R}^p$ and $S \subseteq \mathbb{R}^p$ is a closed convex cone. Similarly, the complementarity problems amounts to solving Equilibrium Problem with

$$F(x,y) =< G(x), y - x >$$

- Stampacchia Variational Inequality Problem demands finding $\bar{x} \in S$ such that

$$< G(\bar{x}), y - \bar{x} > \geq 0, \forall y \in S$$

where $G : \mathbb{R}^p \to \mathbb{R}^p$ and $S \subseteq \mathbb{R}^p$ is a closed set. This problem can be formulated as Equilibrium Problem with

$$F(x,y) =< G(x), y - x >$$

- Nash Equilibrium problems in a noncooperative game with p players, each player i has a set of possible strategies $K_i \subseteq \mathbb{R}^{n_i}$ and aims to minimize a loss function $f_i : K \to \mathbb{R}$ with $K = K_1 \times \ldots \times K_p$. A Nash equilibrium point is any $\bar{x} \in K$ such that no player can reduce its loss by unilaterally changing their strategy, i.e, any $\bar{x} \in K$ such that

$$f_i(\bar{x}) \leq f_i(\bar{x}(y_i))$$

holds for any $y_i \in K_i$ for any $i = 1, \ldots, p$, with $\bar{x}(y_i))$ denoting the vector obtained from $\bar{x}$ by replacing $\bar{x}_i$ with y_i. This problem is equivalent to solving Equilibrium Problem with

$$F(x,y) = \sum_{i=1}^{p} [f_i(x(y_i)) - f_i(x)]$$

The above-mentioned problems are particular cases of the Vector Equilibrium Problem. Hence, it is important to obtain and study the optimality conditions for the solution such more general problem. Moreover, Vector Equilibrium problems are an active branch of non-linear analysis with plenty of publications being made up to this date. In 2003, authors as Iusem and Sosa [16] already studied the relation between equilibrium problems and some auxiliary convex problems.

In the addition to this, over the past century, many physicists such as Albert Einstein already proposed the use the Riemannian spaces to model Unified Field theories, further stressing the importance of the study of this context.

Riemannian manifolds are an extension of Euclidean space where the measurement of distance; normally associated with the dot product $|u|^2 =< u, u >$ is replaced by a metric tensor $|u|^2 = g_{ab}u^a u^b$ (see section Preliminaries for more details). However, manifolds are not a linear flat spaces but posses curvature. In other words, $ax + by \notin M$, $\forall x, y \in M$, $a, b \in \mathbb{R}$, where M is a Riemannian manifold. Hence, the Euclidean line element, shortest connection between two points in a flat surface, is replaced by a geodesic equation. In this sense, the geodesic curves of a curved manifold represent the straight lines of such space. This can be seen from the fact that geodesic curves are solutions to the Euler-Lagrange equations which minimize the functional of the lagrangian of such space. For example, in physics, geodesic curves describe the motion of a falling rock or an orbiting satellite despite not always being straight trajectories.

Even though manifolds usually posses curvature, the minimization of functions on a Hadamard manifold is, at least locally, equivalent to the smoothly constrained optimization problem on a Euclidean space, due to the fact that every C^∞ Hadamard manifold can be isometrically imbedded in an Euclidean space by virtue of John Nash's embedding theorem. Generally, the study of optimization problems on Hadamard manifolds is a powerful tool. This is due to the fact that, generally, solving nonconvex constrained problems in $\mathbb{R}^n$ with the Euclidean metric can be rephrased as solving the unconstrained convex minimization problem in the Hadamard manifold feasible set with the

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affine metric (see [6]). In Colao et al. [6] the existence of solutions for equilibrium problems under some suitable conditions on Hadamard manifolds and their applications to Nash equilibrium for non-cooperative games is studied. In Németh [20] the existence and uniqueness results for variational inequality problems on Hadamard manifolds are obtained.

Moreover, there is a considerable number of optimization problems which cannot be solved in linear spaces and require of Hadamard manifolds structures for their formalization and study. For example, in controlled thermonuclear fusion research (see [1]), signal processing, numerical analysis or computer vision (see [21], [23]). Also, geometrical structures hidden in data sets of machine learning problems are studied in terms of manifolds. In the field of medicine, Hadamard manifolds have been used in analysis of medical images quantifying growths of tumors and consequently deduce the progression of diseases, as it has been shown by Fletcher et al. [11]. Finally, in economics, characterization, existence, and stability of Nash-Stampacchia equilibria are studied using strategy sets based on geodesic convex subsets of Hadamard manifolds taking advantage of the geometrical features of these spaces as shown by Kristály [18].

In convex optimization, the convexity of a set in a linear space is based upon the possibility of connecting any two points of the space. Furthermore, it is known that a convex environment has good properties for the search of optimal points. In Ferreira [9], a characterization for convex functions defined on Hadamard manifolds is presented. A significant generalization of the convex functions are the invex functions, introduced by Hanson [14]. The invexity concept is an extension of differentiable convexity by generalizing the difference $(x - y)$ in the definition of convex function to any function $\eta(x, y)$. A scalar function is invex if and only if every critical point is a global minimum solution. The conditions for optimality that invexity involves are essential to obtain optimal points through practical numerical methods or algorithms due to the coincidence of critical points and solutions are assured. In Barani and Pouryayeli [5] and Hosseini and Pouryayevali [15], the relation between invexity and monotonicity using mean value theorem is studied. Ruiz-Garzón et al. [22] show that the invexity can be characterized in the context of Riemannian manifolds for both scalar and vector cases, in a similar way to Euclidean spaces. Recently, in Ahmad et al. [2] the authors introduce the log-preinvex and log-invex functions on Riemannian manifolds and the mean value theorem on Cartan-Hadamard manifolds.

In the same way, several authors have studied vector equilibrium problems. Ansari and Flores-Bazán [3] considered the generalized vector quasi-equilibrium problem and proved the existence of its solution by using known fixed point and maximal element theorems. Furthermore, the necessary and sufficient conditions for weakly efficient solution for the vector equilibrium problems with constraints under convexity conditions on real Hausdorff topological vector spaces were presented by Gong [12]. In the following years, scalarization results for the solutions to the vector equilibrium problems were also giving by Gong [13]. Later, optimality conditions for weakly efficient solutions to vector equilibrium problems with constraints in real normed spaces were investigated by Wei and Gong [24]. Also, sufficient conditions of weakly efficient solutions on real Banach spaces for vector equilibrium and vector optimization problems with constraints under generalized invexity were obtained by Feng and Qiu [8].

Motivated by Gong's works mentioned above, our objective will focus on extending the KKT necessary and sufficient conditions for constrained vector equilibrium problems obtained in topological or normed spaces to other environments like the Hadamard manifolds, not found in the literature up to date of publication. Hence, we propose a generalization that extends the linear space definition to Hadamard manifolds, substituting line segments by geodesic arcs. We will see that the KKT classic conditions for constrained vector optimization are a particular case of the ones obtained for constrained vector equilibrium problem.

The organization of the paper is as follows: In section 2, we discuss notations, differentials and invex functions' concepts on Hadamard manifolds. Section 3 is devoted to prove the main results obtained in this paper, studying the necessary and sufficient optimality conditions for weakly efficient

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points of constrained vector equilibrium problem. Section 4 dwells on how the previous results can be reduced to classical KKT conditions for constrained vector optimization problems. Finally, some examples are presented as well as the final conclusions.

2. Preliminaries

In this section we recall some notations, definitions and properties of Riemannian manifolds used throughout this paper.

Let M be a C^∞-manifold modeled on a Hilbert space H, either finite or infinite dimensional, endowed with a Riemannian metric g_x on a tangent space T_xM. We denote by T_xM the n-dimensional tangent space of M at x, by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of M, by $\bar{T}M$ an open neighborhood of the submanifold M of TM. The corresponding norm is denoted by $\|.\|_x$ and the length of a piecewise C^1 curve $\alpha : [a, b] \to M$ is defined by

$$L(\alpha) = \int_a^b \|\alpha'(t)\|_{\alpha(t)} dt$$

For any point $x, y \in M$, we define

$$d(x, y) = \inf\{L(\alpha) |\, \alpha \text{ is a piecewise } C^1 \text{ curve joining } x \text{ and } y\}$$

then d is a distance which induces the original topology on M. Any Riemannian manifold (M, g) can be converted into a metric space (M, d), where d is the distance induced by the Riemannian metric g.

Any path α joining x and y in M such that $L(\alpha) = d(x, y)$ is a geodesic and is called a minimal geodesic. The existence theorem for ordinary differential equation implies that for every $V \in TM$, there is an open interval $J(V)$ containing 0 and exactly one geodesic $\alpha_V : J(V) \to M$ with $d\alpha_V(0)/dt = V$. For differentiable manifolds, it is possible to define the derivatives of the curves on the manifold. The derivatives at a point x on the manifold lies in a vector space T_xM. We define as $\exp : \bar{T}M \to M$ defined as $\exp_x(V) = \alpha_V(1)$ for every $V \in \bar{T}M$, where α_V is the geodesic starting at x with velocity V (i.e. $\alpha(0) = x$, $\alpha'(0) = V$).

Assume now that η is a map $\eta : M \times M \to TM$ defined on the product manifold and such that

$$\eta(x, y) \in T_yM, \ \forall x, y \in M$$

Definition 1. *[4] A subset S_1 of M is called totally convex if S_1 contains every geodesic $\alpha_{x,y}$ of M whose endpoints x and y belong to S_1.*

We also recall that a simply connected complete Riemannian manifold of non-positive sectional curvature is called a Cartan-Hadamard manifold. Hadamard manifolds and Euclidean spaces have similar geometrical properties. One of them is the separation theorem (see Ferreira and Oliveira [10]).

In addition, for any two points in M, there exists a minimal geodesic joining these two points. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of M is a global diffeomorphism and exp map is defined on the whole tangent space ([19]).

Example 1. *Let $M = \mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$ be endowed with the Riemannian metric defined by $g(y) = y^{-2}$ is a Hadamard manifold. Hyperbolic spaces and geodesic spaces, more precisely, a Busemann nonposiive curvature (NPC) space are examples of Hadarmard manifolds.*

If we consider M to be a Cartan-Hadamard manifold (either infinite or finite dimensional), then on M there is a map playing the role of $x - y \in \mathbb{R}^n$. We can define the function η as $\eta(x, y) = \alpha'_{x,y}(0)$ for all $x, y \in M$. Here $\alpha_{x,y}$ is the unique minimal geodesic joining y to x as follows

$$\alpha_{x,y} = \exp_y(\lambda \exp_y^{-1} x) \quad \forall \lambda \in [0, 1]$$

We will need an adequate differential concept:

Definition 2 . *to be a differential map along the geodesic* $\alpha_{x,y}$ *at* $y \in M$ *if and only if the limit*

$$f_i'(y) = \lim_{\lambda \to 0} \frac{f_i(exp_y(\lambda\eta(x,y))) - f_i(y)}{\lambda\|\eta(x,y)\|}$$

exists.

The gradient of a real-valued C^∞ *function* $f = (f_1, \ldots f_p) : S_1 \subseteq M \to \mathbb{R}^n$ *on* M *in* x*, denoted by* $grad f_x = (f_1'(x), f_2'(x), \ldots, f_n'(x))$*, is the unique vector in* T_xM *such that* $df_x(X) = \langle grad f_x, X\rangle$ *for all* X *in* T_xM *is the differential of* f *at* $\bar{x}$ *of* X*.*

Remark 1. *The differential of* f *at* $\bar{x}$ *of* X *is similar to the definition of directional derivative in the Euclidean space.*

Let $S_1 \subset M$ be a nonempty open totally convex subset and let $F : S_1 \times S_1 \to \mathbb{R}^p$, $g : S_1 \to \mathbb{R}^p$ be mappings.

Definition 3. *We define the constraint set* $S = \{x \in S_1 : g(x) \in -\mathbb{R}^p_+\}$ *and consider the vector equilibrium problem with constraints (VEPC): find* $x \in S$ *such that*

$$F(x,y) \notin -\mathbb{R}^p_+ \setminus \{0\},\ \forall y \in S$$

where $\mathbb{R}^p_+$ *is the nonnegative orthant of* $\mathbb{R}^p$*.*

We recall the classical concept:

Definition 4. *A vector* $x \in S$ *satisfying* $F(x,y) \notin -int\,\mathbb{R}^p_+$*,* $\forall y \in S$ *is called a weakly efficient solution to the VEPC.*

Notation 1. *Let* $x \in S$*, be given. Denote the mapping* $H : S_1 \to \mathbb{R}^p$ *by*

$$H_x(y) = F(x,y),\ \forall y \in S_1$$

Inspired by the concept of convexity on a linear space, the notion of invexity function concept on Hadamard manifolds has become a successful tool in vector optimization. This definition generalizes given by Hanson [14].

Definition 5. *Let* S_1 *be a nonempty open totally convex subset of a Hadamard manifold* M*. A differentiable* $h : S_1 \to \mathbb{R}^p$ *function is said to be a* $\mathbb{R}^p_+$*-invex at* $\bar{x} \in S_1$ *respect to* $\eta : M \times M \to TM$ *if there exist* $\eta(x,\bar{x}) \in T_{\bar{x}}M$ *such that*

$$h(x) - h(\bar{x}) - dh_{\bar{x}}(\eta(x,\bar{x})) \in \mathbb{R}^p_+$$

In the next section, we will use the assumption of invexity of the functions of the problem to obtain the sufficient conditions of optimality.

3. Main Results

Next we will obtain a characterization of the weakly efficient points of VEPC through two theorems. Let's see the necessary condition:

Theorem 1. *[Necessary KKT-conditions] Let* S_1 *be a nonempty open totally convex subset of Hadamard manifold* M *and let* $F : S_1 \times S_1 \to \mathbb{R}^p$*,* $g : S_1 \to \mathbb{R}^p$ *be mappings. Let* $F(\bar{x},\bar{x}) = H_{\bar{x}}(\bar{x}) = 0$*. Assume*

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that H and g are differentiable at $\bar{x} \in S$. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -int\,\mathbb{R}^p_+$. If $\bar{x}$ is a weakly efficient solution to the VEPC, then there exist $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ such that

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0,\ \forall x \in S_1$$

$$ug(\bar{x}) = 0$$

Proof. Assume that $\bar{x} \in S$ is a weakly efficient solution to the VEPC. We can see that W is a nonempty open totally convex set where the set

$$W = \{(y, z) \in \mathbb{R}^p \times \mathbb{R}^p : \text{ there exists } \quad x \in S_1, \quad \text{such that} \quad y - dH_{\bar{x}}(\eta(x, \bar{x})) \in int\,\mathbb{R}^p_+,$$

$$z - [g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \in int\,\mathbb{R}^p_+\}$$

Step 1. We have to prove that $(0, 0) \notin W$. By reduction ad absurdum. If not, then there exists $x_0 \in S_1$, such that

$$dH_{\bar{x}}(\eta(x_0, \bar{x})) \in -int\,\mathbb{R}^p_+, \quad g(\bar{x}) + dg_{\bar{x}}(\eta(x_0, \bar{x})) \in -int\,\mathbb{R}^p_+$$

From the differentiability

$$dH_{\bar{x}}(\eta(x_0, \bar{x})) = \lim_{\lambda \to 0} \frac{1}{\lambda}[H_{\bar{x}}(exp_{\bar{x}}(\lambda\eta(x_0, \bar{x})) - H_{\bar{x}}(\bar{x})] \in -int\,\mathbb{R}^p_+$$

$$g(\bar{x}) + dg_{\bar{x}}(\eta(x_0, \bar{x})) = g(\bar{x}) + \lim_{\lambda \to 0} \frac{1}{\lambda}[g(exp_{\bar{x}}(\lambda\eta(x_0, \bar{x}))) - g(\bar{x})] \in -int\,\mathbb{R}^p_+$$

Since $-int\,\mathbb{R}^p_+$ is an open set, there exists some $0 < \lambda_0 < 1$ such that

$$\frac{1}{\lambda_0}[H_{\bar{x}}(exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))) - H_{\bar{x}}(\bar{x})] \in -int\,\mathbb{R}^p_+$$

$$g(\bar{x}) + \frac{1}{\lambda_0}[g(exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))) - g(\bar{x})] \in -int\,\mathbb{R}^p_+$$

By hypothesis, from $g(\bar{x}) \in -\mathbb{R}^p_+$, $F(\bar{x}, \bar{x}) = H_{\bar{x}}(\bar{x}) = 0$, and $\frac{1}{\lambda_0} > 1$, then

$$H_{\bar{x}}[exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))] \in -int\,\mathbb{R}^p_+ \quad \text{and} \quad g\left(exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))\right) \in -int\,\mathbb{R}^p_+$$

Since S_1 is a totally convex set

$$exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x})) \in S_1, \quad F\left(\bar{x}, exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))\right) \in -int\,\mathbb{R}^p_+$$

and

$$g\left(exp_{\bar{x}}(\lambda_0\eta(x_0, \bar{x}))\right) \in -int\,\mathbb{R}^p_+$$

Stands in contradiction with $\bar{x} \in S$ is a weakly efficient solution to the VEPC. Thus $(0, 0) \notin W$.

Step 2. We will prove that there exists a multiplier $v \in \mathbb{R}^p_+$. As W is an open set and the separation theorem holds (see Theorem 2.13 and Remark 2.14 in [17]) or [10]), there exists $(v, u) \neq (0, 0) \in \mathbb{R}^p \times \mathbb{R}^p$ such that

$$vy + uz > 0,\ \forall (y, z) \in W \tag{1}$$

Let $(y, z) \in W$ be a point then there exists $x \in S_1$ such that

$$y - dH_{\bar{x}}(\eta(x, \bar{x})) \in int\,\mathbb{R}^p_+, \quad z - [g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \in int\,\mathbb{R}^p_+$$

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And for every $c \in int\,\mathbb{R}^p_+$, $k \in int\,\mathbb{R}^p_+$, $t', t'' > 0$, we have $(y + t'c, z) \in W$ and $(y, z + t''k) \in W$. By (1), we have

$$v(y + t'c) + u(z) > 0,\ \forall c \in int\,\mathbb{R}^p_+,\ t' > 0$$

Then

$$vc > \frac{-uz - vy}{t'}$$

letting $t \to \infty$ we get $vc \geq 0$, $\forall c \in int\,\mathbb{R}^p_+$ and therefore $vc \geq 0$ for all $c \in \mathbb{R}^p_+$, that is $v \in \mathbb{R}^p_+$. In the same way, we can show that $u \in \mathbb{R}^p_+$.

Step 3. We will prove that $v \neq 0$, thus is, $v \in \mathbb{R}^p_+ \setminus \{0\}$. By reduction ad absurdum, if $v = 0$, from (1) we get

$$uz > 0,\ \forall (y, z) \in W$$

By assumption, there exists $x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -int\,\mathbb{R}^p_+$; thus, we have

$$(dH_{\bar{x}}(\eta(x_1, \bar{x})) + c, g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) + k) \in W,\ \forall c \in int\,\mathbb{R}^p_+ \quad \forall k \in int\,\mathbb{R}^p_+$$

Therefore from (1)

$$u[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) + k] > 0, \quad \forall k \in int\,\mathbb{R}^p_+$$

$$uk > -u[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x}))]$$

In particular, we have $[g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x}))] \in -int\,\mathbb{R}^p_+$, and if $k = 0$ then we get $u \cdot 0 = 0 > 0$. This is a contradiction, thus $v \neq 0$.

Step 4. We will prove that first KKT condition.

As

$$(dH_{\bar{x}}(\eta(x, \bar{x})) + c, g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x})) + k) \in W$$

for all $x \in S_1$, $c \in int\,\mathbb{R}^p_+$, $k \in int\,\mathbb{R}^p_+$. By (1), we obtain

$$v[dH_{\bar{x}}(\eta(x, \bar{x})) + c] + u[g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x})) - k] > 0$$

for all $x \in S_1$, $c \in int\,\mathbb{R}^p_+$, $k \in int\,\mathbb{R}^p_+$.

Letting $c \to 0$, $k \to 0$, we get

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + u[g(\bar{x}) + dg_{\bar{x}}(\eta(x, \bar{x}))] \geq 0, \quad \forall x \in S_1$$

Step 5. We will prove that second KKT condition.

It is clear

$$\left(dH_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'c, g(\bar{x}) + dg_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'k\right) \in W$$

for all $c \in int\,\mathbb{R}^p_+$, $k \in int\,\mathbb{R}^p_+$, $t' > 0$. By (1), we obtain

$$v[dH_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'c] + u[g(\bar{x}) + dg_{\bar{x}}(\eta(\bar{x}, \bar{x})) + t'k] = t'vc + ug(\bar{x}) + t'uk > 0$$

Letting $t' \to 0$, we obtain $ug(\bar{x}) \geq 0$. Nothing that $g(\bar{x}) \in -\mathbb{R}^p_+$ and $u \in \mathbb{R}^p_+$, we have that $ug(\bar{x}) \leq 0$ thus us

$$ug(\bar{x}) = 0$$

And therefore there exist $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ such that KKT conditions

$$vdH_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0,\ \forall x \in S_1$$

$$ug(\bar{x}) = 0$$

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hold.

□

Let's see the reciprocal of the previous theorem. To obtain it we need conditions of invexity.

Theorem 2. *[Sufficient KKT-conditions] Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $F : S_1 \times S_1 \to \mathbb{R}^p$, $g : S_1 \to \mathbb{R}^p$ be mappings. Let $F(\bar{x}, \bar{x}) = H(\bar{x}) = 0$. Assume that H and g are differentiable at $\bar{x} \in S$. H and g are $\mathbb{R}^p_+$-invex at $\bar{x}$ respect to η on S_1. If there exist $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ such that*

$$vdH_{\bar{x}}(\eta(x,\bar{x})) + udg_{\bar{x}}(\eta(x,\bar{x})) \geq 0, \ \forall x \in S_1 \tag{2}$$

$$ug(\bar{x}) = 0 \tag{3}$$

then $\bar{x}$ is a weakly efficient solution to the VEPC.

Proof. Since the mappings $H_{\bar{x}}$ and g are differentiable at $\bar{x} \in S$ and H and g are $\mathbb{R}^p_+$-invex at $\bar{x}$ respect to η on S_1 then

$$dH_{\bar{x}}(\eta(x,\bar{x})) \in H_{\bar{x}}(x) - H_{\bar{x}}(\bar{x}) - \mathbb{R}^p_+ = H_{\bar{x}}(x) - \mathbb{R}^p_+, \ \forall x \in S_1$$

$$dg_{\bar{x}}(\eta(x,\bar{x})) \in g(x) - g(\bar{x}) - \mathbb{R}^p_+, \forall x \in S_1$$

From $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ and (2) we obtain that

$$vH_{\bar{x}}(x) + u(g(x) - g(\bar{x})) = vdH_{\bar{x}}(\eta(x,\bar{x})) + udg_{\bar{x}}(\eta(x,\bar{x})) \geq 0, \quad \forall x \in S_1$$

By hypothesis (3), we get on the one hand

$$vH_{\bar{x}}(x) + ug(x) \geq 0, \quad \forall x \in S_1 \tag{4}$$

On the other hand, we will show that $\bar{x}$ is a weakly efficient solution to the VEPC. If not, then by definition there exists $y_0 \in S$ such that

$$F(\bar{x}, y_0) \in -int\, \mathbb{R}^p_+$$

From $v \in \mathbb{R}^p_+ \setminus \{0\}$ and the above statement, we have

$$vF(\bar{x}, y_0) < 0$$

Noticing $y_0 \in S$, we have $g(y_0) \in -\mathbb{R}^p_+$, so $ug(y_0) \leq 0$ because of $u \in \mathbb{R}^p_+$. Hence,

$$vF(\bar{x}, y_0) + ug(y_0) < 0$$

Stands in contradiction with (4) and therefore $\bar{x}$ is a weakly efficient solution to the VEPC. □

Remark 2. *Theorems 1 and 2 extend Theorem 3.1 in [12] on real Hausdorff topological vector spaces, Theorem 3.2 and Theorem 3.4 in [24] on real normed spaces and Theorem 3.1 and 3.3 in [8] on real Banach spaces to Hadamard manifolds.*

To sum up, we obtain the KKT optimality conditions for weakly efficient solutions to the vector equilibrium problems with constraints. This results are not only necessary but also sufficient.

4. Application

As a particular case of the results obtained in the previous section, we will obtain the optimality conditions of KKT for constrained vector optimization problems.

Let us consider the constrained multiobjective programming (CVOP) defined as:

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$$
\text{(CVOP)} \quad \begin{array}{l} \min f(x) \\ \text{subject to:} \\ \quad g(x) \leqq 0 \\ \quad x \in X \subseteq M \end{array}
$$

where $f = (f_1, \ldots f_p) : X \subseteq M \to \mathbb{R}^p$, with $f_i : X \subseteq M \to \mathbb{R}$ for all $i : 1, \ldots, p$, $g = (g_1, \ldots, g_m) : X \subseteq M \to \mathbb{R}^m$ are differentiable functions on the open set $X \subseteq M$ and let M be a Hadamard manifold.

As a consequence of the previous theorems and considering CVOP as a particular case of VEPC we have the KKT classical conditions.

Corollary 1. *Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $f, g : S_1 \to \mathbb{R}^p$ be mappings. Assume that f and g are differentiable at $\bar{x} \in S$. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\bar{x}) + dg_{\bar{x}}(\eta(x_1, \bar{x})) \in -int\, \mathbb{R}^p_+$. If $\bar{x}$ is a weakly efficient solution to the CVOP, then there exist $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ such that*

$$
vdf_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \ \forall x \in S_1
$$

$$
ug(\bar{x}) = 0
$$

Corollary 2. *Let S_1 be a nonempty open totally convex subset of Hadamard manifold M and let $f, g : S_1 \to \mathbb{R}^p$ be mappings. Assume that f and g are differentiable at $\bar{x} \in S$. Assume that f and g are differentiable at $\bar{x} \in S$ and f and g are $\mathbb{R}^p_+$-invex respect at $\bar{x}$ to η on S_1. If there exist $v \in \mathbb{R}^p_+ \setminus \{0\}$, $u \in \mathbb{R}^p_+$ such that*

$$
vdf_{\bar{x}}(\eta(x, \bar{x})) + udg_{\bar{x}}(\eta(x, \bar{x})) \geq 0, \ \forall x \in S_1 \tag{5}
$$

$$
ug(\bar{x}) = 0 \tag{6}
$$

then $\bar{x}$ is a weakly efficient solution to the CVOP.

Proof. The demonstrations are similar to those already shown without further consider CVOP as particular cases of VEPC just by taking $F(x, y) = \max_{i=1,\ldots,p}[f_i(y) - f_i(x)], \forall x, y \in M$. □

Remark 3. *Corollaries 1 and 2 extend Theorem 4.4 in [12] on real Hausdorff topological vector spaces and Corollary 3.3 in [8] on real Banach spaces to Hadamard manifolds and these results coincide with Corollary 3.8 given by Ruiz-Garzón et al. [22].*

We illustrate the previous results with some examples:

Example 2. *[22] Let us consider the set $Pos_2(\mathbb{R})$ of positive definite 2×2 matrices and the $Sym_2(\mathbb{R})$ of symmetric 2×2 matrices endowed with the Frobenius metric $k_X(U, V) = trace(UV)$ where $X \in Pos_2(\mathbb{R})$ and $U, V \in T_X Pos_2(\mathbb{R}) = Sym_2(\mathbb{R})$. Consider the following problem on $Pos_2(\mathbb{R})$:*

Consider the CVOP:

$$
\text{(CVOP)} \quad \begin{array}{l} Max\ f(X) = (f_1, f_2)(X) = (x_1, x_3) \\ \text{subject to:} \end{array}
$$

$$
g_1(X) = x_2 + x_3 - 7 \leq 0
$$

$$
g_2(X) = -x_1 + 1 \leq 0
$$

$$
X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in Pos_2(\mathbb{R})
$$

Given

$$\overline{X} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

using the Riemannian metric k and f, g *is* $\mathbb{R}^2_+$*-invex at* $\bar{x}$ *respect to* $\eta(X, \overline{X}) = X - \overline{X}$ *and there exists* $Q = \eta(X, \overline{X}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *we have that*

$$df_{1(\overline{X})}(Q) = trace\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = trace\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$df_{2(\overline{X})}(Q) = trace\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = trace\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

The

$$dg_{1(\overline{X})}(Q) = trace\left[\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = trace\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

$$dg_{2(\overline{X})}(Q) = trace\left[\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = trace\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

We have

$$df_{\overline{X}}(Q) = (df_{1(\overline{X})}(Q), df_{2(\overline{X})}(Q)) = (0, 1)$$

$$dg_{\overline{X}}(Q) = (dg_{1(\overline{X})}(Q), dg_{2(\overline{X})}(Q)) = (1, 0)$$

and therefore there exist $v = (1, 0)$ *and* $u = (0, 1)$ *such that*

$$vdf_{\bar{x}}(Q) + udg_{\bar{x}}(Q) = 0$$

$$ug(\bar{x}) = 0$$

and then $\overline{X}$ *is a weakly efficient solution to the CVOP.*

Example 3. *Let us consider the set* $\Omega = \{p = (p_1, p_2) \in \mathbb{R}^2 : p_2 > 0\}$*. Let G be a 2x2 matrix defined by* $G(p) = (g_{ij}(p))$ *with*

$$g_{11}(p) = g_{22}(p) = \frac{1}{p_2^2}, \quad g_{12}(p) = g_{21}(p) = 0$$

Endowing Ω *with the Riemannian metric* $\ll u, v \gg = < G(p)u, v >$*, we obtain a complete Riemannian manifold* $\mathbb{H}^2$*, namely, the upper half-plane model of Hyperbolic space and grad* $f(p) = G(p)^{-1}\nabla f(p)$*.*

Consider the CVOP:

$$(CVOP) \qquad Min f(p) = (f_1, f_2)(p) = (p_1, \ln p_2)$$

subject to:

$$g_1(p) = 2p_1 - 2 \geq 0$$

$$g_2(p) = p_2 - 1 \geq 0$$

Given $\overline{p} = (1, 1)$ *using the Riemannian metric k and* f, g *is* $\mathbb{R}^2_+$*-invex at* $\bar{p}$ *respect to* $\eta(p, \overline{p}) = 2p - \overline{p}$ *and there exists* $q = \eta(p, \overline{p}) = (0, 1)$ *we have that*

$$df_{1(\overline{p})}(q) = < \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >= (p_2^2, 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$df_{2(\overline{p})}(q) =< \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ p_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >= (0, p_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_2$$

The

$$dg_{1(\overline{p})}(q) =< \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >= (2p_2^2, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$dg_{2(\overline{p})}(q) =< \begin{pmatrix} p_2^2 & 0 \\ 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >= (0, p_2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_2^2$$

We have

$$df_{\overline{p}}(q) = (df_{1(\overline{p})}(q), df_{2(\overline{p})}(q)) = (0, p_2)$$

$$dg_{\overline{p}}(q) = (dg_{1(\overline{p})}(q), dg_{2(\overline{p})}(q)) = (0, p_2^2)$$

and therefore there exist $v = u = (1, 0)$*such that*

$$vdf_{\bar{x}}(q) + udg_{\bar{x}}(q) = 0$$

$$ug(\bar{x}) = 0$$

and then $\overline{p}$ *is a weakly efficient solution to the CVOP.*

5. Conclusions

In conclusion, we have shown the existence of KKT optimality conditions for weakly efficient solutions to the equilibrium vector problems with constraints on Hadamard manifolds, in particular, to constrained vector optimization problems. Substituting the segments by geodesics, this has implied:

- The need for an extension of the concept of convex set to that of totally convex.
- The use of an adequate definition of differential function in similar terms to those of a directional derivatives in Euclidean space using exponential Riemannian map.
- Generalizing the invexity definition by extending its classical definition given by Hanson [14] in order to obtain sufficient optimality conditions.

Thus, our study provides evidence of the logical continuity of the KKT formulation when extended to other contexts different from Banach spaces or norms given in the literature by Gong [12] and Wei and Gong [24] and Feng and Qiu [8]. Finally, it would be interesting to continue the studies in this order to go further and consider other type of solutions to the vector equilibrium problem with constraints.

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