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Article

On 2-Inner Product Spaces and Reproducing Kernel Property

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Abstract: This paper is devoted to the study of reproducing kernels on 2-inner product Hilbert spaces. We focus on a new structure to produce reproducing kernel Hilbert and Banach spaces. According to multi-variable computing, this structures can be useful in electrocardiographs, machine learning and economy.

Keywords: 2-inner product; vector-valued spaces; 2-semi norm

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1. Introduction

Reproducing Kernel Hilbert Spaces (RKHS) have been found unbelievably useful in several branch of abstract and objective sciences like the machine learning community. This theory developed for quite some time and has been used in especially in statistics for about twenty years. More recently, their application extended in to objective theories and arithmetics algorithms, deeply. distribution theory is the most relation between abstract concepts and objective applications. (See [7,10]).

A reproducing kernel Hilbert space (RKHS in short) is a Hilbert space $\mathcal{H}$ of functions defined on a fixed set X such that for each $x \in X$ the evaluation functional at x, i.e., $E_x(f) := f(x), f \in \mathcal{H}$, is continuous on $\mathcal{H}$. The Riesz representation theorem gives a unique function $K : X \times X \to \mathbb{C}$ such that

- $\{k(., x) : x \in X\} \subset \mathcal{H}$,
- $f(x) = \langle f, k(., x)\rangle_{\mathcal{H}}, \quad x \in X, f \in \mathcal{H}$.

The function K is called the reproducing kernel of $\mathcal{H}$. Similar structure can be defined for Banach spaces with a quick difference. More details about these kind of spaces can be found in [1,8,12].

In the following, in section 2, we prepare some preliminaries of 2-inner product spaces and review some important and useful theorems and lemmas. In section 3, we defined 2-inner product reproducing kernel Hilbert spaces and proved some theorems. finally, we extended the theorems to 2-inner reproducing Banach spaces.

2. Preliminaries

In this section, we review some basic concepts and definitions. we also, recall some important lemmas and theorems. Among these content, Riesz representation theorem on 2-inner product spaces plays a key roll in the next sections. This theorem, is a wise extension of basic type in [9] to 2-inner product spaces. Let V be a linear space of dimension greater than 1 over the field F (either $\mathbb{R}$ or $\mathbb{C}$). The function $\langle ., .; .\rangle : V \times V \times V \to F$ is called a 2-inner product if the following conditions holds,

1. $\langle x, x; z\rangle \geq 0$ and $\langle x, x; z\rangle = 0$ iff x and z are linearly dependent.
2. $\langle x, x; z\rangle = \underline{\langle z, z; x\rangle}$.
3. $\langle y, x; z\rangle = \overline{\langle x, y; z\rangle}$.
4. $\langle \alpha x, y; z\rangle = \alpha\langle x, y; z\rangle$, for all scalars $\alpha \in F$.
5. $\langle x_1 + x_2, y; z\rangle = \langle x_1, y; z\rangle + \langle x_2, y; z\rangle$.

Therefore, the pair $(V, \langle ., .; . \rangle)$ is called a 2-inner product space. Let $(V, \langle ., .; . \rangle)$ be a 2-inner product space and $x, y, b \in X$ then

$$x \perp^b y \Leftrightarrow \langle x, y; b \rangle = 0.$$

We define a 2-norm on $V \times V$ by

$$\|x, y\|^2 = \langle x, x; y \rangle.$$

It is easy to see that, this norm satisfies the following conditions:

1. $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent.
2. $\|x, z\| = \|z, x\|$,
3. $\|\alpha x, z\| = |\alpha| \|x, z\|$ for all $\alpha \in F$,
4. $\|x_1 + x_2, z\| \leq \|x_1, z\| + \|x_2, z\|$.

Therefore the pair $(V, \|., .\|)$ is called a linear 2-normed space.

2-norms have several interesting properties. More details can be found in [5]. If $(V, \langle ., . \rangle)$ is an inner product space, then the standard 2-inner product $\langle ., .; . \rangle$ is defined on V by

$$\langle x, y; z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle.$$

Let $(V, \langle ., .; . \rangle)$ be a 2-inner product space over F. If $\{e_i\}$, $1 \leq i \leq n$ are linearly independent vectors in the 2-inner product space V, then $\{e_i\}$, $1 \leq i \leq n$ is called a b-orthonormal set if for $b \in V$, $\langle e_i, e_j; b \rangle = 0$ for $i \neq j$ and $\langle e_i, e_j; b \rangle = 1$ for $i = j$ where $1 \leq i \leq n$. Let $(V, \langle ., .; . \rangle)$ be a 2-inner product space over F. The subspace of all linearly independent vectors in the 2-inner product space V respect to $x \in V$ is closed. Let $(H, \langle ., .; . \rangle)$ be a 2-inner product space over F, $b \in H$, then

1. A sequence $\{x_n\}$ in H is said to be a b-Cauchy sequence if for every $\epsilon > 0$ there exists $N > 0$ such that for every $m, n \geq N$, $0 < \|x_n - x_m, b\| < \epsilon$.
2. H is said to be b-Hilbert if every b-Cauchy sequence is convergent in the semi-normed space $(H, \|., b\|)$.

Let $(V, \|., .\|)$ be a 2-normed space. Let W be a subspace of V, $b \in V$ be fixed, then a map $T : W \times \langle b \rangle \to F$ is called a b-linear functional on $W \times \langle b \rangle$ whenever for every $x, y \in W$ and $\alpha \in F$ holds

1. $T(x + y, b) = T(x, b) + T(y, b)$,
2. $T(\alpha x, b) = \alpha T(x, b)$.

A b-linear functional $T : W \times \langle b \rangle \to F$ is said to be bounded if there exists a real number $M > 0$ such that $|T(x, b)| \leq M\|x, b\|$ for every $x \in W$.
The norm of the b-linear functional $T : W \times \langle b \rangle \to F$ is defined by

$$\|T\| = inf\{M > 0; |T(x, b)| \leq M|x, b|, \forall x \in W\}.$$

It can be seen that,

$$\|T\| = sup\{|T(x, b)|; \|x, b\| \leq 1\},$$
$$\|T\| = sup\{|T(x, b)|; \|x, b\| = 1\},$$
$$\|T\| = sup\{|T(x, b)|; \|x, b\| \neq 0\}.$$

and $|T(x, b)| \leq \|T\| \|x, b\|$.
For a 2-normed space $(V, \|., .\|)$ and $0 \neq b \in V$, V_b^* denote the Banach space of all bounded b-linear functionals on $V \times \langle b \rangle$, where $\langle b \rangle$ is the subspace of V generated by b. Let V be a vector space over

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F. Let $b \in V$ and $y_1, y_2 \in V$, then y_1 is said to be b-congruent to y_2 iff $(y_1 - y_2) \in \langle b \rangle$, the subspace generated by b. (Riesz Representation Theorem on 2-Inner Product Spaces) Let H be a b-Hilbert space and $T \in H_b^*$ then there exists a unique $y \in H$ up to b-congruence such that $T(x, b) = \langle x, y; b \rangle$ and $\|T\| = \|y, b\|$.

3. Main Theorems

In this section, we try to extend the concept of reproducing kernel to 2-inner product spaces. This extension is along with the previews concepts. The main goal of this extension is to construct a new space with more applicable properties. Let X be a set and H be a linear space of two variable functions $f : X \times X \times X \to \mathbb{R}$ endowed with a 2-inner product. Fix $g \in H$. So we can speak about g-Hilbert space H_g. Let $\mathcal{H}_g$ be a subspace of H_g such that all 3-dimensional evaluation

$$E_{(x,y)}(f) = f(x, y),$$

be bounded. So by theorem (2), there exist a function $k_{x,y}$ such that

$$E_{(x,y)}(f) = f(x, y) = \langle f, k_{(x,y)}; g \rangle.$$

Now we can define a kernel function $K : X \times X \times X \to \mathbb{R}$ by

$$K(x, y, z) = \langle k_{(x,y)}, k_{(y,z)}; g \rangle.$$

In this case

$$K(x, x, x) = \langle k_{(x,x)}, k_{(x,x)}; g \rangle = \|k_{(x,x)}, g\|^2.$$

The pair $(\mathcal{H}_g, K)$ is called a 2-inner product reproducing kernel Hilbert space (2IPRKHS in short). An examples of 2-inner product reproducing kernel Hilbert space is as follows: Let H_2^2 be the two variable Hardy-Hilbert space, consists of all analytic functions having power series representations with square-summable complex coefficients and $\mathcal{H}_I$ be a subspace of H_2^2 such that

$$\mathcal{H}_I = \left\{ f : f(z_1, z_2) = \sum_{n=0}^{\infty} a_n z_1^n z_2^n \quad : a_1 = 0 \quad and \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\},$$

Where I denotes the identity function, $f(z_1, z_2) = \sum_{n=0}^{\infty} a_n z_1^n z_2^n$ and $h(z_1, z_2) = \sum_{n=0}^{\infty} b_n z_1^n z_2^n$. The 2-inner product on $\mathcal{H}_I$ is defined by

$$\langle f, h; I \rangle = a_0 \overline{b_0} + \sum_{n=2}^{\infty} a_n \overline{b_n}.$$

In this case, let $k_{(z_1,z_2)}(t_1, t_2) = 1 + \sum_{n=2}^{\infty} \overline{z_1^n z_2^n} t_1^n t_2^n$, then

$$f(z_1, z_2) = \langle f, k_{(z_1,z_2)}; I \rangle,$$

and

$$K(z_1, z_2, z_3) = \langle k_{(z_1,z_2)}, k_{(z_2,z_3)}; I \rangle = 1 + \sum_{n=2}^{\infty} |z_2|^{2n} z_1^n z_3^n.$$

We follow by explaining some properties of 2-inner product reproducing kernels. If $\mathcal{H}_g$ is a 2IPRKHS on X with reproducing kernel $K(x, y, z)$, then $K(x, y, x) = \overline{K(y, x, y)}$.

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Proof. By properties of 2-inner product we have

$$K(x,y,x) = \langle k_{(x,y)}, k_{(y,x)}; g\rangle = \overline{\langle k_{(y,x)}, k_{(x,y)}; g\rangle} = \overline{K(y,x,y)}.$$

□

Let $\mathcal{H}_g$ be a 2-inner product reproducing kernel Hilbert space and $\{e_i\}_i \in I$ be a basis for this space. Note that each e_i is an element of $\mathcal{H}_g$ and so is a function. We can define matrix $K = (\langle e_i, e_j; g\rangle)$. This matrix is positive. To see that, let $x \in \mathcal{H}_g$. Then

$$x^* K x = (x^* \langle e_i, e_j, g\rangle x) = (\langle e_i, e_j; g\rangle \|x\|^2) \geq 0.$$

In general form and similar to ordinary reproducing kernel Hilbert spaces, 2-inner product reproducing kernel Hilbert space corresponding to a 2-inner product kernel, is unique. Convergence of a sequence of elements of $\mathcal{H}_g$ is similar to RKHS. Let $\mathcal{H}_g$ be a 2IPRKHS and let $\{f_n\} \subseteq \mathcal{H}_g$. If $\lim_n \|f_n - f\|_g = 0$, then $f(x) = lim_n f_n(x)$ for every x. Uniqueness of kernel function corresponding to a 2IPRKHS is an other important problems. Naturally, it seems that the kernel function should be unique. Let $\mathcal{H}_{gi}, i = 1,2$ be 2IPRKHSs with kernels, $K_i(x,y,z)$, $i = 1,2$. If $K_1(x,y,z) = K_2(x,y,z)$ for all x,y,z then $\mathcal{H}_{g1} = \mathcal{H}_{g2}$ and $\|f\|_{g1} = \|f\|_{g2}$ for every f where $\|.\|_{gi}$ is the norm corresponding to $\mathcal{H}_{gi}$.

Proof. It is easy to see that $\mathcal{H}_g$ is the closure of a set contains all of functions $k_{(x,y)}$ and also it is a linear space. So let $\{k_{(x_i,y_j)}\}_{i,j}$ be a basis of kernels for $\mathcal{H}_{g_1}$ and $\{k'_{(x_i,y_j)}\}_{i,j}$ be a basis of kernels for $\mathcal{H}_{g2}$. By properties of 2-inner product kernels and equality of kernel functions we have

$$\begin{aligned}\|f\|_{g1}^2 &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \langle k_{(x_i,y_j)}, k_{(y_j,z_t)}; g\rangle = \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K_1(x,y,z) \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K_2(x,y,z) = \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \langle k'_{(x_i,y_j)}, k'_{(y_j,z_t)}; g\rangle = \|f\|_{g2}^2.\end{aligned}$$

Moreover, by the previews lemma, boundary elements are the same. Equality of norms on $\mathcal{H}_{g1}$ and $\mathcal{H}_{g2}$ is a direct consequence of equality on the mentioned dense subspaces. □

Motivated to extend these concepts to wavelets, we try to define Parseval frames in 2IPRKHS's. Let $\mathcal{H}_g$ be a 2IPRKHS with 2 inner product, $\langle .,.;.\rangle$. A set of vectors $\{f_s : s \in S\} \subseteq \mathcal{H}_g$ is called a Parseval frame for $\mathcal{H}_g$ provided that

$$\|h\|_g^2 = \sum_{s\in S} |\langle h, f_s; g\rangle|^2.$$

for every $h \in \mathcal{H}_g$. Let $\mathcal{H}_g$ be a 2IPRKHS on X with reproducing kernel $K(x,y,z)$. Then $\{f_s : s \in S\} \subseteq \mathcal{H}_g$ is a Parseval frame for $\mathcal{H}_g$ if and only if

$$K(x,y,z) = \sum_{s\in S} f_s(x,y)\overline{f_s(y,z)}.$$

Where the series converges point-wise.

Proof. Let $\{f_s : s \in S\}$ be parseval frame. then we have

$$\begin{aligned}K(x,y,z) &= \langle k_{(x,y)}, k_{(y,z)}; g\rangle \\ &= \sum_{s\in S} \langle k_{(x,y)}, f_s; g\rangle \langle f_s, k_{(y,z)}; g\rangle = \sum_{s\in S} f_s(x,y)\overline{f_s(y,z)}.\end{aligned}$$

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Conversely, let α_{ij} are scalars and $h = \sum_{ij} \alpha_{ij} k_{(x_i,y_j)}$ is any finite linear combination of kernel functions, then

$$\|h\|_g^2 = \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \langle k_{(x_i,y_j)}, k_{(y_j,z_t)} \rangle = \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K(x_i, y_j, z_t)$$
$$= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \sum_{s\in S} \overline{f_s(y_j)} f_s(y_i) = \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \sum_{s\in S} \langle k_{(x_i,y_j)}, f_s; g\rangle \langle f_s, k_{(y_j,z_t)}; g\rangle$$
$$= \sum_{s\in S} \langle \sum_{i,j} \alpha_{ij} k_{(x_i,y_j)}, f_s; g\rangle \langle f_s, \sum_{j,t} \alpha_{jt} k_{(y_j,z_t)}; g\rangle = \sum_{s\in S} |\langle h, f_s; g\rangle|^2.$$

Now it is easy to see that if we take a limit of a norm convergent sequence of vectors on both sides of this identity, then we obtain the identity for the limit vector, too and the proof is complete. □

Next, we try to construct sum of two 2IPRKHS in inner form. First, we recall a lemma of RKHS in 2-inner product case as the following lemma. Let $\mathcal{H}_{gi}$, $i = 1, 2$ be 2IPRKHSs with reproducing kernels, K_i and norms, $\|.\|_{gi}$, $i = 1, 2$. If $K = K_1 + K_2$ and $\mathcal{H}$ denotes the corresponding 2IPRKHS with norm, $\|.\|$, then

$$\mathcal{H} = \{f_1 + f_2 : f_i \in \mathcal{H}_{gi}, i = 1, 2\},$$

and for $f \in \mathcal{H}$, we have

$$\|f\|^2 = \min\{\|f_1\|_{g1}^2 + \|f_2\|_{g2}^2 : f = f_1 + f_2, f_i \in \mathcal{H}_{gi}, i = 1, 2\}.$$

Proof. Proof of this lemma is similar to proof of theorem 5.7 in [8] with substitution of inner product by 2-inner product. Be care that in this case we have

$$\langle f, h, g_{12}\rangle = \langle f_1, h_1; g_1\rangle + \langle f_2, h_2; g_2\rangle, \quad g_{12} = g_1.g_2.$$

□

This lemma helps us to think about vector-valued case. In the following, we want to extend vector-valued forms to 2IPRKHS's. Given a set X and a normed vector space $\mathcal{V}$, a map

$$K : X \times X \times X \to \mathcal{L}(\mathcal{V}),$$

is called a 2-inner product $\mathcal{V}$-reproducing kernel if

$$\sum_{i,j,t=1}^{n} \langle K(x_i, x_j)y_j, K(x_j, x_t)y_i; g\rangle \geq 0, \tag{1}$$

for any $x_1, \ldots, x_n$ in X, $y_1, \ldots, y_n$ in $\mathcal{V}$ and $n \geq 1$. Given $x_1, x_2 \in X$, $K_{(x_1,x_2)} : \mathcal{V} \to \mathcal{F}(X, \mathcal{V})$ denotes the linear operator whose action on a vector $y \in \mathcal{V}$ is the function $K_{[(x_1,x_2),y]} \in \mathcal{F}(X, \mathcal{V})$ defined by

$$(K_{[(x_1,x_2),y]})(t) = K(t, x_1, x_2)y, \quad t \in X. \tag{2}$$

Next theorem investigates the preserving of reproducing property between two vector spaces under bounded operators. Let K be a 2-inner product $\mathcal{V}$-reproducing kernel on $\mathcal{H}_g$. Let $\mathcal{V}'$ be another 2-inner product Hilbert space and $w : \mathcal{V} \to \mathcal{V}'$ be a bounded operator. Define

$$K_w : X \times X \times X \to L(\mathcal{V}'), \quad K_w(x_1, x_2, x_3) = wK(x_1, x_2, x_3)w^*.$$

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then K_w is a 2-inner product $\mathcal{V}'$-reproducing kernel.

Proof. By definition of K we have

$$K_w(x_1, x_2, x_3) = w\langle K(x_1, x_2), K(x_2, x_3); g\rangle w^*.$$

Since K is a positive function, K_w is well-defined. The reproducing properties naturally inherited from K. $\square$

The previews theorem describes that reproducing property is invariant under continuous operator.

Recall that two normed vector spaces V_1 and V_2 are said to be isometric if there is a bijective linear norm-preserving mapping between them. We call such V_1 and V_2, an identification of each other. We would like the dual space B^* of an reproducing kernel Banach space B on X to be isometric to a Banach space of functions on X. As it explained in [11,12], we like a reproducing kernel Banach space B, to be reflexive in the sense that $(B^*)^* = B$.

A reproducing kernel Banach space (RKBS in short) on a set X is a reflexive Banach space of functions on X such that its topological dual $B^\dagger$ is isometric to a Banach space of functions on X and the point evaluations are continuous linear functionals on both B and $B^\dagger$. More details can be found on [3,4]. Let B be a Banach space. The mapping $[.,.;.] : B \times B \times B \to F$ will be called the 2-semi-inner product if the following properties are satisfied:

1. $[x_1 + x_2, y; z] = [x_1, y; z] + [x_2, y; z], \quad \forall x, y, z \in B;$
2. $[\lambda x, y; z] = \lambda[x, y; z], \quad \forall x, y, z \in B, \lambda \in F;$
3. $[x, x; z] \geq 0$ and $[x, x; z] = 0$ iff x and z are linearly dependent.
4. $|[x, y; z]|^2 \leq [x, x; z][y, y; z], \quad \forall x, y, z \in B;$
5. $[x, \lambda y; z] = \bar{\lambda}[x, y; z], \quad \forall x, y \in B, \lambda \in F.$

A Banach space B endowed with a 2-semi-inner product, is called a 2-semi-inner product Banach space (2SIPBS in short). 2-inner product spaces have several interesting properties. We use some of these properties in the following and refer the readers to [6] for more details. A 2-semi inner product $[.,.;.]_V$ on a complex vector space V is an 2-inner product if and only if

$$[x, y + z; b]_V = [x, y; b]_V + [x, z; b]_V, \quad \textit{for all } x, y, z, b \in V.$$

In other hand, a 2-semi inner product is a 2-inner product without linear property of second variable. Let B be a linear space and $[.,.;.]$ a 2-semi-inner product on B. Then the following mapping is a 2-norm on $B \times B$;

$$\|.,.\| : X \times X \to \mathbb{R}, \quad \|x, y\| = [x, x; y]^{1/2}.$$

Like in Hilbert spaces, we define a b-Banach space. Let $(B, [.,.;.])$ be a 2SIPBS over a field F, $b \in B$, then

1. A sequence $\{x_n\}$ in B is said to be a b-Cauchy sequence if for every $\epsilon > 0$ there exists $N > 0$ such that for every $m, n \geq N$, $0 < \|x_n - x_m, b\| < \epsilon$.
2. B is said to be b-Banach if every b-Cauchy sequence is convergent in the semi-normed space $(B, \|., b\|)$.

Definitions 2 and 2 completely holds for Banach spaces. So we can speak about b-linear functionals and bounded functionals. To try for define a 2 semi inner product Banach space, we need a theorem similar to "Riesz Representation Theorem". It is well known that such a theorem holds for semi inner product Banach spaces. for more details we refer the reader to [2,3]. A uniform 2-semi inner product Banach space M is a uniformly continuous 2SIPBS where the induced normed vector space

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is uniformly convex and complete respect to its norm. It is easy to see that for a uniform 2-semi inner product Banach space B, the dual space B^* is a uniform 2SIPBS with respect to the 2-semi inner product defined by $[f^*, h^*; g^*]_{B^*} = [f, h; g]_B$. Recall that B is a Banach space of functions. Suppose B is a uniform 2-semi inner product g-Banach space. Then

1. (RIESZ REPRESENTATION THEOREM) For each $s \in B^*$, there exists a unique $h \in B$ up to g-congruence such that $s = h^*$, i.e.,

$$s(f, g) = [f, h; g]_B, \quad \textit{and} \quad \|s, g\|_{B^*} = \|h\|_B, \quad f \in B.$$

2. B^* is an 2SIPBS with respect to the 2SIP defined by

$$[h^*, f^*; g^*]_{B^*} := [f, h; g]_B, \quad \textit{and} \quad \|h^*, g^*\|_{B^*} := [h^*, h^*; g^*]_{B^*}^{1/2}, \quad f, h \in B.$$

Proof. It is sufficient to see that If $s(f, g) = 0$ for all $f \in B$, then by lemma 2, the null space N of g is a proper closed vector subspace of B. Other parts of the proof is similar to theorem 6 in [2] by replacing 2-semi inner product instead of semi inner product. Proof of second part is a direct corollary of remark 3. □

Let X be a set. A 2-semi inner product g-Banach space $\mathcal{B}_g$ of functions on X is called a 2-semi inner product reproducing kernel Banach space (2SIPRKBS in short) provided that

1. $\mathcal{B}$ is reflexive Banach space of functions on X.
2. Every point evaluation on $X \times X$ is a bounded linear functional on both $\mathcal{B}$ and $\mathcal{B}^\dagger$.
3. $\mathcal{B}$ is uniformly convex and uniformly Frechet differentiable on X respect to its new norm.

Let $\mathcal{B}_g$ be an 2SIPRKBS on X and K its reproducing kernel. Then there exists a unique function $G : X \times X \to \mathbb{C}$ such that $\{G(x, .) : x \in X\} \subseteq \mathcal{B}$ and

$$f(x, y) = [f, G(x, .); g]_{\mathcal{B}}, \quad \textit{for all } f \in \mathcal{B}, x \in X. \tag{3}$$

Moreover, there holds the relationship

$$K(., x) = (G(x, .))^*, \quad x \in X, \tag{4}$$

and

$$f^*(x, y) = [K(., x), f; g]_{\mathcal{B}}, \quad \textit{for all } f \in B, x \in X. \tag{5}$$

Proof. Since $\mathcal{B}_g$ is a 2SIPRKBS, by theorem 3 for each $x \in X$ there exists a function $G_x \in \mathcal{B}$ such that $f(x, y) = [f, G_x; g]_{\mathcal{B}}$ for all $f \in \mathcal{B}$. Define $G : X \times X \to \mathbb{C}$ by $G(x, y) := G_x(y)$, $x, y \in X$. It is easy to see that $G(x, .) \in \mathcal{B}$, $x \in X$, and 3 holds. By the uniqueness in the theorem 3, such a function G is unique. It is easy to verify 4 and 5 while they are direct consequences of properties of a 2-semi inner product. □

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