Article

Infinitesimal transformations of locally conformal Kähler manifolds

Yevhen Cherevko[1], **Volodymyr Berezovski**[2], **Irena Hinterleitner**[3] **and Dana Smetanová**[4]

[1] Odesa National Economic University, Ukraine; cherevko@usa.com
[2] Uman National University of Horticulture, Ukraine; berez.volod@gmail.com
[3] Brno University of Technology, Czech Republic; Hinterleitner.Irena@seznam.cz
[4] Institute of Technology and Business in České Budějovice, Czech Republic; smetanova@mail.vstecb.cz

Abstract: The article is devoted to infinitesimal transformations. We have obtained that LCK-manifolds do not admit nontrivial infinitesimal projective transformations. Then we study infinitesimal conformal transformations of LCK-manifolds. We have found the expression for the Lie derivative of a Lee form. Also we have obtained the system of partial differential equations for the transformations, and explored its integrability conditions. Hence we have got the necessary and sufficient conditions in order that the an LCK-manifold admits a group of conformal motions. Also we have calculated the number of parameters which the group depends on. We have proved that a group of conformal motions admitted by an LCK-manifold is isomorphic to a homothetic group admitted by corresponding Kählerian metric. We also established that an isometric group of an LCK-manifold is isomorphic to a some subgroup of homothetic group of the coresponding local Kählerian metric.

Keywords: Hermitian manifold; locally conformal Kähler manifold; Lee form; diffeomorphism; conformal transformation; Lie derivative.

MSC: 53C55, 53C80, 35M10

---

1. Introduction

Kählerian manifolds, because of their properties, have been used for modeling of physical processes for a long time, for instance in supersymmetric theories [19], in a string theory as so called Calabi-Yau manifolds (e.g. [9, p. 411]). A manifold is called locally conformal Kähler manifold (for brevity, LCK-manifolds) if its metric is conformal to some local Kahlerian metric in the neighborhood of each point of the manifold. On other hand one knows that conformal mappings preserve the Petrov type of a manifold [6]. The LCK-manifolds are also used for physical modeling. For instance, in [11] authors offered a Kaluza-Klein model with spontaneous compactification, using a generalized Hopf manifold. Also, explorers use locally conformally Calabi-Yau manifold to build M-theory models. According to [14] locally conformally Calabi-Yau manifold is LCK-manifold with a Ricci-flat metric. For example eight-dimensional Hopf manifold admits a Ricci-flat metric, hence it may be used in a model of eleven-dimensional Supergravity.

The objects under consideration in the article are the LCK-manifolds for which $dim(M) = n = 2m > 2$. LCK-manifolds were explored by [15], [2], [4]. Also the book [10] is worth to note as one of the most distinguished in this realm. Infinitesimal conformal transformations were explored in [7], [5]. Infinitesimal conformal transformations of complex manifolds were studied by Yano [16]. Transformations of LCK-manifolds were explored in [13]. The main goal of the article is also to explore transformations of LCK-manifolds.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

2. Locally conformal Kähler manifolds

A Hermitian manifold (M^{2m}, J, g) is called a *locally conformal Kähler manifold (LCK - manifold)* if there is an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of M^{2m} and a family $\{\sigma_\alpha\}_{\alpha\in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK - manifold is endowed with some form ω, so called a *Lee form* which can be calculated as [3]

$$\omega = \frac{1}{m-1}\delta\Omega \circ J \quad \text{or} \quad \omega_i = -\frac{2}{n-2} J^\alpha_{\beta,\alpha} J^\beta_i, \tag{1}$$

The form should be closed:

$$d\omega = 0.$$

One can compute covariant derivative an almost complex structure with respect to the Levi-Civita connection of (M^{2m}, J, g) using the formula

$$J^k_{i,j} = \frac{1}{2}(\delta^k_j J^\alpha_i \omega_\alpha - \omega^k J_{ij} - J^k_j \omega_i + J^k_\alpha \omega^\alpha g_{ij}). \tag{2}$$

3. Infinitesimal transformations of manifolds

Definition 1. *Transformation of a manifold M^n*

$$\overline{x}^h = x^h + \epsilon\xi^h(x^1, x^2, \ldots, x^n), \tag{3}$$

is called infinitesimal transformation *of a manifold M^n. Vector $\xi(x^1, x^2, \ldots, x^n)$ is often refered as* a generator of transformation. *An arbitrary small parameter ϵ is independent on x^i.*

Lie derivative of a tensor of type (p, q) $T^{i_1\ldots i_p}_{j_1\ldots j_q}$ with respect to a vector field ξ may be calculated by te equation [1, p. 196]:

$$\mathcal{L}_\xi T^{i_1\ldots i_p}_{j_1\ldots j_q} = T^{i_1\ldots i_p}_{j_1\ldots j_q,s}\xi^s + T^{i_1\ldots i_p}_{kj_2\ldots j_q}\xi^k{}_{,j_1} + \ldots + T^{i_1\ldots i_p}_{kj_k\ldots j_q}\xi^k{}_{,j_1} - T^{li_2\ldots i_p}_{j_1\ldots j_q}\xi^{i_1}{}_{,l} - \ldots - T^{i_1 i_2\ldots l}_{j_1\ldots j_q}\xi^{i_p}{}_{,l}. \tag{4}$$

In particular, for a metric tensor g we get

$$\mathcal{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} \tag{5}$$

If a manifold M^n was transformed then its metric tensor $\overline{g}$ of the transformed $\overline{M}^n$ is

$$\overline{g}_{ij} = g_{ij} + h_{ij}\varepsilon, \tag{6}$$

where $h_{ij} = \mathcal{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i}$ [7, p. 275]. For the Christoffel symbols we have also[17, p. 8]:

$$\mathcal{L}_\xi \Gamma^h_{jk} = \nabla_k \nabla_j \xi^h + \xi^m R^h_{jmk} \tag{7}$$

Transvecting (7) with g_{hi} we get:

$$\xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + g_{hi}\mathcal{L}_\xi \Gamma^h_{jk} \tag{8}$$

The item $g_{hi}\mathcal{L}_\xi \Gamma^h_{jk}$ depends on transformation type. We are interested primarily in the case when a vector field $\xi(x^1, x^2, \ldots, x^n)$ generates a transformation preserving the complex structure [16]:

$$\mathcal{L}_\xi J^i_j = J^i_{j,k}\xi^k - J^\alpha_j \xi^i{}_{,\alpha} + J^i_\alpha \xi^\alpha{}_{,j} = 0. \tag{9}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

The field is called a *contravariant analytic* vector field, and the infinitesimal transformation is refered as a *holomorphic* one. It is worth to note that since exterior differentiation and the Lie derivation with respect to ξ are commutative

$$d\mathcal{L}_\xi \omega = \mathcal{L}_\xi d\omega \tag{10}$$

hence any infinitesimal transformation preserves the closeness property of Lee form.

3.1. Projective transformations and LCK-manifolds

If a transformation (3) does not change geodesics of a manifold, it is called a projective transformation. Mikeš an Radulovich in [4] proved that LCK-manifolds ($n > 2$) do not admit nontrivial finite geodesic mappings onto Hermitian manifolds if a preserving complex structure is requared. We have to explore whether nontrivial projective transformations preserving a complex structure are admitted on LCK-manifolds. Hence let us suppose that such transformation is admitted. Then

$$L_\xi \Gamma^h_{ij} = \psi_i \delta^h_j + \psi_j \delta^h_i,$$

where ψ is a scalar whose gradient $\psi_i = \partial_i \psi$ and a vector ξ generate the transformation. Then combining (8) and its conditions of integrability, we obtain:

$$\begin{cases} \xi_{i,j} = \xi_{ij}; \\ \psi_{,i} = \psi_i; \\ \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \psi_k g_{ij} + \psi_j g_{ik} \\ \psi_{ij} = \frac{1}{n-1}\left(\xi^\alpha R_{\alpha i,j} + \xi^\alpha{}_{,i} R_{\alpha j} + \xi^\alpha{}_{,j} R_{\alpha i} + \xi^\alpha R^\beta_{ij\alpha,\beta}\right) \end{cases}$$

Also the equation

$$h_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} \tag{11}$$

is satisfied [7, p. 275]. Since the metric g_{ij} is Hermitian, we get:

$$J^t_i g_{tj} + J^t_j g_{ti} = 0. \tag{12}$$

Also, since deformed metric $\overline{g}_{ij}$ is Hermitian and the complex structure is preserved, hence on the deformed manifold $\overline{M}^n$, the identity

$$J^t_i \overline{g}_{tj} + J^t_j \overline{g}_{ti} = 0. \tag{13}$$

is satisfied. Taking into account (6) and (12), from (13) we obtain:

$$J^t_i h_{tj} + J^t_j h_{ti} = 0. \tag{14}$$

Differentiating covariantly (14) with respect to the Levi-Civita connection which is compatible with a metric g_{ij}, we get:

$$J^t_{i,k} h_{tj} + J^t_i h_{tj,k} + J^t_{j,k} h_{ti} + J^t_j h_{ti,k} = 0.$$

Then we use (2) and (11):

$$\begin{aligned} &\frac{1}{2}\left(\delta^t_k J^\alpha_i \omega_\alpha - \omega^t J_{ik} - J^t_k \omega_i + J^t_\alpha \omega^\alpha g_{ik}\right) h_{tj} + J^t_i \left(2\psi_k g_{tj} + \psi_t g_{jk} + \psi_j g_{tk}\right) \\ &+ \frac{1}{2}\left(\delta^t_k J^\alpha_j \omega_\alpha - \omega^t J_{jk} - J^t_k \omega_j + J^t_\alpha \omega^\alpha g_{jk}\right) h_{ti} + J^t_j \left(2\psi_k g_{ti} + \psi_t g_{ik} + \psi_i g_{tk}\right) = 0. \end{aligned}$$

 doi:10.20944/preprints201906.0232.v1

Then, let us regroup the items:

$$\begin{aligned}(\psi_j - \frac{1}{2}\omega^t h_{tj})J_{ik} + (\psi_i - \frac{1}{2}\omega^t h_{ti})J_{jk}\\ -\frac{1}{2}h_{sj}J^s_k\omega_i - \frac{1}{2}h_{si}J^s_k\omega_j + (J^t_j\psi_t + \frac{1}{2}h_{tj}J^t_s\omega^s)g_{ik}\\ +(J^t_i\psi_t + \frac{1}{2}h_{ti}J^t_s\omega^s)g_{jk} + \frac{1}{2}h_{kj}J^s_i\omega_s + \frac{1}{2}h_{ki}J^s_j\omega_s = 0.\end{aligned} \tag{15}$$

Using symmetrization of (15), and taking into account that according to (12) and (14), the sum of the first four items in left hand side of (15) is equal to zero, we get

$$\begin{aligned}(J^t_j\psi_t + \frac{1}{2}h_{tj}J^t_s\omega^s)g_{ik} + (J^t_i\psi_t + \frac{1}{2}h_{ti}J^t_s\omega^s)g_{jk}\\ +(J^t_k\psi_t + \frac{1}{2}h_{tk}J^t_s\omega^s)g_{ij} + \frac{1}{2}J^s_i\omega_s h_{kj} + \frac{1}{2}J^s_j\omega_s h_{ki} + \frac{1}{2}J^s_k\omega_s h_{ij} = 0,\end{aligned}$$

or, for brevity

$$\chi_j g_{ik} + \chi_i g_{jk} + \chi_k g_{ij} + \theta_j h_{ki} + \theta_i h_{jk} + \theta_k h_{ij} = 0, \tag{16}$$

where $\chi_i = (J^t_i\psi_t + \frac{1}{2}h_{ti}J^t_s\omega^s)$, $\theta_i = \frac{1}{2}J^s_i\omega_s$. If $dim(M^n) > 2$ then it's possible to choose a vector η^i that $\eta^i\chi_i = \eta^i\theta_i = 0$. Transvecting (16) with η^i, we get:

$$\chi_j\eta_k + \chi_k\eta_j + \theta_j h_{ki}\eta^i + \theta_k h_{ij}\eta^i = 0. \tag{17}$$

Transvecting (17) with η^j produces:

$$\chi_k||\eta||^2 + \theta_k h_{ij}\eta^i\eta^j = 0. \tag{18}$$

It follows from (18) that $\chi_k = \alpha\theta_k$. Hence,

$$\theta_j(\alpha g_{ik} + h_{ik}) + \theta_k(\alpha g_{ij} + h_{ij}) + \theta_i(\alpha g_{jk} + h_{jk}) = 0. \tag{19}$$

It follows from (19) that one of the equations holds, namely $\theta_i = 0$, or $\alpha g_{jk} + h_{jk} = 0$. In the former case we have that the manifold M^n is Kählerian since $\omega_i = 0$ and the transformation is trivial because $\psi_i = 0$. In the latter case the equation $h_{jk} = -\alpha g_{jk}$ means that the transformations is a conformal one. But one knows that if a transformation is simultaneously conformal and projective then it is a trivial one. Hence we obtain the theorem.

Theorem 1. *An LCK-manifold M^n, $dim(M^n) > 2$ does not admit nontrivial projective transformations with respect to a Levi-Civita connection preserving its complex structure.*

Note that proving the theorem we use methods offered in [4].

3.2. Conformal infinitesimal transformations of locally conformal Kähler manifolds

Infinitesimal transformations are called conformal if the equations hold [16, p. 275]:

$$\mathcal{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} = \varphi g_{ij}. \tag{20}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

It is known, if a vector field ξ generates conformal infinitesimal transformations, the field and invariant φ satisfy the system [6], [5]:

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \varphi_{,i} = \varphi_i; \\ 3) & \xi_{i,j} + \xi_{j,i} = \varphi g_{ij}; \\ 4) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}(\varphi_k g_{ij} + \varphi_j g_{ik} - \varphi_i g_{jk}); \\ 5) & \varphi_{i,j} = \frac{2}{n-2}\left(\xi^\alpha R_{ij,\alpha} + \xi_{\alpha,i} R^\alpha_j + \xi_{\alpha,j} R^\alpha_i - \frac{g_{ij}}{2(n-1)}\left(\xi^\alpha R_{,\alpha} + \varphi R\right)\right). \end{array} \tag{21}$$

3.3. Nijenhuis tensor and Lee form under conformal infinitesimal transformations

Taking into account (9) and (10), we have that a necessary and sufficient condition that under conformal infinitesimal transformation an LCK-manifold remains also locally conformal Kählerian is that the Lie derivative of Nijenhuis tensor

$$N^k_{ij} = J^\alpha_i\left(J^k_{\alpha,j} - J^k_{j,\alpha}\right) - J^\alpha_j\left(J^k_{\alpha,i} - J^k_{i,\alpha}\right)$$

must be equal to zero:

$$\mathcal{L}_\xi N^k_{ij} = 0.$$

The Lie derivative of a Nijenhuis tensor is

$$\mathcal{L}_\xi N^k_{ij} = J^\alpha_i\left(\mathcal{L}_\xi J^k_{\alpha,j} - \mathcal{L}_\xi J^k_{j,\alpha}\right) - J^\alpha_j\left(\mathcal{L}_\xi J^k_{\alpha,i} - \mathcal{L}_\xi J^k_{i,\alpha}\right). \tag{22}$$

because of(9).

It is known the identity [16, p. 159]:

$$\mathcal{L}_\xi J^k_{i,j} - \left(\mathcal{L}_\xi J^k_i\right)_{,j} = J^\beta_i \mathcal{L}_\xi \Gamma^k_{j\beta} - J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{ji}, \tag{23}$$

where Γ^k_{ji} are components of a symmetric affine connection which is compatible with a metric g_{ij}. Because of (9), from (23) we get

$$\mathcal{L}_\xi J^k_{i,j} = J^\beta_i \mathcal{L}_\xi \Gamma^k_{j\beta} - J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{ji}. \tag{24}$$

Let us calculate the Lie derivative of a Nijenhuis tensor with respect to the vector field ξ, taking into account (24)

$$\begin{aligned} \mathcal{L}_\xi N^k_{ij} &= J^\alpha_i\left(\mathcal{L}_\xi J^k_{\alpha,j} - \mathcal{L}_\xi J^k_{j,\alpha}\right) - J^\alpha_j\left(\mathcal{L}_\xi J^k_{\alpha,i} - \mathcal{L}_\xi J^k_{i,\alpha}\right) = \\ &= J^\alpha_i (J^\beta_\alpha \mathcal{L}_\xi \Gamma^k_{j\beta} - J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{j\alpha} - J^\beta_j \mathcal{L}_\xi \Gamma^k_{\alpha\beta} + J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{\alpha j}) - \\ &\quad - J^\alpha_j (J^\beta_\alpha \mathcal{L}_\xi \Gamma^k_{i\beta} - J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{i\alpha} - J^\beta_i \mathcal{L}_\xi \Gamma^k_{\alpha\beta} + J^k_\beta \mathcal{L}_\xi \Gamma^\beta_{\alpha i}). \end{aligned} \tag{25}$$

Removing the parentheses and collecting similar terms in (25) we obtain that the Lie derivative of a Nijenhuis tensor is equal to zero

$$\mathcal{L}_\xi N^k_{ij} = 0.$$

Taking into account that any infinitesimal transformation preserves the closeness property of its Lee form we obtain the theorem.

Theorem 2. *Any infinitesimal transformation of an LCK-manifold preserving its complex structure, transforms it into an locally conformal Kählerian one.*

 doi:10.20944/preprints201906.0232.v1

Proof. Let us calculate a Lie derivative of a Lee form. Because of (9), from (1) we have

$$\mathcal{L}_\xi \omega_i = -\frac{2}{n-2}\mathcal{L}_\xi (J^\alpha_{\beta,\alpha} J^\beta_i) = -\frac{2}{n-2}\mathcal{L}_\xi (J^\alpha_{\beta,\alpha}) J^\beta_i. \tag{26}$$

On other hand, since a Lie derivative and contraction are commutative hence contracting for k and j from (24) we obtain

$$\begin{aligned} \mathcal{L}_\xi J^\alpha_{i,\alpha} = \frac{1}{2}\left(nJ^\beta_i \varphi_\beta - \varphi^\alpha J_{i\alpha} - J^\alpha_\alpha \varphi_i + J^\alpha_\beta \varphi^\beta g_{i\alpha}\right) = \\ = \frac{1}{2}\left(nJ^\beta_i \varphi_\beta - \varphi^\alpha J_{i\alpha} + J_{\beta i}\varphi^\beta\right) = \\ = \frac{1}{2}\left(nJ^\beta_i \varphi_\beta - \varphi^\alpha J_{i\alpha} - J_{i\beta}\varphi^\beta\right) = \frac{n-2}{2} J^\beta_i \varphi_\beta. \end{aligned} \tag{27}$$

Substituting (27) into (26) we find that

$$\mathcal{L}_\xi \omega_i = -\frac{2}{n-2}\cdot\frac{n-2}{2} J^\beta_\gamma \varphi_\beta J^\gamma_i = \varphi_i. \tag{28}$$

□

Theorem 3. *If a vector field ξ generates a conformal infinitesimal transformation of an LCK-manifold, then components of Lie derivatives of the Lee form are equal to the partial derivatives of the invariant φ defined by the system (21)*

$$\mathcal{L}_\xi \omega_i = \varphi_i.$$

Proof. It is worth to note that according to (4)

$$\mathcal{L}_\xi \omega_i = \omega_{i,\alpha}\xi^\alpha + \omega_\alpha \xi^\alpha{}_{,i}.$$

On other hand,

$$\frac{\partial}{\partial x^i}(\omega_\alpha \xi^\alpha) = \omega_{\alpha,i}\xi^\alpha + \omega_\alpha \xi^\alpha{}_{,i}.$$

Since the Lee form is closed then $\omega_{i,j} = \omega_{j,i}$, and hence from (4) it follows that

$$\frac{\partial}{\partial x^i}(\omega_\alpha \xi^\alpha) = (\omega_\alpha \xi^\alpha)_{,i} = \varphi_i. \tag{29}$$

Hence the scalar φ mentioned in may be expressed by the equation

$$\varphi = \omega_\alpha \xi^\alpha + C,$$

where C is an arbitrary constant. Hence taking into account the conditions (9) the PDE system (21) becomes

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \xi_{i,j} + \xi_{j,i} = (\omega_\alpha \xi^\alpha + C) g_{ij}; \\ 3) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}\left((\omega_\alpha \xi^\alpha)_{,k} g_{ij} + (\omega_\alpha \xi^\alpha)_{,j} g_{ik} - (\omega_\alpha \xi^\alpha)_{,i} g_{jk}\right); \\ 4) & J^i_{j,k}\xi^k - J^\alpha_j \xi^i{}_{,\alpha} + J^i_\alpha \xi^\alpha{}_{,j} = 0. \end{array} \tag{30}$$

Let us find the conditions of integrability of (30). According to [17, p. 17] for the Levi-Civita connection the conditions are

$$\mathcal{L}_\xi R^h_{ijk} = \nabla_j \mathcal{L}_\xi \Gamma^h_{ik} - \nabla_k \mathcal{L}_\xi \Gamma^h_{ij}. \tag{31}$$

For the present case we have

$$\mathcal{L}_\xi \Gamma^h_{ij} = \frac{1}{2}\left((\delta^h_i \omega_\alpha \xi^\alpha)_{,j} + \delta^h_j (\omega_\alpha \xi^\alpha)_{,i} g_{ik} - g^{th}(\omega_\alpha \xi^\alpha)_{,t} g_{ij}\right). \tag{32}$$

Since for the conformal transformations the equations

$$\omega_j \mathcal{L}_\xi g_{ik} = \omega^\alpha (\mathcal{L}_\xi g_{j\alpha}) g_{ik},$$

are satisfied hence (32) can be presented in the form

$$\mathcal{L}_\xi \Gamma^h_{ij} = \mathcal{L}_\xi B^h_{ij}, \tag{33}$$

where

$$B^h_{ij} = \frac{1}{2}(\delta^h_i \omega_j + \delta^h_j \omega_i - \omega^h g_{ij}).$$

Also there is identity [17, p. 16] that for the present case is

$$\mathcal{L}_\xi \nabla_k B^h_{ij} - \nabla_k \mathcal{L}_\xi B^h_{ij} = \mathcal{L}_\xi \Gamma^h_{tk} B^t_{ij} - \mathcal{L}_\xi \Gamma^t_{ik} B^h_{tj} - \mathcal{L}_\xi \Gamma^t_{jk} B^h_{it}. \tag{34}$$

Taking account of (32), (33), (32), from (31) we obtain

$$\mathcal{L}_\xi R^h_{ijk} = \mathcal{L}_\xi \nabla_j B^h_{ik} - \mathcal{L}_\xi \nabla_k B^h_{ij} + \mathcal{L}_\xi (B^h_{tk} B^t_{ij}) - \mathcal{L}_\xi (B^h_{tj} B^t_{ik}).$$

Finally we have

$$\begin{aligned}\mathcal{L}_\xi R^h_{ijk} = \mathcal{L}_\xi \Big(& \frac{1}{2}\delta^h_k \omega_{i,j} - \frac{1}{2}\nabla_j \omega^h g_{ik} - \frac{1}{2}\delta^h_j \omega_{i,k} + \frac{1}{2}\nabla_k \omega^h g_{ij} \\ & + \frac{1}{4}\delta^h_k \omega_i \omega_j - \frac{1}{4}\delta^h_j \omega_i \omega_k + \frac{1}{4}||\omega||^2 \delta^h_j g_{ik} \\ & - \frac{1}{4}||\omega||^2 \delta^h_k g_{ij} + \frac{1}{4}\omega^h \omega_k g_{ij} - \frac{1}{4}\omega^h \omega_j g_{ik}\Big),\end{aligned}$$

or

$$\mathcal{L}_\xi Q^h_{ijk} = 0, \tag{35}$$

where Q^h_{ijk} is defined as

$$\begin{aligned} Q^h_{ijk} = R^h_{ijk} &+ \delta^h_j \Big(\frac{1}{2}\omega_{i,k} + \frac{1}{4}\omega_i \omega_k - \frac{1}{8}||\omega||^2 g_{ik}\Big) - \delta^h_k \Big(\frac{1}{2}\omega_{i,j} + \frac{1}{4}\omega_i \omega_j - \frac{1}{8}||\omega||^2 g_{ij}\Big) \\ &+ \Big(\frac{1}{2}{\omega^h}_{,j} + \frac{1}{4}\omega^h \omega_j - \frac{1}{8}||\omega||^2 \delta^h_j\Big) g_{ik} - \Big(\frac{1}{2}{\omega^h}_{,k} + \frac{1}{4}\omega^h \omega_k - \frac{1}{8}||\omega||^2 \delta^h_k\Big) g_{ij}. \end{aligned} \tag{36}$$

Differentiating several times (35) we get a system of differential prolongations. For convenience we use the identity for Lie derivative of tensor covariant derivative [17, p.16] and we obtain first differential prolongation for (35)

$$\mathcal{L}_\xi \nabla_l Q^h_{ijk} = \mathcal{L}_\xi \Gamma^h_{tl} Q^t_{ijk} - \mathcal{L}_\xi \Gamma^t_{il} Q^h_{tjk} - \mathcal{L}_\xi \Gamma^t_{jl} Q^h_{itk} - \mathcal{L}_\xi \Gamma^t_{kl} Q^h_{ijt}, \tag{37}$$

where $\mathcal{L}_\xi \Gamma^h_{jk}$ and Q^h_{ijk} are defined (32) and (36) respectively. We can continue the process until it turn out that the new equations are satisfied identically or the system have became inconsistent.

The equations (30_1) are solvable for $n = 2m$ unknown functions, and the equations (30_3) are solvable for $n^2 = 4m^2$ unknown functions. The equations (30_2) include $\frac{n(n+1)}{2} = \frac{2m(2m+1)}{2}$ restrictions. It is easy to see that (30_4) determines $2m^2$ independent restrictions. Since an LCK-manifold is a Hermitian one, then it follows from integrability of its almost complex structure that there exists a system of complex coordinate neighbourhoods. In the complex coordinate system $(z^\alpha, z^{\hat\alpha})$ the conditions (30_4) are presented in the form

$$\partial_{\hat\beta} \xi^\alpha = 0$$
$$\partial_\beta \xi^{\hat\alpha} = 0$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

Hence we have

$$\nabla_{\hat{\beta}}\xi^{\alpha} = \Gamma^{\alpha}_{\hat{\beta}\delta}\xi^{\delta} = \frac{\sqrt{-1}}{2}J^{\alpha}_{\hat{\beta},\delta}\xi^{\delta} \qquad \text{and} \qquad \nabla_{\beta}\xi^{\hat{\alpha}} = \Gamma^{\hat{\alpha}}_{\beta\hat{\delta}}\xi^{\hat{\delta}} = \frac{\sqrt{-1}}{2}J^{\hat{\alpha}}_{\beta,\hat{\delta}}\xi^{\hat{\delta}}.$$

Lowering the indices we obtain

$$\nabla_{\hat{\beta}}\xi_{\hat{\alpha}} = \frac{\sqrt{-1}}{2}J_{\hat{\beta}\hat{\alpha},\delta}\xi^{\delta} \qquad \text{and} \qquad \nabla_{\beta}\xi_{\alpha} = \frac{\sqrt{-1}}{2}J_{\beta\alpha,\hat{\delta}}\xi^{\hat{\delta}}.$$

Hence we find that the equations (30_2) include $m(m+1)$ restrictions which involve (30_4). It follows that solution of the system (30) involves not more then

$$4m^2 + 2m - \frac{2m(2m+1)}{2} + m(m+1) - 2m^2 + 1 = (m+1)^2$$

constants. □

Theorem 4. *In order that an LCK-manifold (M^n, J, g) admits a group of conformal transformations, it is necessary and sufficient that the equations*

$$\begin{array}{l} \xi_{i,j} + \xi_{j,i} = (\omega_{\alpha}\xi^{\alpha} + C)g_{ij}; \\ J^{i}_{j,k}\xi^{k} - J^{\alpha}_{j}\xi^{i}_{\ ,\alpha} + J^{i}_{\alpha}\xi^{\alpha}_{\ ,j} = 0, \end{array}$$

the conditions of integrability (35), their differential prolongations (37),... etc, be algebraically consistent with respect to ξ^i and ξ^i_j. If there are, among the equations (35), (37),..., exactly k equations which are linearly independent among themselves and of

$$\begin{array}{l} \xi_{i,j} + \xi_{j,i} = (\omega_{\alpha}\xi^{\alpha} + C)g_{ij}; \\ J^{i}_{j,k}\xi^{k} - J^{\alpha}_{j}\xi^{i}_{\ ,\alpha} + J^{i}_{\alpha}\xi^{\alpha}_{\ ,j} = 0, \end{array}$$

then the LCK-manifold admits a $r = (m+1)^2 - k$ parameter group of conformal transformations.

Considering the system (30) we can find that if $\omega_{\alpha}\xi^{\alpha} = 0$, then the system may also be written in the form

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \xi_{i,j} + \xi_{j,i} = Cg_{ij}; \\ 3) & \xi_{i,jk} = \xi_{\alpha}R^{\alpha}_{kji}; \\ 4) & J^{i}_{j,k}\xi^{k} - J^{\alpha}_{j}\xi^{i}_{\ ,\alpha} + J^{i}_{\alpha}\xi^{\alpha}_{\ ,j} = 0. \end{array}$$

thus we have the following theorem

Theorem 5. *If on an LCK-manifold (M^n, J, g) Lie algebra of conformal vector fields includes such subalgebra that everywhere on (M^n, J, g) $\omega_{\alpha}\xi^{\alpha} = 0$ holds, then the subalgebra generates group of homothetic transformations.*

Proof. The Theorem follows immediately from the Frobenius Theorem [1, p. 201]. □

3.4. Local isomorphism between conformal group of an LCK-manifold and homothetic group of the corrisponding Kählerian metric

Let Kählerian metric $\hat{g}$ be locally conformal to the metric of an LCK-manifold (M^n, J, g). According to the definition $g_{ij} = g_{ij}e^{-2\sigma}$, $\omega_i = 2\sigma_{,i}$. Then

$$\hat{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} - \frac{1}{2}\delta^{k}_{i}\omega_{j} - \frac{1}{2}\delta^{k}_{j}\omega_{i} + \frac{1}{2}\omega^{k}g_{ij}, \tag{38}$$

is the Levi-Civita connection which is compatible with the metric $\hat{g}$. Let us define a contravariant vector field ξ^i on (M^n, J, g). Let us denote

$$\xi_i = \xi^\alpha g_{\alpha i}; \quad \hat{\xi}_i = \xi^\alpha \hat{g}_{\alpha i} = \xi_i e^{-2\sigma}.$$

Then we differentiate covariantly $\hat{\xi}_i$ with respect to the Levi-Civita connection which is compatible with the metric $\hat{g}$. Covariant derivative with respect to the connection $\hat{\Gamma}^k_{ij}$ is denoted as $''|''$. Covariant derivative with respect to the connection Γ^k_{ij} is denoted as usual by comma. We get

$$\begin{aligned}
\hat{\xi}_{i|j} &= \hat{\xi}_{i,j} + (\frac{1}{2}\delta^\alpha_i \omega_j + \frac{1}{2}\delta^\alpha_j \omega_i - \frac{1}{2}\omega^\alpha g_{ij})\hat{\xi}_\alpha = \\
&= \left(\xi_i e^{-2\sigma}\right)_{,j} + \frac{1}{2}\hat{\xi}_i\omega_j + \frac{1}{2}\hat{\xi}_j\omega_i - \frac{1}{2}\omega^\alpha\hat{\xi}_\alpha g_{ij} = \\
&= \xi_{i,j}e^{-2\sigma} - \xi_i e^{-2\sigma}\omega_j + \frac{1}{2}\hat{\xi}_i\omega_j + \frac{1}{2}\hat{\xi}_j\omega_i - \frac{1}{2}\omega^\alpha\hat{\xi}_\alpha g_{ij} = \\
&= (\xi_{i,j} - \frac{1}{2}\xi_i\omega_j + \frac{1}{2}\xi_j\omega_i - \frac{1}{2}\omega^\alpha\xi_\alpha g_{ij})e^{-2\sigma} = \\
&= (\xi_{i,j} - \frac{1}{2}\xi_i\omega_j + \frac{1}{2}\xi_j\omega_i)e^{-2\sigma} - \frac{1}{2}\omega^\alpha\xi_\alpha\hat{g}_{ij}
\end{aligned} \tag{39}$$

Suppose that a field ξ^i generates a homothetic group of the metric $\hat{g}$. Then it must satisfy equations

$$\hat{\xi}_{i|j} + \hat{\xi}_{j|i} = C\hat{g}_{ij} \tag{40}$$

Substituting (39) into (40) we obtain

$$\begin{aligned}
e^{-2\sigma}(\xi_{i,j} + \xi_{j,i}) - \omega^\alpha\xi_\alpha\hat{g}_{ij} &= C\hat{g}_{ij}; \\
e^{-2\sigma}(\xi_{i,j} + \xi_{j,i}) &= C\hat{g}_{ij} + \omega^\alpha\xi_\alpha\hat{g}_{ij}; \\
e^{-2\sigma}(\xi_{i,j} + \xi_{j,i}) = e^{-2\sigma}&(\omega^\alpha\xi_\alpha g_{ij} + C g_{ij});
\end{aligned}$$

Since $e^{-2\sigma} \neq 0$ holds, (30_2) are necessarily satisfied

$$\xi_{i,j} + \xi_{j,i} = (\omega_\alpha\xi^\alpha + C)g_{ij}.$$

Let us differentiate covariantly $\hat{\xi}_{i|j}$ with respect to the connection $\hat{\Gamma}^k_{ij}$. Since (38) holds, we obtain

$$\begin{aligned}
\hat{\xi}_{i|jk} = e^{-2\sigma}(\xi_{i,jk} &+ \frac{1}{2}(\xi_{i,j} + \xi_{j,i})\omega_i - \frac{1}{2}\omega^\alpha(\xi_{\alpha,j}g_{ik} + \xi_{i,\alpha}g_{jk}) \\
&+ \frac{1}{4}\omega_k(\xi_j\omega_i - \xi_i\omega_j) - \frac{1}{4}\xi^\alpha\omega_\alpha(\omega_i g_{jk} - \omega_j g_{ik}) \\
+\frac{1}{2}(\xi_j\omega_{i,k} - \xi_i\omega_{j,k}) &+ \frac{1}{4}||\omega||^2(\xi_i g_{jk} - \xi_j g_{ik}) - \frac{1}{2}(\omega_\alpha\xi^\alpha)_{,k}g_{ij};
\end{aligned}$$

On other hand

$$\begin{aligned}
&\frac{1}{2}(\xi_{i,j} + \xi_{j,i})\omega_i - \frac{1}{2}\omega^\alpha(\xi_{\alpha,j}g_{ik} + \xi_{i,\alpha}g_{jk}) \\
= \frac{1}{2}(\omega_i\mathcal{L}_\xi g_{jk} &- \nabla_j(\omega_\alpha\xi^\alpha)g_{ik} + \xi^\alpha\omega_{\alpha,j}g_{ik} - \omega^\alpha(\mathcal{L}_\xi g_{i\alpha} - \xi_{\alpha,i})) \\
&= \frac{1}{2}(\omega_i\mathcal{L}_\xi g_{jk} - \omega^\alpha(\mathcal{L}_\xi g_{i\alpha})g_{jk} - \nabla_j(\omega_\alpha\xi^\alpha)g_{ik} \\
&+\xi^\alpha\omega_{\alpha,j}g_{ik} + \nabla_i(\omega_\alpha\xi^\alpha)g_{jk} - \xi^\alpha\omega_{\alpha,i}g_{jk})
\end{aligned}$$

hence

$$\begin{aligned}\hat{\xi}_{i|jk} = e^{-2\sigma}\big(\xi_{i,jk} + \frac{1}{4}\omega_k(\xi_j\omega_i - \xi_i\omega_j) - \frac{1}{4}\xi^\alpha\omega_\alpha(\omega_i g_{jk} - \omega_j g_{ik})\\ +\frac{1}{2}(\xi_\alpha\omega^\alpha{}_{,j}g_{ik} - \xi_\alpha\omega^\alpha{}_{,i}g_{jk}) + \frac{1}{2}(\xi_j\omega_{i,k} - \xi_i\omega_{j,k})\\ +\frac{1}{4}||\omega||^2(\xi_i g_{jk} - \xi_j g_{ik}) + \frac{1}{2}(\omega_i\mathcal{L}_\xi g_{jk} - \omega^\alpha(\mathcal{L}_\xi g_{i\alpha})g_{jk})\\ -\frac{1}{2}((\omega_\alpha\xi^\alpha)_{,k}g_{ij} + (\omega_\alpha\xi^\alpha)_{,j}g_{ik} - (\omega_\alpha\xi^\alpha)_{,i}g_{jk})\big)\end{aligned} \tag{41}$$

Since according to (21_2) in the case of conformal transformations we have $\mathcal{L}_\xi g_{jk} = \varphi g_{jk}$, hence $\omega_i\mathcal{L}_\xi g_{jk} - \omega^\alpha(\mathcal{L}_\xi g_{i\alpha})g_{jk} = 0$, and (41) can be written as

$$\begin{aligned}\hat{\xi}_{i|jk} = e^{-2\sigma}\big(\xi_{i,jk} + \frac{1}{4}\omega_k(\xi_j\omega_i - \xi_i\omega_j) - \frac{1}{4}\xi^\alpha\omega_\alpha(\omega_i g_{jk} - \omega_j g_{ik})\\ +\frac{1}{2}(\xi_\alpha\omega^\alpha{}_{,j}g_{ik} - \xi_\alpha\omega^\alpha{}_{,i}g_{jk}) + \frac{1}{2}(\xi_j\omega_{i,k} - \xi_i\omega_{j,k})\\ +\frac{1}{4}||\omega||^2(\xi_i g_{jk} - \xi_j g_{ik}) - \frac{1}{2}((\omega_\alpha\xi^\alpha)_{,k}g_{ij} + (\omega_\alpha\xi^\alpha)_{,j}g_{ik} - (\omega_\alpha\xi^\alpha)_{,i}g_{jk})\big),\end{aligned}$$

or

$$\begin{aligned}\hat{\xi}_{i|jk} = e^{-2\sigma}\big(\xi_{i,jk} + \xi_\alpha\big(\frac{1}{4}\omega_k(\delta^\alpha_j\omega_i - \delta^\alpha_i\omega_j) - \frac{1}{4}\omega^\alpha(\omega_i g_{jk} - \omega_j g_{ik})\\ +\frac{1}{2}(\omega^\alpha{}_{,j}g_{ik} - \omega^\alpha{}_{,i}g_{jk}) + \frac{1}{2}(\delta^\alpha_j\omega_{i,k} - \delta^\alpha_i\omega_{j,k}) + \frac{1}{4}||\omega||^2(\delta^\alpha_i g_{jk} - \delta^\alpha_j g_{ik})\big)\\ -\frac{1}{2}((\omega_\alpha\xi^\alpha)_{,k}g_{ij} + (\omega_\alpha\xi^\alpha)_{,j}g_{ik} - (\omega_\alpha\xi^\alpha)_{,i}g_{jk})\big),\end{aligned} \tag{42}$$

where $||\omega||^2 = \omega_i\omega_j g^{ij}$. On other hand, it follows from (38) that the curvature tensor $\hat{R}$ of a Kähler metric $\hat{g}$ and the curvature tensor R of an LCK-metric are related by the following expression

$$\begin{aligned}\hat{R}^h_{ijk} = R^h_{ijk} + \delta^h_j\big(\frac{1}{2}\omega_{i,k} + \frac{1}{4}\omega_i\omega_k - \frac{1}{4}||\omega||^2 g_{ik}\big) -\\ -\delta^h_k\big(\frac{1}{2}\omega_{i,j} + \frac{1}{4}\omega_i\omega_j - \frac{1}{4}||\omega||^2 g_{ij}\big) +\\ +\big(\frac{1}{2}\omega^h{}_{,j} + \frac{1}{4}\omega^h\omega_j\big)g_{ik} - \big(\frac{1}{2}\omega^h{}_{,k} + \frac{1}{4}\omega^h\omega_k\big)g_{ij},\end{aligned} \tag{43}$$

It is known that if a field ξ^i generates homothetic transformation of metric $\hat{g}$ then the field satisfies also the equation [7]

$$\hat{\xi}_{i|jk} = \hat{\xi}_\alpha\hat{R}^\alpha_{kji}. \tag{44}$$

Substituting (42) and (43) into (44), taking into account that $\hat{\xi}_i = \xi_i e^{-2\sigma}$, we get

$$e^{-2\sigma}\xi_{i,jk} = e^{-2\sigma}\big(\xi_\alpha R^\alpha_{kji} + \frac{1}{2}((\omega_\alpha\xi^\alpha)_{,k}g_{ij} + (\omega_\alpha\xi^\alpha)_{,j}g_{ik} - (\omega_\alpha\xi^\alpha)_{,i}g_{jk})\big).$$

Again, it follows from $e^{-2\sigma} \neq 0$ that (30_3) is satisfied

$$\xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}\big((\omega_\alpha\xi^\alpha)_{,k}g_{ij} + (\omega_\alpha\xi^\alpha)_{,j}g_{ik} - (\omega_\alpha\xi^\alpha)_{,i}g_{jk}\big).$$

The condition that for a Kähler metric $\hat{g}$ a vector field ξ^i satisfies

$$\mathcal{L}_\xi J^i_j = J^i_{j|k}\xi^k - J^\alpha_j\xi^i{}_{|\alpha} + J^i_\alpha\xi^\alpha{}_{|j} = 0.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

if and only if the similar conditions(9) is satisfied. Hence if a vector field ξ^i satisfies the system (30), then it satisfies the system

$$\begin{aligned}
&1) \quad \hat{\xi}_{i,j} = \hat{\xi}_{ij};\\
&2) \quad \hat{\xi}_{i,j} + \hat{\xi}_{j,i} = C\hat{g}_{ij};\\
&3) \quad \hat{\xi}_{i,jk} = \hat{\xi}_{\alpha}\hat{R}^{\alpha}_{kji};\\
&4) \quad J^i_{j|k}\xi^k - J^{\alpha}_j\xi^i{}_{|\alpha} + J^i_{\alpha}\xi^{\alpha}{}_{|j} = 0.
\end{aligned}$$

We obtain the theorem

Theorem 6. *If an LCK-manifold (M^n, J, g), $n = 2m$ admits a group G_r of infinitesimal conformal transformations preserving the complex structure, then the group G_r is isomorphic to the group of homothetic transformations of the Kähler metric $\hat{g}$ conformally corresponding to the LCK-metric.*

It is worth to note that the obtained theorem is very similar to the results obtained by R. F. Bilyalov ([6], p. 274) for real Lorenzian manifolds. Namely, let G_r be a group of conformal transformations of a Lorenzian manifold (M^n, g) which is not conformally flat. Then we can find a manifold $(\hat{M}^n, \hat{g})$, conformally corresponding to (M^n, g) whose homothetic group is isomorphic to the group of conformal transformations of the (M^n, g). But our result does not require that the manifold needs not to be conformally flat.

Applying the Theorems 6 and 4 to conformally flat manifolds, in particularly to a Hopf manifold, equipped by the Boothby metric, we obtain that conformal groups of the manifolds depend on $(m+1)^2$ parameters, where $m = \dim_{\mathbb{C}}(M^n)$.

3.5. Conformal infinitesimal transformations on compact LCK-manifolds

Let (M^n, J, g) be a compact LCK-manifold, vector field ξ generates conformal transformations (30_2). Transvecting (30_3) with g^{jk} we have

$$\nabla^t\nabla_t\xi_i - \xi_{\alpha}R^{\alpha}_i = \frac{2-n}{2}\nabla_i(\omega_{\alpha}\xi^{\alpha}). \tag{45}$$

Then we raise the index i in (45)

$$\nabla^t\nabla_t\xi^i - \xi^{\alpha}R^i_{\alpha} = \frac{(2-n)g^{it}}{2}\nabla_t(\omega_{\alpha}\xi^{\alpha}). \tag{46}$$

On other hand, it'known [18], that a necessary and sufficient condition for a vector field ξ in a compact almost Hermitian space to be contravariant almost analytic is

$$\nabla^t\nabla_t\xi_i - \xi_{\alpha}R^{\alpha}_i = -J^i_{\alpha}(\mathcal{L}_{\xi}\nabla_{\beta}J^{\beta}_{\gamma}g^{\alpha\gamma}) + \frac{1}{2}(\nabla_j J^{\alpha}_k + \nabla_k J^{\alpha}_j)J^i_{\alpha}\mathcal{L}_{\xi}g^{jk}. \tag{47}$$

For LCK-manifolds, taking account of (2) and (1), we have

$$\begin{aligned}
-J^i_{\alpha}(\mathcal{L}_{\xi}\nabla_{\beta}J^{\beta}_{\gamma}g^{\alpha\gamma}) + \frac{1}{2}(\nabla_j J^{\alpha}_k + \nabla_k J^{\alpha}_j)J^i_{\alpha}\mathcal{L}_{\xi}g^{jk}\\
= \frac{(2-n)g^{it}}{2}\nabla_t(\omega_{\alpha}\xi^{\alpha}).
\end{aligned} \tag{48}$$

Comparing (46) and (47), taking account of (48) we obtain the theorem.

Theorem 7. *In a compact LCK-manifold (M^n, J, g) any vector field ξ which generates nontrivial conformal transformations is contravariant almost analytic.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

3.6. Isometries of LCK-manifolds

Let a vector field ξ generate one-parameter continuous group of isometries of an LCK-manifold. Then the vector field ξ satisfies Killing equtions.

$$\xi_{i,j} + \xi_{i,j} = 0. \tag{49}$$

Note, that we denote by comma covariant differentiation with respect to the Levi-Civita connection of (M^n, J, g). Taking account of (39), expressing (49) with respect to the Levi-Civita connection which is compatible with the Kählerian metric $\hat{g}$, we obtain

$$\hat{\xi}_{i|j} + \hat{\xi}_{j|i} = -\xi^\alpha \omega_\alpha \hat{g}_{ij}. \tag{50}$$

But it follows from the Theorem 3 that Kählerian metric does not admit nontrivial conformal transformations. Hence $\xi^\alpha \omega_\alpha = \text{const}$, and we obtain the theorem.

Theorem 8. *Isometric Group of an LCK-manifold (M^n, J, g) is isomorphic to a some subgroup of homothetic group of the coresponding local Kählerian metric. In particular, vector fields orthogonal to the Lee field which are Killing respect to the LCK-metric g are also Killing respect to the local Kählerian metric $\hat{g}$.*

3.7. Transformations generated by the Lee fields and anti-Lee fields on pseudo-Vaisman manifolds

Let us consider a pseudo-Vaisman manifold [8] i. e. the LCK-manifold whose Lee form satisfies the equation

$$\Phi_4(\nabla\omega(X, Y)) = \frac{||\omega||^2}{2} g(X, Y), \tag{51}$$

where Φ_4 is the fourth Obata projector. It follows from (51) that, Lie derivative with respect to the vector field $B = \omega^\#$ satisfies the equations

$$\mathcal{L}_B g_{ij} + J_i^s (\mathcal{L}_B g_{st}) J_j^t = 2||\omega||^2 g_{ij}.$$

Let us find a Lie derivative of a fundamental form $\Omega_{ij} = J_i^s g_{sj}$. According to [10, p. 4] on an LCK-manifold, covariant derivative of the complex structure in the directions of B or A is equal to zero

$$\nabla_B J = \nabla_A J = 0. \tag{52}$$

Here $A = -JB$ is so called anti-Lee fields. Hence

$$\mathcal{L}_B \Omega_{ij} = \Omega_{tj} \nabla_i \omega^t + \Omega_{it} \nabla_j \omega^t = -J_j^t \omega_{t,i} + J_i^t \omega_{t,j}. \tag{53}$$

Since (51) is equivalent to

$$\omega_{t,j} J_i^t - \omega_{t,i} J_j^t - ||\omega||^2 J_{ij} = 0,$$

it follows from (53) that

$$\mathcal{L}_B \Omega_{ij} = ||\omega||^2 \Omega_{ij}. \tag{54}$$

Let us find a Lie derivative of the fundamental form with respect to the anti-Lee field $A = -JB = \theta^\#$. Since (52) holds, we have

$$\begin{aligned} \mathcal{L}_A \Omega_{ij} &= \Omega_{tj} \nabla_i \theta^t + \Omega_{it} \nabla_j \theta^t = J_i^t \theta_{t,j} - J_j^t \theta_{t,i} \\ &= J_i^t \nabla_j (J_t^s \omega_s) - J_j^t \nabla_i (J_t^s \omega_s) = J_i^t \omega_s J_{t,j}^s - J_j^t \omega_s J_{t,i}^s + J_i^t J_t^s \omega_{s,j} - J_j^t J_t^s \omega_{s,i} \\ &= -\omega_{i,j} - \omega_{j,i} + \frac{1}{2} J_i^t \omega_s (\delta_j^s J_t^u \omega_u - \omega^s J_{tj} - J_j^s \omega_t + J_u^s \omega^u g_{tj}) \\ &\quad - \frac{1}{2} J_j^t \omega_s (\delta_i^s J_t^u \omega_u - \omega^s J_{ti} - J_i^s \omega_t + J_u^s \omega^u g_{ti}). \end{aligned} \tag{55}$$

 doi:10.20944/preprints201906.0232.v1

Removing the parentheses in (55), and taking into account that Lee form is closed, we have

$$\mathcal{L}_A\Omega_{ij} = 0.$$

91 We obtain the theorem.

Theorem 9. *On a pseudo-Vaisman manifold i. e. on an LCK-manifold whose Lee form satisfies the condition*

$$\Phi_4(\nabla\omega(X,Y)) = \frac{||\omega||^2}{2}g(X,Y),$$

Lie derivatives of the fundamental form with respect to the Lee field $B = \omega^\#$ and anti-Lee field $A = -JB = \theta^\#$ satisfy the equations

$$1)\mathcal{L}_B\Omega_{ij} = ||\omega||^2\Omega_{ij},$$
$$2)\mathcal{L}_A\Omega_{ij} = 0.$$

Let us find a Lie derivative of the complex structure with respect to the Lee field B and the anti-Lee field A taking account of (52).

$$\mathcal{L}_B J_i^k = J_s^k\nabla_i\omega^s - J_i^t\nabla_t\omega^k. \tag{56}$$

$$\begin{aligned}\mathcal{L}_A J_i^k &= J_t^k\nabla_i\theta^t - J_i^t\nabla_t\theta^k = -J_t^k\nabla_i(J_s^t\omega^s) + J_i^t\nabla_t(J_s^k\omega^s)\\ &= -J_t^k J_s^t\nabla_i\omega^s + J_i^t J_s^k\nabla_t\omega^s - J_t^k\omega^s J_{s,i}^t + J_i^t\omega^s J_{s,t}^k\\ &= \nabla_i\omega^k + J_i^t J_s^k\nabla_t\omega^s - \frac{1}{2}J_t^k\omega^s(\delta_i^t J_s^u\omega_u - \omega^t J_{si} - J_i^t\omega_s + J_u^t\omega^u g_{si})\\ &+\frac{1}{2}J_i^t\omega^s(\delta_t^k J_s^u\omega_u - \omega^k J_{st} - J_t^k\omega_s + J_u^k\omega^u g_{st}).\end{aligned} \tag{57}$$

Removing the parentheses in (57) and collecting similar terms, we obtain that

$$\mathcal{L}_A J_i^k = \nabla_i\omega^k + J_i^t J_s^k\nabla_t\omega^s. \tag{58}$$

Let us find a Lie derivative of the LCK-metric with respect to the anti-Lee field A

$$\begin{aligned}\mathcal{L}_A g_{ij} &= \theta_{i,j} + \theta_{j,i} = \nabla_j(J_i^t\omega_t) + \nabla_i(J_j^t\omega_t)\\ &= J_{i,j}^t\omega_t + J_i^t\omega_{t,j} + J_{j,i}^t\omega_t + J_j^t\omega_{t,i}\\ &= \frac{1}{2}\omega_t(\delta_j^t J_i^u\omega_u - \omega^t J_{ij} - J_j^t\omega_i + J_u^t\omega^u g_{ij}) + J_i^t\omega_{t,j}\\ &+\frac{1}{2}\omega_t(\delta_i^t J_j^u\omega_u - \omega^t J_{ji} - J_i^t\omega_j i + J_u^t\omega^u g_{ji}) + J_j^t\omega_{t,i}\end{aligned}$$

Finally, we get

$$\mathcal{L}_A g_{ij} = J_i^t\nabla_j\omega_t + J_j^t\nabla_i\omega_t. \tag{59}$$

Now let us consider the case when the Lee form satisfies strong pseudo-Vaisman condition

$$\nabla\omega(X,Y) = \frac{||\omega||^2}{2}g(X,Y)$$

Hence the Lee field satisfies the equations

$$\omega_{i,j} + \omega_{j,i} = ||\omega||^2 g_{ij}$$

Comparing the equations with (30_2)

$$\xi_{i,j} + \xi_{j,i} = (\omega_\alpha\xi^\alpha + C)g_{ij},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2019 doi:10.20944/preprints201906.0232.v1

we obtain, that the Lee field $\omega^\#$ generates on the LCK-manifold one-parameter conformal group for which in (30_2) the condition $C = 0$ holds. We get

$$\omega_{i,j} + \omega_{j,i} = (\omega_\alpha \omega^\alpha) g_{ij}.$$

Taking account of (39) we obtain that for the connection which is compatible with the Kählerian metric $\hat{g}_{ij} = e^{-2\sigma} g_{ij}$ the equations

$$\hat{\omega}_{i|j} + \hat{\omega}_{j|i} = 0, \tag{60}$$

are satisfied. Here we note $\hat{\omega}_i = \omega_t g^{st} \hat{g}_{ti} = e^{-2\sigma}\omega_i$. It follows from (60) that the vector field $\omega^\#$ generates one-parameter isometry group of the Kählerian metric $\hat{g}_{ij}$. Also it follows from (56) that if the Lee form satisfies strong pseudo-Vaisman condition, the we have

$$\mathcal{L}_B J_i^k = 0.$$

Hence The Lee field is contravariant analytic, i. e. a transformation generated by the field preserves the complex structure. Also, substituting the strong pseudo-Vaisman condition into (57), we obtain

$$\mathcal{L}_A J_i^k = 0,$$

It means that anti-Lee field is also contravariant analytic. Hence we write (59) in the form

$$\mathcal{L}_A g_{ij} = 0.,$$

That mean also that anti-Lee field $\theta^\#$ is a Killing field. Taking into account Theorem 8 we make the following deductions.

Theorem 10. *If on an LCK-manifold (M^n, J, g), $n = 2m$ Lee form satisfies strong pseudo-Vaisman condition*

$$\nabla\omega(X,Y) = \frac{||\omega||^2}{2} g(X,Y),$$

then the vector the Lee and anti-Lee fields (respectively $\omega^\#$ and $\theta^\#$) are contravariant analitic. On the manifold (M^n, J, g) the Lie field $\omega^\#$ generates one-parameter conformal group, and anti-Lee field $\theta^\#$ generates one-parameter group of isometry. Both fields generate one-parameter isometric groups of the Kählerian metric $\hat{g}_{ij}$ conformally corresponding to the LCK-metric g.

4. Conclusion

The manifolds under consideration are LCK-manifolds. The investigations use local coordinates. We assume that all functions under consideration are sufficiently differentiable, and use tensor methods (c.f. [12]).

Complex geometry deals primarily with Kählerian manifolds i.e. manifolds carrying some Kählerian metric. But some complex manifolds, such as complex Hopf manifolds admit no global Kählerian metrics at all. But we can often find for every map of atlas a multiplyer which transforms a metric into a Kählerian one. One can say that a metric g is a locally conformal Kähler (LCK) metric if g is conformal to some local Kählerian metric in the neighborhood of each point of a manifold. Actually the locally Conformal Kähler manifolds was introduced by W. Westlake in 1954, some publications was soon made by P. Libermann, but mainly through the works of Vaisman since the 1970s the geometry of LCK-manifolds has developed. Mappings and transformations of LCK-manifolds were explored by V. F. Kirichenko, J. Mikeš, A. Moroianu, L. Ornea.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

 doi:10.20944/preprints201906.0232.v1

The following abbreviation is used in this manuscript:

LCK-manifolds locally Conformal Kähler manifolds

References

1. Dubrovin, B. A.; Fomenko, A. T.; Novikov, S. P. Modern geometry-methods and applications. Part I. The geometry of surfaces, transformation groups, and fields. Second edition. Translated from the Russian by Robert G. Burns. Graduate Texts in Mathematics, 93. Springer-Verlag, New York, 1992; xvi+468 pp. ISBN:0-387-97663-9.
2. Kirichenko, V. F. Locally conformallity Kählerian manifolds of constant holomorphic sectional curvature, *Math. USSR-Sb.* **1992**, *72(2)*, 333-342, DOI:10.1070/SM1992v072n02ABEH002142.
3. Kirichenko, V. F. Conformally flat and locally conformal Kahler manifolds, *Mathematical Notes* **1992**, *51(5)*, 462-468, DOI:10.1007_BF01262178.
4. Mikeš, J.; Radulovich, Zh. Geodesic mappings of locally conformal Kähler spaces, *Russian Mathematics (Izvestiya VUZ. Matematika)* **1994**, *38(3)*, 48-50.
5. Mikeš, J.; Moldobaev, D. Distribution of the orders of groups of conformal transformations of Riemannian spaces, *Izv. Vyssh. Uchebn. Zaved. Mat.* **1991**, *12*, 24-29. In Russian.
6. Petrov, A. Z. New methods in the general theory of relativity. *Nauka, Moscow*, 1966, 496 pp.
7. Eisenhart, L. P. Riemannian geometry. *Princeton University Press, Princeton, NJ*, 8. 1997. x+306 pp. ISBN: 0-691-02353-0.
8. Cherevko, Y.; Chepurna, O. Complex and Real Hypersurfaces of Locally Conformal Kähler Manifolds In: *Proceedings of the Eighteenth International Conference on Geometry, Integrability and Quantization, Ivanlo M. Mladenov, Guowu Meng and Akira Yoshioka, eds.* Sofia: Avangard Prima, 2017, pp. 117-129, DOI:10.7546/giq-18-2017-117-129.
9. Dine, M. Supersymmetry and String Theory. Beyond the Standard Model. *Cambridge University Press, Cambridge*, 2007, xx+515 pp. ISBN:978-0-521-85841-0.
10. Dragomir, S.; Ornea, L. Locally conformal Kähler geometry. *Birkhäuser: Boston ; Basel ; Berlin*, 1998, xiii+330 pp. ISBN:978-1-4612-2026-8.
11. Januş, S.; Visineşcu, M. Kaluza - Klein theory with scalar fields on generalized Hopf manifolds, *Clas. Quantum Grav.* **1987**, *V(4)*, 1317-1325, DOI:10.1088/0264-9381/4/5/026.
12. Mikeš, J.; Vanžurová, A.; Hinterleitner, I. Geodesic mappings and some generalizations. *Palacky Univ. Press, Olomouc*, 2009, 304 pp. ISBN: 978-80-244-2524-5.
13. Moroianu, A.; Ornea, L. Transformations of locally conformally Kahler manifolds, *Manuscripta Mathematica* **2009**, *130(1)*, 93-100, DOI:10.1007/s00229-009-0278-z.
14. Shahbazi, C. S. M-theory on non-Kähler eight-manifolds. *Journal of High Energy Physics* **2015**, *9*, 1-31, DOI:10.1007/JHEP09(2015)178.
15. Vaisman, I. A geometric condition for an l.c.K. manifold to be Kähler *Geometriae Dedicata* **1981**, *10*, 129-134, DOI:10.1007/BF01447416.
16. Yano, K. Differential geometry on complex and almost complex spaces. *Pergamon Press Book: New York*, 1965, ISBN: 008010259X.
17. Yano, K. The Theory Of Lie Derivatives And Its Applications. *North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York*, 1957, x+299 pp. ISBN: 1296032183.
18. Yano, K.; Ako, M. Almost analytic vectors in almost complex spaces. *Tohoku Math. J.* **1961**, *13(1)*, 24-45, DOI:10.2748/tmj/1178244350.
19. Zumino, B. Supersymmetry and Kähler Manifolds. *Phys. Lett. B* **1979**, *87(3)*, 203-206, DOI:10.1016/0370-2693(79)90964-X.