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Article

A Theory of Inertia Based on Mach's Principle

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Abstract: A non-relativistic theory of inertia based on Mach's principle is presented as has been envisaged but not achieved by Ernst Mach in 1872. Central feature is a space-dependent, anisotropic, symmetric inert mass tensor. The contribution of a mass element dm to the inertia of a particle m_0 experiencing an acceleration from rest is proportional to $\cos^2 \alpha$, where α is the angle between the line connecting m_0 and dm and the direction of the acceleration. Apsidal precession for planets circling around a central star is not a consequence of this theory, thereby avoiding the prediction of an apsidal precession with wrong sign as is done by Mach-like theories with isotropic inert mass.

Keywords: inertia; Mach's principle; history of physics

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1. Introduction

In 1872 Ernst Mach formulated his famous principle about the inert mass of a particle, which he assumed to be induced somehow by the presence of other masses in the universe [4]. Thereby, inertia is the result of gravitational forces. Unfortunately, he never formulated this as a physical theory.

In this paper, I want to present a theory consistent with Mach's principle. The theory is non-relativistic. Hence it is more of a historical value, giving what Mach did not (or could not) deliver nearly 150 years ago, when Mach's ideas were not yet overgrown by Einstein's theories (which, contrary to Einstein's hopes and struggles, do not fully incorporate the Mach principle).

A central aspect of the theory is the anisotropy of the inert mass, which therefore has to become a tensor. In the 1950s, experiments in search of an anisotropy of inertia have been performed with negative result [3]. Later, these experiments have been recognized as not suitable, ignoring the fact that locally not only the test particle but all masses (including those of the experimental setup) exhibit the same anisotropy of inertia [1]. Different experiments where this effect does not matter are still to be performed.

2. Results

2.1. Assumptions

We start with the following assumptions, which are all very simple and plausible:

- **Assumption 1:** A single point mass in an otherwise empty universe has no defined movement, *i.e.* no inertia, no acceleration, no momentum or kinetic energy.
- **Assumption 2:** Two point masses in an otherwise empty universe can have a defined movement only along their connecting line. Any movement at constant distance between the masses cannot be defined. This comprises rotation about each other (no tangential inertia) and collective translation. No kinetic energy or momentum arises due to such movement: Inertia, acceleration, momentum or kinetic energy can only be defined with respect to increasing/reducing the distance between the masses, *i.e.* for radial movement. We would like to have a theory where the sign of this radial movement does not matter.

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- **Assumption 3:** Since the inertia of a test particle is 'created' by the gravitational forces of other masses, it will depend on the direction. Therefore the angle between the direction of the test particle movement and the connecting line to any inducing mass will be crucial.
- **Assumption 4:** The size of the induced inertia should decrease with increasing distance r between test particle and inducing mass as $1/r^a$ with $0 < a < 2$. Here $0 < a$ guarantees vanishing inertia at infinite distance (which, of course, is an assumption itself). $a \geq 2$ would lead to large effects perceivable in daily life, since anisotropy of inertia would balance or even excel the effects of Newton's gravity. In fact, $a = 1$ is a standard assumption in literature, see *e.g.* [5], where, however, isotropy of inertia is postulated.
- **Assumption 5:** The inertia of a test particle within a hollow sphere of constant mass density is isotropic and constant. If the mass distribution is largly isotropic within our universe, this assumption guarantees that this theory is very similar to our daily experience.

2.2. Theory

A definition of the kinetic energy of a test particle with mass m_0 within a universe with other (inertia inducing) masses m^α, which is consistent with all assumptions above is given by:

$$E_{kin} = \frac{1}{2} \sum_\alpha v_i^\alpha M_{ij}^\alpha v_j^\alpha \,, \tag{1}$$

$$M_{ij}^\alpha = M_{ji}^\alpha = \gamma m_0 m^\alpha \frac{r_i^\alpha r_j^\alpha}{|\vec{r}^{\,\alpha}|^3} \,. \tag{2}$$

γ is a natural constant, presumably connected with Newton's gravitational constant. $\vec{r}^{\,\alpha} = \vec{x}_0 - \vec{x}^{\,\alpha}$ is the distance between m_0 and m^α, with $\vec{x}_0$ and $\vec{x}^{\,\alpha}$ the absolute positions of m_0 and m^α in space, respectively. Similarly, $\vec{v}^{\,\alpha} = \dot{\vec{x}}_0 - \dot{\vec{x}}^{\,\alpha}$ is the *relative* (not only radial!) velocity between m_0 and m^α (as will be the relative acceleration $\dot{\vec{v}}^{\,\alpha} = \vec{a}^{\,\alpha}$ later on). The sum is over all masses m^α in the universe with the exception of m_0.

The corresponding components of the momentum and force of the test particle m_0 are then given by

$$p_i = \frac{\partial E_{kin}}{\partial (\vec{v}_0)_i} = \sum_\alpha M_{ij}^\alpha v_j^\alpha \,, \tag{3}$$

$$F_i = \frac{dp_i}{dt} = \sum_\alpha \left(\dot{M}_{ij}^\alpha v_j^\alpha + M_{ij}^\alpha a_j^\alpha \right) \,. \tag{4}$$

If and only if we are in a coordinate system where *all* inducing masses m^α are at rest, we have $\vec{v}^{\,\alpha} = \vec{v}_0$ and the formulae simplify to

$$E_{kin} = \frac{1}{2} \vec{v}_0{}^T \overleftrightarrow{M}\, \vec{v}_0 \,, \quad \vec{p} = \overleftrightarrow{M}\, \vec{v}_0 \,, \quad \overleftrightarrow{M} = \sum_\alpha m^\alpha \frac{\vec{r}^{\,\alpha} \left(\vec{r}^{\,\alpha}\right)^T}{|\vec{r}^{\,\alpha}|^3} \tag{5}$$

2.3. Check of Assumptions

Let us now investigate how this definitions fullfill assumptions 1-5. For this, we do not consider Newtonian gravitation beyond induction of inertia.

Assumption 1: Since there exists only m_0 and not one single m^α, the M_{ij}^α are all zero. There is no kinetic energy and no momentum and also no force.

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Assumption 2: For translations at constant distance $\vec{v}^{\,\alpha} = 0$. For (possibly non-uniform) rotations of a mass m_0 with constant radius R around a single point mass m^α (or vice versa), we assume $\vec{x}^{\,\alpha} = (0,0,0) \Rightarrow \vec{r}^{\,\alpha} = \vec{x}_0 - \vec{0} = (R\cos\omega t, R\sin\omega t, 0)$, with ω possibly time dependent. Thus we have

$$\overleftrightarrow{M} = \frac{\gamma m_0 m^\alpha}{R} \begin{pmatrix} \cos^2\omega t & \sin\omega t\cos\omega t & 0 \\ \sin\omega t\cos\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

$$\vec{v}^{\,\alpha} = R(\dot{\omega}t + \omega) \begin{pmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{pmatrix}. \tag{7}$$

With our definitions above one sees easily that $\vec{p} = \vec{0},\ E = 0$ and again there is no force (this is also true if m^α and m_0 rotate synchronously around some other rotational center, *e.g.* both being fixed onto one single watch hand). Especially, there are no centripetal or centrifugal forces. (Therefore, the gravitational attraction between the two masses will eventually lead to collision, regardless if there is a rotation or not.)

Assumption 3: Anisotropy of inertia is realized in our definition. As an illustration let us consider a simple experiment involving three masses, see Figure 1. m_0 is at rest in $(0,0,0)$. We have two masses m^α, with mass $m^1 = M$ at $(0,-R,0)$ and mass $m^2 = 2M$ at $(-R,0,0)$.

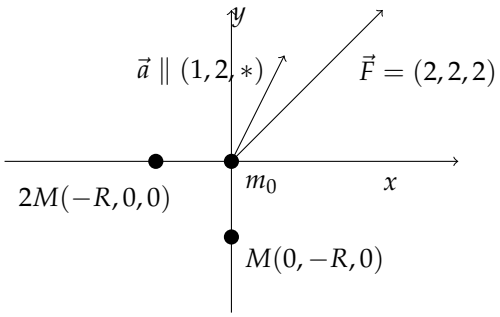

Figure 1. Simple example for the check of Assumption 3.

After a short calculation, we find (with $\vec{p}_0 = 0$)

$$\overleftrightarrow{M} = \frac{\gamma m_0 M}{R} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \vec{F} = \overleftrightarrow{M}\vec{a}\,. \tag{8}$$

If a force $\vec{F} = (2,2,2)$ is applied, this will lead to an acceleration of $\vec{a} = (1,2,*)$, which is clearly not the same direction as $\vec{F}$. The * signals the not defined acceleration in the here inertia-free z-direction.

If $\vec{p}_0 \neq \vec{0}$ at the beginning, things become more complicated, since the change of $\vec{r}^{\,1}$ and $\vec{r}^{\,2}$ (the position of m_0 relative to the two other masses) gives extra terms leading to $\vec{a} \nparallel (1,2,*)$.

Assumption 4: One sees easily that the contribution of each mass m^α to $\overleftrightarrow{M}$ is proportional (besides the dependence on direction) to $1/|\vec{r}^{\,\alpha}|$.

All these considerations can easily be generalized to systems with more masses m^α, where one also has *e.g.* the absence of kinetic energy and momentum if all masses move uniformly or the absence of centrifugal forces if all masses rotate uniformely without change of relative distances. This corresponds perfectly to Mach's famous bucket-gedankenexperiment.

Assumption 5 is perhaps the most crucial and definitely the most complicated to prove. In Appendix A, we will give the detailed calculation (A2). The result for the inert mass tensor of a test

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particle m_0 (resting or moving with constant velocity relative to the sphere) within a thin sphere of radius R and constant area mass density σ (and hence $4\pi R^2\sigma$ its overall mass) is:

$$\overleftrightarrow{M} = \frac{\gamma}{3}\frac{4\pi R^2\sigma}{R} m_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9}$$

This corresponds to a constant, isotropic inertia independent of the spatial position (as long as within the sphere) and would therefore be the physical setting for the emergence of usual Newtonian inertia. The calculation in Appendix A shows that the contribution of a mass element dm to the inertia of a particle m_0 at rest (experiencing an acceleration) is proportional to $\cos^2\alpha$, where α is the angle between the line connecting m_0 and dm and the direction of the acceleration. So if the angle is 90°, there is no contribution of dm to the inertia, whereas the contribution is maximal if both directions are in parallel. Thus, apsidal precession for planets circling around a central star is not a consequence of this theory (leaving room for additional effects due to general relativity), but it avoids the prediction of an apsidal precession with wrong sign as is done by Mach-like theories with isotropic inert mass (proportional e.g. to the local graviational potential), [5].

3. Conclusions

In this paper, a non-relativistic theory of inertia based on Mach's principle was presented. The theory is by no means intended to replace Einstein's general theory of relativity, e.g. it will not explain perihelion precession of planets since the gravitation of the central star (at this scale nearly a point mass) has barely any influence on the tangential inertia of surrounding objects. However, this theory may come to life at the scale of galaxies and their movement. In any case, the next step will be to (try to) incorporate this theory into general relativity.

The most crucial task would be a proper testing of inert (an)isotropy using experiments which do not rely only on masses (*i.e.* gravitation and inertia but also *e.g.* electric fields. Since induced inertia and Newtonian gravitation always come together, interference of both effects may complicate things.

Appendix A. Calculation of M within a sphere

We start with a test particle at the origin of our coordinate system. This particle is within a thin sphere of constant area mass density σ with center $(0|0|-a)$ and radius R ($R > a$), see Figure A1.

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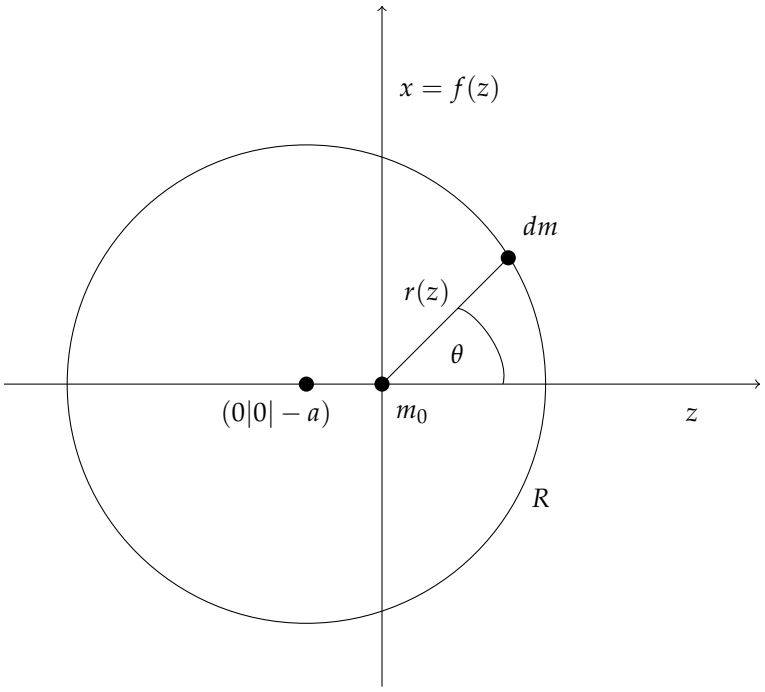

Figure A1. Simple example for the check of Assumption 5.

The mass element dm of the sphere is at point $(z, x = f(z), 0)$, where

$$\begin{aligned} f(z) &= \sqrt{R^2 - (z+a)^2}\,, \\ f'(z) &= -\frac{z+a}{\sqrt{R^2-(z+a)^2}}\,, \\ f(z)\sqrt{1+f'^2(z)} &= \sqrt{R^2-(z+a)^2}\sqrt{1+\frac{(z+a)^2}{R^2-(z+a)^2}} = R\,, \\ r(z) &= \sqrt{z^2+f^2(z)} = \sqrt{R^2-a^2-2az}\,. \end{aligned}$$

For later use, we want to calculate the gravitational potential Φ_{in} within the sphere:

$$\begin{aligned} m_0\Phi_{in} &= \int_0^{2\pi} d\varphi \int_{-R-a}^{R-a} dz\, \frac{m_0\gamma\sigma}{r(z)} \underbrace{f(z)\sqrt{1+f'^2(z)}}_{R} \qquad \text{(A1)} \\ &= 2\pi R m_0\gamma\sigma \int_{-R-a}^{R-a} \frac{dz}{\sqrt{R^2-a^2-2az}} \\ &= 2\pi R m_0\gamma\sigma\, \frac{-1}{a}\sqrt{R^2-a^2-2az}\Big|_{-R-a}^{R-a} \\ &= \frac{2\pi R m_0\gamma\sigma}{a}\Big(\underbrace{\sqrt{R^2-a^2+2aR+2a^2}}_{R+a} - \underbrace{\sqrt{R^2-a^2-2aR+2a^2}}_{R-a}\Big) \\ &= 4\pi R m_0\gamma\sigma\,. \end{aligned}$$

Of course, this result is not at all surprising: The gravitational potential within a hollow sphere of homogeneous mass density is constant, *i.e.* independent of the position.

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We now want to calculate $\overleftrightarrow{M}$ at the coordinate origin (where m_0 is located):

$$\begin{aligned}\overleftrightarrow{M} &= \int_{\text{sphere}} d^3r\, m_0\gamma\sigma\, \frac{\vec{r}\,\vec{r}^{\,T}}{|\vec{r}|^3} \\ &= \int_0^{2\pi} d\varphi \int_{-R-a}^{R-a} dz\, \frac{m_0\gamma\sigma}{r^3(z)} \underbrace{f(z)\sqrt{1+f'^2(z)}}_{R} \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix} .\end{aligned}$$

With $(x,y,z) = (r(z)\sin\theta\cos\phi,\ r(z)\sin\theta\sin\phi,\ r(z)\cos\theta)$ we see immediately that after the ϕ-integration the off-diagonal terms are all zero and the diagonal terms yield 2π for the z^2-term and $2\pi/2$ for the other two:

$$\overleftrightarrow{M} = 2\pi R m_0\gamma\sigma \int_{-R-a}^{R-a} dz \frac{1}{r(z)} \begin{pmatrix} \frac{1}{2}\sin^2\theta & 0 & 0 \\ 0 & \frac{1}{2}\sin^2\theta & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix} .$$

We start with the zz-term proportional to $\cos^2\theta$. We rewrite:

$$\cos^2\theta = \left(\frac{z}{r(z)}\right)^2 = \frac{z^2}{R^2-a^2-2az}$$

and get:

$$M_{zz} = 2\pi R m_0\gamma\sigma\, I_z = 2\pi R m_0\gamma\sigma \int_{-R-a}^{R-a} dz \frac{z^2}{(R^2-a^2-2az)^{3/2}} .$$

Here we use the well known formula [2]

$$\begin{aligned}&\int dz \frac{z^2}{(Az+B)^{3/2}} \\ &= \frac{2}{A^3}\left(\frac{1}{3}(Az+B)^{3/2} - 2B(Az+B)^{1/2} - B^2(Az+B)^{-1/2}\right) ,\end{aligned}$$

where in our case $A=-2a$ and $B=R^2-a^2$ (note that the denominator is $r^3(z)>0$):

$$\begin{aligned}\int_{-R-a}^{R-a} \frac{z^2}{(R^2-a^2-2az)^{3/2}} = &-\frac{1}{4a^3}\Big(\frac{1}{3}(R^2-a^2-2az)^{3/2} \\ &-2(R^2-a^2)(R^2-a^2-2az)^{1/2} \\ &-(R^2-a^2)^2(R^2-a^2-2az)^{-1/2}\Big)\Big|_{-R-a}^{R-a} .\end{aligned}$$

We notice again

$$\begin{aligned}(R^2-a^2-2a(R-a))^{1/2} &= R-a\,, \\ (R^2-a^2+2a(R+a))^{1/2} &= R+a\,,\end{aligned}$$

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and find:

$$\begin{aligned} I_z &= -\frac{1}{4a^3}\Big(\frac{1}{3}(R-a)^3 - 2(R^2-a^2)(R-a) - \frac{(R^2-a^2)^2}{R-a}\Big) \\ &\quad + \frac{1}{4a^3}\Big(\frac{1}{3}(R+a)^3 - 2(R^2-a^2)(R+a) - \frac{(R^2-a^2)^2}{R+a}\Big) \\ &= \frac{1}{4a^3}\cdot\frac{8}{3}a^3 = \frac{2}{3}\,, \\ M_{zz} &= 2\pi R m_0\gamma\sigma\, I_z = \frac{1}{3}\cdot 4\pi R m_0\gamma\sigma\,. \end{aligned}$$

This is exactly one third of what we had before (A1), independent of the test particle's position within the sphere.

For the xx- and yy-component, the integration is easy using $\frac{1}{2}\sin^2\theta = \frac{1}{2}(1-\cos^2\theta)$ with the results from above:

$$2\pi R m_0\gamma\sigma I_{x,y} = \frac{1}{2}(1-1/3)\cdot 4\pi R m_0\gamma\sigma = \frac{1}{3}\cdot 4\pi R m_0\gamma\sigma = I_z\,.$$

Again, it is exactly $1/3$ of what we had before. In sum, we end up with

$$\overleftrightarrow{M} = \frac{\gamma}{3}\frac{4\pi R^2\sigma}{R} m_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A2}$$

Conflicts of Interest: The authors declare no conflict of interest.

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