Article

Effect of Geometric Nonlinearity on Aerodynamic Stability of Membrane Roofs

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Abstract: Membrane materials are most widely applied in construction engineering with small mass and high flexibility, it presents strong geometric nonlinearity in the process of vibration. In the paper, an improved multi-scale perturbation method is used to solve the aeroelastic stability of closed and open membrane roofs for quantify the effect of geometric nonlinearity on the single-mode aeroelastic instability wind speed of membrane roofs. The results show that the critical wind speed values of the two models are small when the geometrical nonlinearity of membrane material is neglected. In addition, under normal wind load, the influence of geometrical nonlinearity of membrane on the aerodynamic stability of roof can be neglected, However, under strong wind load, when the roof deformation reaches 3% of the span, the influence of geometric nonlinearity should be considered and the influence increases with the decrease of transverse and downwind span of membrane roof. The results obtained in this paper have important theoretical reference value for the design the membrane structures.

Keywords: geometric nonlinearity; improved multis-cale method; orthotropic membrane; aeroelastic instability

1. Introduction

Fabric membrane is the most widely used membrane material in construction engineering. It has the characteristics of high tensile strength and good flexibility. Fabric membranes are mainly composed of substrates and coatings. The substrates are usually braided by orthogonal fibers, which results in the orthotropic properties of the membranes, That is to say, the elastic modulus and Poisson's ratio in the two orthogonal directions are different. The building which is made up of membrane material covered on the structure skeleton or tensioned as a whole has beautiful appearance, good transparency, environmental protection and energy saving [1, 2]. Therefore, it is widely used in large-scale stadiums, exhibition venues and other public buildings. Because of the small mass and flexibility , it is easy to vibration under external disturbance. and the stiffness of the membrane material is small, which results in the large vibration deformation of the membrane structure under wind load, showing strong geometric non-linearity. Many research results shows that the single-mode aeroelastic instability can easily occur in membrane structures when the pre-tension of membrane materials is small [3, 4].

In the mathematical analysis of aeroelastic instability of flexible membrane structures, Yang et al. [5, 6] established the wind-induced dynamic coupling equation of hyperbolic parabolic membrane roofs with small sag by using elastic shallow shell theory and ideal fluid potential flow theory in 2006, and determined the critical wind speed of aeroelastic instability according to Routh-Hurwitz stability criterion. The influence of geometric nonlinearity of membranes was not taken into account when establishing the mathematical model. In 2011, Zheng et al. [7, 8] studied the non-linear aerodynamic stability of orthotropic tensioned membrane structures in rectangular

plane and hyperbolic paraboloid respectively. The critical wind velocities of single-mode instability of two membrane structures were determined by assuming the solution of vibration equation. In 2017, Liu et al. [9] studied the aerodynamic stability of closed tensioned membrane structures by Galerkin method. The geometric nonlinearity of membrane vibration is weakened, and the critical wind speed of instability is obtained by using the weak nonlinearity solution method.

In order to investigate the influence of geometric nonlinearity on the aeroelastic stability of membrane materials, the nonlinear wind-induced dynamic equations of membrane roofs are established based on Von Kamen's large deflection theory and Darumbel's principle, taking the flat rectangular orthotropic tensioned membrane roofs with fixed supports on the four sides of open and closed structures as analytical models and considering the effects of geometric nonlinearity and air damping of membrane materials. An improved multi-scale method which suitable for strong geometric nonlinearity is used to solve the vibration equation. The critical wind speed of instability obtained is compared with the results without considering geometric nonlinearity. The effect of geometric nonlinearity on the wind speed of single-mode aeroelastic instability of membrane material is obtained quantitatively.

2. Analytical deduction of single-mode instability of orthotropic membrane aeroelasticity

2.1. Establishment of Basic Equations

Let the length and width of the orthotropic rectangular flexible membrane with four sides fixed be a and b respectively; the pre-tension along the length direction is N_{0x} and the width direction is N_{0y}. The wind blows parallel to the roof and toward the membrane surface, which makes the membrane surface vibrate. For flexible membranes, the research results show that the shear stress has little influence on the vibration process of the membranes and can be considered as zero[10]. Assuming that the planar membrane is in the xoy plane when equilibrium, and the pre-tension in the x direction is N_x and in the Y direction is N_y, When the membrane is disturbed by external forces perpendicular to the xoy plane, it will deform and then produce transverse vibration perpendicular to the membrane surface under the action of tension. Taking a vibrating micro-units on the vibrating membrane surface as shown in Figure 1.

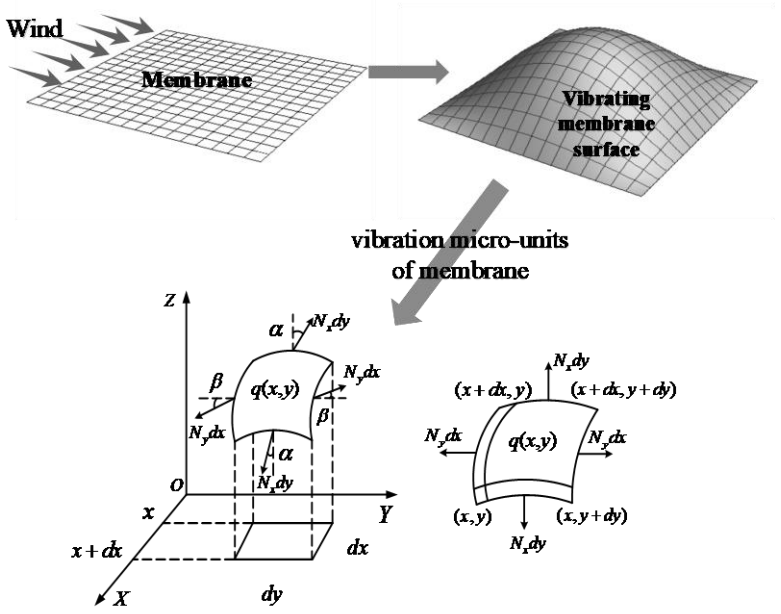

Figure 1. Vibration micro-units of membrane

Taking the element $dxdy$ on the membrane surface. When the micro-facets are deformed, the edges of the micro-facets are subjected to the tension of the adjacent facets. In the

X-direction, we can regard the surface element as composed of countless chord elements with length dx and width of one unit. The tension acting on the chord element is consistent with its tangent direction. The tension N_x is at an angle α with the x coordinate axis. Therefore, the vertical component of the tension acting on the chord element at one end of X is $N_x \sin\alpha$. because α is small, $\sin\alpha \approx \tan\alpha$. Let w be the vertical displacement of a point on the membrane away from the equilibrium position. Therefore:

$$N_x \sin\alpha = N_x \tan\alpha = N_x \left(\frac{\partial w}{\partial x}\right)_x \tag{1}$$

The vertical force acting on the dy edge is: $N_x \left(\frac{\partial w}{\partial x}\right)_x dy$; and the vertical force at the x edge should be: $N_x \left(\frac{\partial w}{\partial x}\right)_{x+dx} dy$. Thus, the resultant force in the vertical direction on the x and x+dx sides of the panel is as follows:

$$N_x \left(\frac{\partial w}{\partial x}\right)_{x+dx} dy - N_x \left(\frac{\partial w}{\partial x}\right)_x dy = N_x \frac{\partial^2 w}{\partial x^2} dxdy \tag{2}$$

Similarly, the resultant force of the vertical component of the tension acting on the Y direction can be obtained as follows:

$$N_y \left(\frac{\partial w}{\partial y}\right)_{y+dy} dx - N_y \left(\frac{\partial w}{\partial y}\right)_y dx = N_y \frac{\partial^2 w}{\partial y^2} dxdy \tag{3}$$

So the total vertical force acting on the whole panel is:

$$F_z = N_x \frac{\partial^2 w}{\partial x^2} dxdy + N_y \frac{\partial^2 w}{\partial y^2} dxdy + +q(x,y)dxdy \tag{4}$$

Where, N_x is the tension in the x direction (longitude), N_y is the tension in the y direction (latitude), w is the deflection of the membrane, and $q(x,y)$ is the external load acting on the unit area of the projection surface of the membrane. According to the force balance, we can obtained that:

$$N_x \frac{\partial^2 w}{\partial x^2} dxdy + N_y \frac{\partial^2 w}{\partial y^2} dxdy + q(x,y)dxdy = 0 \tag{5}$$

$$q(x,y) + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \tag{6}$$

The generalized external loads of flexible membrane roof under wind load include the wind load acting on the membrane surface, structural damping force and inertial force [10]. If the aerodynamic term is defined as p, then the generalized external load per unit area $q(x,y)$ is:

$$q(x,y) = p(x,y,t) - 2\rho c \frac{\partial w(x,y,t)}{\partial t} - \rho \frac{\partial^2 w(x,y,t)}{\partial t^2} \tag{7}$$

For membrane material, the stiffness of membrane surface comes from the initial pre-tension of membrane material, so the initial pre-tension should be added in formula 6. Finally, the differential equations of motion of flexible membranes are obtained as follows:

$$p(x,y,t)-2\rho c\frac{\partial w(x,y,t)}{\partial t}-\rho\frac{\partial^2 w(x,y,t)}{\partial t^2}+\left(N_{0x}+N_{xt}\right)\frac{\partial^2 w}{\partial x^2}+\left(N_{0y}+N_{yt}\right)\frac{\partial^2 w}{\partial y^2}=0 \quad (8)$$

Introducing the stress function $\varphi(x,y)$, $N_x=h\frac{\partial^2\varphi}{\partial y^2}$, $N_y=h\frac{\partial^2\varphi}{\partial x^2}$, Then equation (6) becomes:

$$\left(N_{0x}+h\frac{\partial^2\varphi}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2}+\left(N_{0y}+h\frac{\partial^2\varphi}{\partial x^2}\right)\frac{\partial^2 w}{\partial y^2}+p-2\rho c\frac{\partial w}{\partial t}-\rho\frac{\partial^2 w}{\partial t^2}=0 \quad (9)$$

After deformation, the membrane surface strain is composed of linear and non-linear parts. The linear strain is caused by in-plane displacement u and v, and the non-linear strain is caused by deflection w. After ignoring shear stress, the total strain is as follows:

$$\left.\begin{aligned}\varepsilon_x&=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y&=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\end{aligned}\right\} \quad (10)$$

Where, ε_x is the strain in the X direction, ε_y is the strain in the Y direction.

By eliminating u and v in equation (10), the continuous deformation conditions satisfying the strain and deflection of the film surface can be obtained.

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \quad (11)$$

The membrane is orthotropic, and the direction of the fiber is the main direction of elasticity, so that it is consistent with the direction of coordinate system X and Y. Assuming that the direction of fiber is the same as the direction of coordinate system X and Y. The Young's modulus of elasticity in X and Y directions is E_1 and E_2, respectively. The longitudinal Poisson's ratio and the latitudinal Poisson's ratio is μ_1 and μ_2, respectively. The relationship between elastic modulus and Poisson's ratio is as follows.

$$\frac{\mu_1}{E_1}=\frac{\mu_2}{E_2} \quad (12)$$

The stress-strain relationship is as follows:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix}=\begin{pmatrix}\frac{E_1}{1-\mu_1\mu_2} & \frac{\mu_1E_2}{1-\mu_1\mu_2}\\ \frac{\mu_2E_1}{1-\mu_1\mu_2} & \frac{E_2}{1-\mu_1\mu_2}\end{pmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix} \quad (13)$$

Where σ_x and σ_y is the normal stresses in the X direction and Y direction respectively. h is the thickness of membrane.

Letting $N_x=h\cdot\sigma_x$, $N_y=h\cdot\sigma_y$, substituting Equation (13) into Equation (11), the compatibility equation is obtained as follows:

$$\frac{1}{E_1 h}\frac{\partial^2 N_x}{\partial y^2}-\frac{\mu_2}{E_2 h}\frac{\partial^2 N_y}{\partial y^2}-\frac{\mu_1}{E_1 h}\frac{\partial^2 N_x}{\partial x^2}+\frac{1}{E_2 h}\frac{\partial^2 N_y}{\partial x^2} =\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{14}$$

By substituting the stress function into Equation (14), it can be transformed into:

$$\frac{1}{E_1}\frac{\partial^4 \varphi}{\partial y^4}-\frac{\mu_2}{E_2}\frac{\partial^4 \varphi}{\partial x^2\partial y^2}-\frac{\mu_1}{E_1}\frac{\partial^4 \varphi}{\partial x^2\partial y^2}+\frac{1}{E_2}\frac{\partial^4 \varphi}{\partial x^4} =\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{15}$$

2.2. Modified multiscale solutions of governing equations

The initial surface function of rectangular planar membrane $z_0(x,y)=0$, then the surface equation of flexible membrane under wind load is as follows:

$$z(x,y,t)=w(x,y,t) \tag{16}$$

According to the Bubnov-Galerkin method, assuming the solution of the governing equation is [9, 10].

$$\begin{cases} w(x,y,t)=\sum_{i=1}^{n} T_i(t)W_i(x,y) \\ \varphi(x,y,t)=\sum_{i=1}^{n} U_i(t)\phi_i(x,y) \end{cases} \tag{17}$$

Where $W_i(x,y)$ is the mode function, $\phi_i(x,y)$ is the unknown stress function about the coordinates, $T_i(t)$ and $U_i(t)$ are the time-dependent function.

Because the membrane is fixed on four sides, the vertical deflection at the boundary of the membrane is zero, and the vibration mode function satisfying the conditions is assumed to be:

$$W(x,y)=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{18}$$

where m and n are positive integer, which denote sinusoidal half wave number.

substituting Equation (18) into Equation (17), the following equation is obtained:

$$w(x,y,t)=T(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{19}$$

Substituting Equation (19) into Equation (15), yields:

$$\frac{1}{E_1}\frac{\partial^4 \varphi}{\partial y^4}+\frac{1}{E_2}\frac{\partial^4 \varphi}{\partial x^4}=\frac{m^2n^2\pi^4}{2a^2b^2}T^2(t)\left(\cos\frac{2m\pi x}{a}+\cos\frac{2n\pi y}{b}\right) \tag{20}$$

Assuming that the solution of stress function in formula (20) is

$$\begin{cases} \varphi(x,y,t)=T^2(t)\phi(x,y) \\ \phi(x,y)=\alpha\cos\frac{2m\pi x}{a}+\beta\cos\frac{2n\pi y}{b} \end{cases} \tag{21}$$

substituting Equation (21) into Equation (20), yields:

$$\alpha = \frac{E_2 a^2 n^2}{32 b^2 m^2}, \qquad \beta = \frac{E_1 b^2 m^2}{32 a^2 n^2} \tag{22}$$

2.1.1. Solution of Flexible Membrane Roof on Closed Structure

Flexible membrane covers the top of closed structure as roof, and rigid wall around the structure as vertical bearing member. Because its stiffness is far greater than the membrane's stiffness, it is assumed that the stiffness of vertical component is infinite in the process of theoretical derivation in this paper. The membrane roof on closed structure is shown in Figure 2.

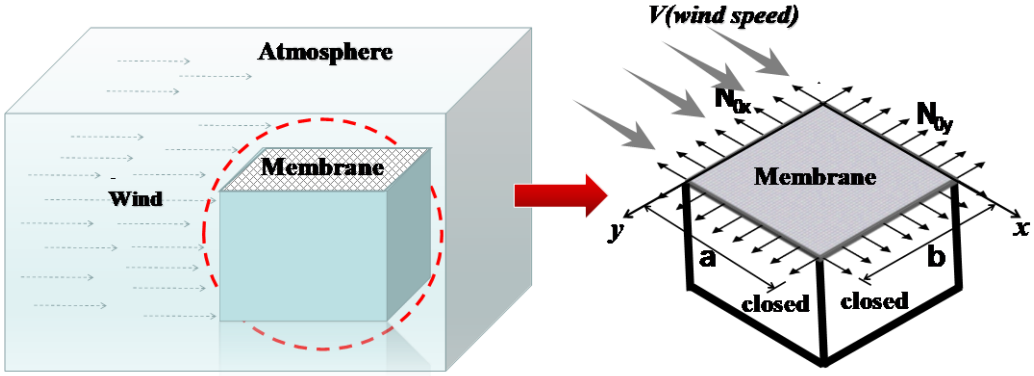

Figure 2. The membrane roof on closed structure

For the membrane roof on closed structure, the aerodynamic force acting on the unit area of the membrane projection surface is expressed as follows[8]:

$$p = -\frac{\rho_0}{2\pi}\left(-V \iint_{Ra} \frac{\left(V\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \right)_{\substack{x=\xi \\ y=\eta}} (x-\xi)}{\left(\sqrt{(x-\xi)^2 + (y-\eta)^2} \right)^3} d\xi d\eta + \iint_{Ra} \frac{\left(V\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 w}{\partial t^2} \right)_{\substack{x=\xi \\ y=\eta}}}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta \right) \tag{23}$$

Substituting Equation (23) into Equation (9), yields:

$$\begin{aligned} &(h\frac{\partial^2 \varphi}{\partial y^2} + N_{0x})\frac{\partial^2 w}{\partial x^2} + (h\frac{\partial^2 \varphi}{\partial x^2} + N_{0y})\frac{\partial^2 w}{\partial y^2} - 2\rho c\frac{\partial w}{\partial t} \\ &-\rho\frac{\partial^2 w}{\partial t^2} - \frac{\rho_0}{\pi}\iint_{Ra}\frac{1}{r}\left(\frac{\partial^2 w}{\partial t^2}\right)_{\substack{x=\xi \\ y=\eta}} d\xi d\eta - \frac{\rho_0 V}{2\pi}\iint_{Ra}\frac{1}{r}\left(\frac{\partial^2 w}{\partial x \partial t}\right)_{\substack{x=\xi \\ y=\eta}} d\xi d\eta \\ &+\frac{\rho_0 V^2}{2\pi}\iint_{Ra}\frac{1}{r^3}\left(\frac{\partial w}{\partial x}\right)_{\substack{x=\xi \\ y=\eta}} (x-\xi) d\xi d\eta + \frac{\rho_0 V}{2\pi}\iint_{Ra}\frac{1}{r^3}\left(\frac{\partial w}{\partial t}\right)_{\substack{x=\xi \\ y=\eta}} (x-\xi) d\xi d\eta = 0 \end{aligned} \tag{24}$$

Where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, the integral region $Ra \in \{0 \le \xi \le a, 0 \le \eta \le b\}$.

Substituting Equations (19), (21) and (22) into Equation (24), yields:

$$\begin{aligned} &\left(\rho W + \frac{\rho_0}{\pi}\gamma_1\right)\frac{d^2 T(t)}{dt^2} + \left[\frac{\rho_0 V}{2\pi}(\gamma_2 - \gamma_4) + 2\rho c W\right]\frac{dT(t)}{dt} \\ &-\left(N_{0x}\frac{\partial^2 W}{\partial x^2} + N_{0y}\frac{\partial^2 W}{\partial y^2} + \frac{\rho_0 V^2}{2\pi}\gamma_3\right)T(t) - h\left(\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)T^3(t) = 0 \end{aligned} \tag{25}$$

Where:

$$\gamma_1 = \iint_{Ra} \frac{1}{r}(W)_{\substack{x=\xi \\ y=\eta}} d\xi d\eta = \iint_{Ra} \frac{1}{r} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta$$

$$\gamma_2 = \iint_{Ra} \frac{1}{r}\left(\frac{\partial W}{\partial x}\right)_{\substack{x=\xi \\ y=\eta}} d\xi d\eta = \frac{m\pi}{a} \iint_{Ra} \frac{1}{r} \cos\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta$$

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$$\gamma_3 = \iint_{Ra} \frac{1}{r^3}\left(\frac{\partial W}{\partial x}\right)_{\substack{x=\xi \\ y=\eta}} (x-\xi) d\xi d\eta = \frac{m\pi}{a} \iint_{Ra} \frac{1}{r^3}(x-\xi) \cos\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta$$

$$\gamma_4 = \iint_{Ra} \frac{1}{r^3}(W)_{\substack{x=\xi \\ y=\eta}} (x-\xi) d\xi d\eta = \iint_{Ra} \frac{1}{r^3}(x-\xi) \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta$$

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Using Bubnov-Galerkin method to integral Equation (25), yields:

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$$\iint_S \left\{ \begin{array}{l} \left(\rho W + \frac{\rho_0}{\pi}\gamma_1\right)\frac{d^2T(t)}{dt^2} + \left[\frac{\rho_0 V}{2\pi}(\gamma_2 - \gamma_4) + 2\rho c W\right]\frac{dT(t)}{dt} \\ -\left(N_{0x}\frac{\partial^2 W}{\partial x^2} + N_{0y}\frac{\partial^2 W}{\partial y^2} + \frac{\rho_0 V^2}{2\pi}\gamma_3\right)T(t) - h\left(\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)T^3(t) \end{array} \right\} W(x,y) dxdy = 0 \quad (26)$$

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179 Where $S \in \{0 \le x \le a, 0 \le y \le b\}$.

Simplifing Formula (26), yields:

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$$A\frac{d^2T(t)}{dt^2} + B\frac{dT(t)}{dt} - CT(t) - DT^3(t) = 0 \quad (27)$$

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Where:

$$A = \iint_S \left(\rho W + \frac{\rho_0}{\pi}\gamma_1\right) W dxdy$$

$$= \rho \iint_S \left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 dxdy + \frac{\rho_0}{\pi}\iint_{Ra}\left(\iint_{Ra}\frac{1}{r}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

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$$= \frac{\rho ab}{4} + \frac{\rho_0}{\pi}\alpha_1$$

$$\alpha_1 = \iint_S \left(\iint_{Ra}\frac{1}{r}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$B = \frac{\rho_0 V}{2\pi}\iint_S (\gamma_2 - \gamma_4) W dxdy + 2\rho c \iint_S W^2 dxdy$$

$$= \frac{\rho_0 mV}{2a}\iint_S \left(\iint_{Ra}\frac{1}{r}\cos\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

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$$-\frac{\rho_0 V}{2\pi}\iint_S \left[\iint_{Ra}\frac{1}{r^3}(x-\xi)\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$+2\rho c\iint_S \left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 dxdy$$

$$= \frac{\rho_0 mV}{2a}\alpha_2 - \frac{\rho_0 V}{2\pi}\alpha_4 + \frac{\rho cab}{2}$$

$$\alpha_2 = \iint_S \left(\iint_{Ra} \frac{1}{r} \cos\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta \right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy$$

$$\alpha_4 = \iint_S \left[\iint_{Ra} \frac{1}{r^3} (x-\xi) \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta \right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy$$

$$C = \iint_S \left(N_{0x} \frac{\partial^2 W}{\partial x^2} + N_{0y} \frac{\partial^2 W}{\partial y^2} + \frac{\rho_0 V^2}{2\pi} \gamma_3 \right) W dxdy$$

$$= \iint_S N_{0x} \frac{\partial^2 W}{\partial x^2} W dxdy + \iint_{Ra} N_{0y} \frac{\partial^2 W}{\partial y^2} W dxdy + \frac{\rho_0 V^2}{2\pi} \iint_{Ra} \gamma_3 W dxdy$$

$$= -\frac{m^2\pi^2 b N_{0x}}{4a} - \frac{n^2\pi^2 a N_{0y}}{4b} + \frac{\rho_0 m V^2}{2a} \alpha_3$$

$$\alpha_3 = \iint_S \left(\iint_{Ra} \frac{1}{r^3} (x-\xi) \cos\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} d\xi d\eta \right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy$$

$$D = \iint_S h \left(\frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) W dxdy$$

$$= -\frac{hm^2 n^2 \pi^4 (\alpha + \beta)}{2ab}$$

It can be obtained from numerical calculation that only when $b/a \ll 0.1$, $A \le 0$; this will not happen in practical engineering.Letting $u = B/(\varepsilon A)$, $\omega_0^2 = -C/A$, $\varepsilon = -D/A$. Then the equation (27) is transformed into:

$$\ddot{T} + \omega_0^2 T + \varepsilon\left(\mu\dot{T} + T^3\right) = 0 \tag{28}$$

Letting ω is the vibration frequency of membrane material and expanding ω^2 to the power series of ε near ω_0^2 as follows:

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{29}$$

The transformation parameters are introduced as follows.

$$\alpha = \frac{\varepsilon\omega_1}{\omega_0^2 + \varepsilon\omega_1} \tag{30}$$

$$\varepsilon = \frac{\omega_0^2 \alpha}{\omega_1(1-\alpha)}$$

$$\omega_0^2 + \varepsilon\omega_1 = \frac{\omega_0^2}{1-\alpha} \tag{31}$$

Expanding ω^2 to the power series of ε as follows:

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 \left[1 + \frac{1}{\omega_0^2 + \varepsilon\omega_1} \left(\varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots \right) \right] = \frac{\omega_0^2}{1-\alpha} \left(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots \right) \tag{32}$$

$$\omega = \omega_0 \left[1 + \frac{1}{2}\alpha + \left(\frac{3}{8} + \frac{\delta^2}{2}\right)\alpha^2 + \cdots \right] \tag{33}$$

The form of perturbation solution of the equation (28) can be:

$$T(t,\alpha) = T_0(t_0,t_1) + \alpha T_1(t_0,t_1) + \alpha^2 T_2(t_0,t_1) + \cdots \tag{34}$$

Where $t_0 = t$, $t_1 = \alpha t$.

The differential operators are obtained as follows.

$$\frac{d}{dt} = D_0 + \alpha D_1 + \alpha^2 D_2 + ..., \qquad \frac{d^2}{dt^2} = D^2{}_0 + 2\alpha D_0 D_1 + \alpha^2 (D_1^2 + 2D_0 D_2) + ... \tag{35}$$

Substituting Equations (30), (31) , (32) and (34) into Equation (28), yields:

$$\begin{aligned} &(1-\alpha)\left[D^2{}_0 + 2\alpha D_0 D_1 + \alpha^2 (D_1^2 + 2D_0 D_2)\right](T_0 + \alpha T_1 + \alpha^2 T_2 + \cdots) \\ &+(1-\alpha)\omega_0^2 (T_0 + \alpha T_1 + \alpha^2 T_2 + \cdots) + \frac{\alpha\omega_0^2}{\omega_1^2}\left[\begin{matrix}(T_0 + \alpha T_1 + \alpha^2 T_2 + \cdots)^3 + (D_0 + \alpha D_1 + \alpha^2 D_2)\cdot \\ (T_0 + \alpha T_1 + \alpha^2 T_2 + \cdots)\end{matrix}\right] \\ &= 0 \end{aligned} \tag{36}$$

$$\begin{aligned} &\alpha^0 \quad D_0^2 T_0 + \omega_0^2 T_0 = 0 \\ &\alpha^1 \quad D_0^2 T_1 + \omega_0^2 T_1 + 2D_0 D_1 T_0 + \frac{\omega_0^2}{\omega_1}(D_0 T_0 + T_0^3) = 0 \\ &\alpha^2 \quad D_0^2 T_2 + T_2 + 2D_0 D_1 T_1 + (D_1^2 + 2D_0 D_2) T_0 + \frac{\omega_0^2}{\omega_1} 3T_0^2 T_1 = 0 \end{aligned} \tag{37}$$

The solution of the first equation in the system of equations (37) can be as follows:

$$T_0 = A(t_1)e^{i\omega_0 t_0} + \bar{A}(t_1)e^{-i\omega_0 t_0} \tag{38}$$

Substituting Equation (38) into The Second Equation in Equation (37), yields:

$$D_0^2 T_1 + \omega_0^2 T_1 + \left(2i\omega_0 D_1 A + 3\frac{\omega_0^2}{\omega_1} A^2 \bar{A} + i\mu\frac{\omega_0^3}{\omega_1} A\right)e^{it_0} + \frac{1}{\omega_1} A^3 e^{3it_0} + cc = 0 \tag{39}$$

Where cc is the conjugate complex term. Eliminating the term of immortality in equation (39) , yields:

$$2D_1 A + 3\frac{\omega_0}{\omega_1} A^2 \bar{A} + i\mu\frac{\omega_0^3}{\omega_1} A = 0 \tag{40}$$

Solving equation (40), we can obtain:

$$T_1 = \frac{1}{8\omega_1}(A^3 e^{3it_0} + \bar{A}^3 e^{-3it_0}) \tag{41}$$

Letting $A = \frac{1}{2} f e^{i\phi}$, bring it into equation (41) and separating the imaginary part from the real part.

$$\frac{df}{dt_1} = -\frac{1}{2}\mu\frac{\omega_0^2}{\omega_1} f, \quad f\frac{d\phi}{dt_1} = -\frac{3\omega_0}{8\omega_1} f^3 \tag{42}$$

Substituting $A = \frac{1}{2} f e^{i\phi}$ into Equation (38), yields:

$$T_0 = f\cos(\omega_0 t_0 + \phi) = f\cos(\omega t + \phi_0) \tag{43}$$

Comparing the angular frequency in equation (33) with equation (43), we can get the equation under the first order approximation condition as follows.

$$\frac{d\phi}{dt_1} = \frac{\omega_0}{2} \tag{44}$$

According to the equations (42) and (43), yields:

$$\omega_1 = \frac{3}{4} f^2 \tag{45}$$

Substituting Equation (45) into Equation (42) and omitting higher order terms, yields:

$$\omega = \sqrt{\omega_0^2 + \frac{3}{4}\varepsilon f^2} \tag{46}$$

The results show that when the critical wind speed of single mode instability is reached, the frequency of the characteristic equation of the system approaches zero, which is equivalent to static equilibrium instability [9]. That is:

$$\omega_0^2 + \frac{3}{4}\varepsilon f^2 = 0 \tag{47}$$

Where f is the vibration amplitude of membrane,

$$\omega_0^2 = \frac{2\pi}{\rho ab\pi + 4\rho_0\alpha_1}\cdot\left(\frac{m^2\pi^2b^2N_{0x} + n^2\pi^2a^2N_{0y}}{2ab} - \frac{\rho_0 m\alpha_3}{a}V^2\right)$$

$$\alpha_1 = \iint_S\left(\iint_{Ra}\frac{1}{r}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$\alpha_3 = \iint_S\left(\iint_{Ra}\frac{1}{r^3}(x-\xi)\cos\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$\varepsilon = \frac{2hm^2n^2\pi^5(\alpha+\beta)}{ab\left(\rho ab\pi + 4\rho_0\alpha_1\right)}$$

Solving the equation, the critical wind speed for single mode instability of closed membrane roof is obtained as follows:

$$V_{cr} = \sqrt{\frac{4\pi\left(m^2\pi^2b^2N_{0x} + n^2\pi^2a^2N_{0y}\right) + 3\varepsilon f^2\left(\rho\,\pi a^2b^2 + 4\rho_0 ab\alpha_1\right)}{8\pi b\rho_0 m\alpha_3}} \tag{48}$$

where f is the vibration amplitude corresponding to the instability critical wind speed. It shows that the critical wind speed is related to the vibration amplitude. Considering the influence of geometric nonlinearity of membrane, the stiffness of membrane will change with the change of amplitude in the process of vibration, which will affect the aerodynamic stability of membrane roof to a certain extent, which is consistent with the conclusions of previous research. When $f \to 0$, the critical wind speed of instability can be obtained according to the theory of small deflection.

2.1.2. Solution of Flexible Membrane Roof on Open Structure

For membrane roofs open open structures, air flows from both sides of the membrane surface due to the smaller thickness of the membrane, which can be approximately determined by the thin-airfoil theory[ ].The membrane roof on open structure is shown in Figure 2.

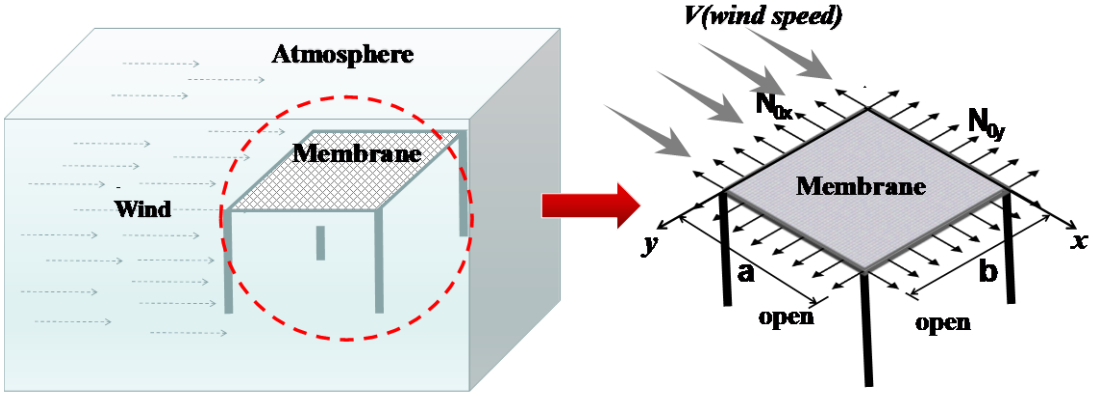


Figure 3. The membrane roof on open structure

For the membrane roof on open structure, the aerodynamic force acting on the unit area of the membrane projection surface is expressed as follows[7]:

$$p = \rho_0 \frac{\partial}{\partial t}\int_0^x \gamma_c(\xi, y, t)\, d\xi + \rho_0 V \gamma_c \tag{49}$$

Where γ_c is the density of vortices.

$$\gamma_c = aV\gamma_j = aV\left(a_{1j}T(t) + a_{2j}\frac{T'(t)}{V}\right), \qquad (j = 1, 2, \cdots M \times N) \tag{50}$$

Substituting Equation (50) into Equation (9) , yields:

$$(h\frac{\partial^2 \varphi}{\partial y^2} + N_{0x})\frac{\partial^2 w}{\partial x^2} + (h\frac{\partial^2 \varphi}{\partial x^2} + N_{0y})\frac{\partial^2 w}{\partial y^2} - 2\rho\xi_0\frac{\partial w}{\partial t} + \rho_0 V\gamma_c + \rho_0\int_0^y \frac{\partial \gamma_c}{\partial t}d\eta = \rho\frac{\partial^2 w}{\partial t^2} \tag{51}$$

Solutions of stress functions in compatible equations such as Equation (21), Substituting Equations (19) and (21) into Equation (51) , yields:

$$\begin{aligned} &\rho W\frac{d^2T(t)}{dt^2} + 2\rho\xi_0 W\frac{dT(t)}{dt} - \left(N_{0x}\frac{\partial^2 W}{\partial x^2} + N_{0y}\frac{\partial^2 W}{\partial y^2}\right)T(t) \\ &-h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)T^3(t) - \rho_0 V\gamma_c - \rho_0\int_0^y \frac{\partial \gamma_c}{\partial t}d\eta = 0 \end{aligned} \tag{52}$$

Using Bubnov-Galerkin method to integral Equation (52), yields:

$$\iint_S \left[\begin{array}{l} \rho W\frac{d^2T(t)}{dt^2} + 2\rho\xi_0 W\frac{dT(t)}{dt} - \left(N_{0x}\frac{\partial^2 W}{\partial x^2} + N_{0y}\frac{\partial^2 W}{\partial y^2}\right)T(t) \\ -h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)T^3(t) - \rho_0 V\gamma_c - \rho_0\int_0^y \frac{\partial \gamma_c}{\partial t}d\eta \end{array}\right] W(x,y)dxdy = 0 \tag{53}$$

Where $S \in \{0 \le x \le a, 0 \le y \le b\}$.

Integrating Equation (53) and simplifing it, yields:

$$A\frac{d^2T(t)}{dt^2} + B\frac{dT(t)}{dt} - CT(t) - DT^3(t) - E = 0 \tag{54}$$

Where:

$$A = \iint_S \rho W^2 dxdy = \frac{\rho ab}{4}$$

$$B = 2\rho\xi_0\iint_S W^2 dxdy = 2\rho\xi_0\iint_S \left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 dxdy = \frac{\rho\xi_0 ab}{2}$$

$$C = \iint_S \left(N_{0y}\frac{\partial^2 W(x,y)}{\partial y^2} + N_{0x}\frac{\partial^2 W(x,y)}{\partial x^2}\right)W(x,y)dxdy = \iint_S N_{0x}\frac{\partial^2 W}{\partial x^2}Wdxdy + \iint_S N_{0y}\frac{\partial^2 W}{\partial y^2}Wdxdy$$

$$= -\frac{m^2\pi^2 bN_{0x}}{4a} - \frac{n^2\pi^2 aN_{0y}}{4b}$$

$$D = \iint_S h\left(\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)Wdxdy = -\frac{hm^2n^2\pi^4(\alpha+\beta)}{2ab}$$

$$E = \iint_S \left(\rho_0 V\gamma + \rho_0\int_0^y \frac{\partial \gamma}{\partial t}d\eta\right)W(x,y)dxdy$$

Substituting Equation (50) into Equation (54) , yields:

$$A_1\frac{d^2T(t)}{dt^2} + B_1\frac{dT(t)}{dt} - C_1T(t) - DT^3(t) = 0 \tag{55}$$

Where:

$$A_1 = A - \rho_0 a \iint_S \left(\int_0^y a_{2j} d\eta \right) W(x,y) dxdy$$

$$B_1 = B - \rho_0 aV \iint_S \left(\int_0^y a_{1j} d\eta \right) W(x,y) dxdy - \rho_0 aV \iint_S a_{2j} W(x,y) dxdy$$

$$C_1 = C + \rho_0 aV^2 \iint_S a_{1j} W(x,y) dxdy$$

The composition of equation (55) is consistent with the vibration control equation of closed membrane roof, and the solving process is consistent with the above, which is not discussed here. Thus, the expression of critical wind speed for single mode instability of open flexible membrane roof can be obtained as follows.

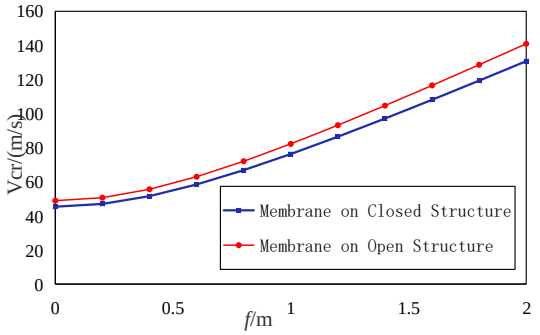
$$V_{cr} = \pi \sqrt{\frac{m^2 b N_{0x} / 4a + n^2 a N_{0y} / 4b + 3hm^2 n^2 \pi^2 (\alpha + \beta) f^2 / 8ab}{\rho_0 a \frac{ab}{MN} \sum_{j=1}^{M \times N} a_{1j} \sin \frac{m\pi x_j}{a} \sin \frac{n\pi y_j}{b}}} \tag{56}$$

3. Analysis of the Effect of Geometric Nonlinearity on Critical Wind Speed

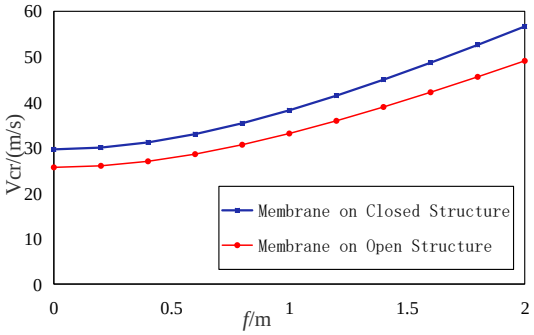
Assuming that the wind speed is in the X direction, $m = n = 1$, $b = 20m$, $N_{0x} = 2kN/m$, $N_{0y} = 2kN/m$. Next, the difference between the results of considering and not considering the geometrical nonlinearity of thin films is discussed, and the necessity of considering the geometrical nonlinearity of thin films in the design of such structures is given.

In this paper, the expression of critical wind speed derived from mathematics is related to the amplitude of membrane vibration. Because the geometric nonlinearity of membrane is considered, the stiffness of membrane will change with the change of amplitude in the process of vibration, which will affect the aerodynamic stability of membrane roof to a certain extent.

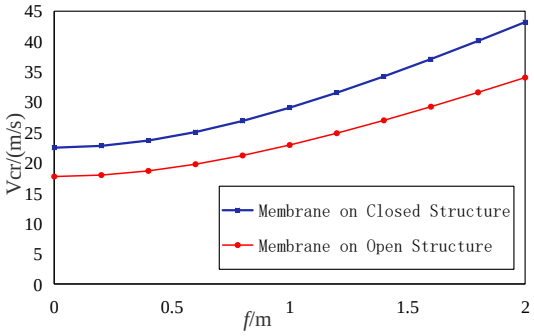
Definiting λ is the ratio of cross (Y) to along (X) wind direction span ratio. The variation of critical wind speed with vibration amplitude for two types of membrane roofs with different span ratios is shown in Figure 4.



(a) λ=0.5



(b) λ=1



(c) λ=2

Figure 4. Curves of critical wind speed with amplitude for two types roof models

Taking membrane roof on closed structure as an example, considering the geometric nonlinearity of membrane, the critical wind speed for single mode instability of closed membrane roof is obtained by solving the equation.

$$V_{cr}=\sqrt{\frac{4\pi\left(m^2\pi^2b^2N_{0x}+n^2\pi^2a^2N_{0y}\right)+3\varepsilon f^2\left(\rho\pi a^2b^2+4\rho_0 ab\alpha_1\right)}{8\pi b\rho_0 m\alpha_3}} \tag{57}$$

By substituting f=0 into Equation (57), the results of critical wind speed calculation without considering geometric nonlinearity can be obtained.

$$V_{cr,L}=\pi\sqrt{\frac{a\left(m^2bN_{0x}/a+n^2aN_{0y}/b\right)}{2\rho_0 m\alpha_3}} \tag{58}$$

Comparing the critical wind speed of membrane roof on closed structure with and without considering the geometric nonlinearity of membrane, the following results are obtained:

$$\frac{V_{cr}}{V_{cr,L}}=\sqrt{1+\frac{3\varepsilon f^2\left(\rho\pi a^2b^2+4\rho_0 ab\alpha_1\right)}{4\pi\left(m^2\pi^2b^2N_{0x}+n^2\pi^2a^2N_{0y}\right)}} \tag{59}$$

4. Discussion

According to Equation (59) , the critical wind speed of single mode instability obtained by considering the geometrical nonlinearity of membrane is larger than the linear one. It shows that the critical wind speed of instability obtained by neglecting the geometrical nonlinearity of membrane is on the small side. It is conservative for structural design. By substituting the specific values, the critical wind speed ratio $\frac{V_{cr}}{V_{cr,L}}$ of the membrane roof on closed structure can be obtained as shown in Table 1.

Table 1. Critical wind speed ratio of membrane roof on closed structure with and without geometric nonlinearity.

	f=0.1m	f=0.2m	f=0.4m	f=0.6m	f=0.8m	f=1m	f=1.2m	f=1.4m
λ=0.5	1.00	1.04	1.14	1.29	1.47	1.68	1.90	2.14
λ=1	1.00	1.01	1.05	1.11	1.19	1.29	1.40	1.52
λ=2	1.00	1.01	1.05	1.11	1.20	1.29	1.40	1.52

Similarly, the critical wind speed ratio $\frac{V_{cr}}{V_{cr,L}}$ of the membrane roof on open structure is obtained as shown in Table 2.

Table 2. Critical wind speed ratio of membrane roof on open structure with and without geometric nonlinearity.

	f=0.1m	f=0.2m	f=0.4m	f=0.6m	f=0.8m	f=1m	f=1.2m	f=1.4m
λ=0.5	1.00	1.04	1.14	1.29	1.47	1.68	1.90	2.14
λ=1	1.00	1.01	1.05	1.11	1.19	1.29	1.40	1.52
λ=2	1.00	1.01	1.05	1.11	1.20	1.29	1.40	1.52

From Table 1 and Table 2, it can be seen that when the cross-wind span ratio is small, the geometric nonlinearity has a greater impact on the critical wind speed of single-mode instability of membrane roof, and increases with the increase of vibration amplitude. When the span ratio is greater than 1, the effect is relatively small. Article 5.3.4 of China's Technical Regulations for Membrane Structures (CECS 158:2015)[12] stipulates that for integral tensioned and cable-supported membrane structures, the deformation of the membrane structure should not be greater than 1/200 of the span when considering the combination of wind load effects. In this example, the normal displacement is limited to 0.1m, and the critical wind speed ratio is 1.00 when geometric nonlinearity is considered or not. Therefore, under normal wind loads, the influence of membrane geometric nonlinearity on the aerodynamic stability of roofs can be neglected. However, under strong wind loads, the deformation of roofs will exceed the norm limit. At this time, the influence of geometric nonlinearity should be considered.

5. Conclusions

In this paper, the aerodynamic stability of orthotropic rectangular planar membranes on closed and open structures is studied by mathematical analytic method. The governing equations of wind-induced nonlinear vibration of tensioned membrane roofs are established by using the theory of large deflection of membrane and Darumbell's principle.According to Bubnov-Galerkin method, the governing equations of aerodynamic coupling are transformed into second-order non-linear differential equations with constant coefficients and their periodic solutions are obtained by using the improved multi-scale method. By judging the stability of the periodic solutions of the equations, critical wind speed for single mode instability of the membrane roof considering geometric nonlinearity is obtained.The influence of geometrical nonlinearity on the critical wind speed of single-mode aeroelastic instability of membrane material is quantitatively obtained by comparing the results with those without consideration. The main conclusions can be summarized as follows:

Considering the geometric nonlinearity of membrane vibration, the critical wind speed of single mode instability of membrane roof increases nonlinearly with the increase of transverse vibration displacement of membrane.

The critical wind speed of single mode instability with the geometrical nonlinearity of membrane considerd is larger than that of linear results. It shows that the critical wind speed of membrane is small when the geometrical nonlinearity of membrane is neglected. For structural design, it is conservative. When the span along the wind direction is small, the geometrical nonlinearity has a great influence on the critical wind speed of single mode instability of membrane roof, and with the amplitude of vibration. When the span ratio is greater than 1, the influence is relatively small.

Under normal wind loads, the influence of membrane geometric nonlinearity on the aerodynamic stability of roofs can be neglected. However, under strong wind loads, the deformation of roofs may exceed the norm limit and reach about 3% of the span, the influence of geometric nonlinearity should be considered.

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