Article

On Partial Sums of Analytic Univalent Functions

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Abstract: Partial sums of analytic univalent functions and partial sums of starlike have been investigated extensively by several researchers. In this paper, we investigate a partial sums of convex harmonic functions that are univalent and sense preserving in the open unit disk.

Keywords: Harmonic; Univalent, Convex; Partial sums

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1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\Omega \subset \mathbb{C}$ if both u and v are real harmonic in Ω.
In any simply connected domain $\Omega \subset \mathbb{C}$, we may write $f = h + \overline{g}$, where h and g are analytic in Ω. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h^{'}(z)| > |g^{'}(z)|$ in Ω. (See [2]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the analytic functions h and g can be expressed as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

A function f of the form (1) is harmonic convex of order $\alpha, 0 \leq \alpha < 1$, denoted by $K_H(\alpha)$, if it satisfies

$$\frac{\partial}{\partial\theta}\Big\{arg\big(\frac{\partial}{\partial\theta} f(re^{i\theta})\big)\Big\} = Re\Big\{\frac{z\big(zh'(z)\big)' + \overline{z\big(zg'(z)\big)'}}{zh'(z) - \overline{zg'(z)}}\Big\} \geq \alpha,$$

where $0 \leq \theta \leq 2\pi, |z| = r < 1$.
As shown recently by Jahangiri [7] a sufficient condition for a function of the form (1) to be in $K_H(\alpha)$ is that

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big) \leq 2. \tag{2}$$

In 1985, Silvia[13] studied the partial sums of convex functions of order α. Later, Silverman [12], Abubaker and Darus[1], Dixit and Porwal[4], Frasin[5,6], Raina and Bansal[10], Rosy et al.[11] and Porwal and Dixit[9] exhibited some results on partial sums for various classes of analytic functions.

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Here, we investigate a partial sums of convex harmonic functions.
Now, we let the sequences of partial sums of functions of the form (1) with $b_1 = 0$, have forms

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k z^k},$$
$$f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k},$$
$$f_{m,n}(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k},$$

and study some some bounds for them.

2. Main Results

The main aims of this paper is the following theorems which determine sharp lower bounds for some classes of univalent functions. In the following theorem, we try it for $Re\{f(z)/f_m(z)\}$. If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}, \quad z \in \mathbb{D}. \tag{3}$$

The result (3) is sharp with the function

$$f(z) = z + \frac{1-\alpha}{(m+1)(m+1-\alpha)} z^{m+1}. \tag{4}$$

Proof. By setting

$$A_1(z) = \sum_{k=2}^{m} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=2}^{\infty} r^{k-1} e^{-i(k+1)\theta} \overline{b_k} + \frac{(m+1)(m+1-\alpha)}{1-\alpha} \Big[\sum_{k=m+1}^{\infty} r^{k-1} e^{i(k-1)\theta} a_k \Big],$$

and

$$B_1(z) = \sum_{k=2}^{m} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=2}^{\infty} r^{k-1} e^{-i(k+1)\theta} \overline{b_k}.$$

We have

$$\frac{(m+1)(m+1-\alpha)}{1-\alpha} \Big[\frac{f(z)}{f_m(z)} - \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)} \Big] = \frac{1+A_1(z)}{1+B_1(z)}.$$

Let $\frac{1+A_1(z)}{1+B_1(z)} = \frac{1+\omega(z)}{1-\omega(z)}$, then $\omega(z) = \frac{A_1+(z)-B_1(z)}{2+A_1(z)+B_1(z)}$. It is easy to see that

$$|\omega(z)| \leq \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} |a_k|\Big)}{2-2\Big(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{\infty} |b_k|\Big) - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} |a_k|\Big)}.$$

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We claim that the right hand side of the above inequality is bounded above by 1, if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k|\Big)\leq 1, \tag{5}$$

It suffices to show that the L. H. S. of (5) is bounded above by

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\Big),$$

which is equivalent to

$$\sum_{k=2}^{m}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k|+\sum_{k=2}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k| \\ +\sum_{k=m+1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}-\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big)|a_k|\geq 0.$$

To complete the proof we show that $f(z)=z+\frac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}$ gives the sharp result. Let $z=re^{i\frac{\pi}{m}}$ we have

$$\frac{f(z)}{f_m(z)}=1+\frac{1-\alpha}{(m+1)(m+1-\alpha)}z^m \\ \Longrightarrow 1-\frac{1-\alpha}{(m+1)(m+1-\alpha)}=\frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)},$$

when $r\to 1^-$. □

In next theorem, we try it on the inverse of the fraction. If $f(z)$ of the form (1) with $b_1=0$ satisfies condition (2), then

$$Re\Big\{\frac{f_m(z)}{f(z)}\Big\}\geq\frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)},\quad z\in\mathbb{D}. \tag{6}$$

The result (6) is sharp with the function given by (4).

Proof. Let

$$A_2(z)=\sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k} \\ -\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k\Big),$$

and $B_2(z)$ be as the $B_1(z)$ on the previews proof. We may write

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\frac{f_m(z)}{f(z)}-\frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}\Big] \\ =\frac{1+A_2(z)}{1+B_2(z)},$$

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where

$$|\omega(z)| \leq \frac{\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\left[\sum\limits_{k=m+1}^{\infty}|a_k|\right]}{2-2\left(\sum\limits_{k=2}^{m}|a_k|+\sum\limits_{k=2}^{\infty}|b_k|\right)-\frac{m(m+2-\alpha)}{1-\alpha}\left(\sum\limits_{k=m+1}^{\infty}|a_k|\right)} \leq 1.$$

Equivalently

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{m(m+2-\alpha)+(1-\alpha)}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k|\right) \leq 1. \tag{7}$$

since the left hand side of (7) is bounded above by $\sum\limits_{k=1}^{\infty}\left(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\right)$, the proof is complete. □

As two corollaries, we are interesting to know about these type of inequalities on derivatives of f. If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq \frac{m}{m+1-\alpha}, \quad z \in \mathbb{D}. \tag{8}$$

The result (8) is sharp with the function given by (4).

Proof. Let

$$A_3 = \sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k} + \frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k\right],$$

and

$$B_3 = \sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+1-\alpha}{1-\alpha}\left[\frac{f'(z)}{f'_m(z)} - \frac{m}{m+1-\alpha}\right] = \frac{1+A_3}{1+B_3}.$$

Then

$$\omega(z) = \frac{\frac{m+1-\alpha}{1-\alpha}\left[\sum\limits_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k\right]}{2+2A_3}.$$

Similarly to the Theorem 2. □

If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{m+1-\alpha}{m+2(1-\alpha)}, \quad z \in \mathbb{D}. \tag{9}$$

The result (9) is sharp with the function given by (4).

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Proof. Since

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+2(1-\alpha)}{1-\alpha}\left[\frac{f'_m(z)}{f'(z)} - \frac{m+1-\alpha}{m+2(1-\alpha)}\right],$$

in a simillar way of the proof of Corollary 2, we obtain the results. □

In the following, we prove the same theorems for $f_n(z)$. If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}, \quad z \in \mathbb{D}. \tag{10}$$

The result (10) is sharp with the function

$$f(z) = z + \frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}. \tag{11}$$

Proof. Let

$$A_5 = \sum_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\left[\sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\right],$$

and

$$B_5 = \sum_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(n+1)(n+1+\alpha)}{1-\alpha}\left[\frac{f(z)}{f_n(z)} - \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}\right] = \frac{1+A_5}{1+B_5}.$$

where

$$\omega(z) = \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\left[\sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\right]}{2+2A_5}.$$

Then

$$|\omega(z)| \leq \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\left[\sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{\infty}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\left(\sum_{k=n+1}^{\infty}|b_k|\right)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{\infty}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\left(\sum_{k=n+1}^{\infty}|b_k|\right) \leq 1. \tag{12}$$

It suffices to show that the left hand side of (12) is bounded above by

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\Big),$$

which is equivalent to

$$\sum_{k=2}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k|+\sum_{k=2}^{n}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k| + \sum_{k=n+1}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha}-\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|b_k|\geq 0.$$

To complete the proof, we see that $f(z)=z+\frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}$ gives the sharp result. Let $z=re^{i\frac{\pi}{n+2}}$ we have

$$\frac{f(z)}{f_n(z)}=1+\frac{1-\alpha}{(n+1)(n+1+\alpha)}r^n e^{-\frac{i\pi}{n+2}(n+2)}$$
$$\Longrightarrow 1-\frac{1-\alpha}{(n+1)(n+1+\alpha)}=\frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)},$$

when $r\to 1^-$. This completes the proof. □

If $f(z)$ of the form (1) with $b_1=0$ satisfies condition (2), then

$$Re\Big\{\frac{f_n(z)}{f(z)}\Big\}\geq\frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2},\quad z\in\mathbb{D}. \tag{13}$$

The result (13) is sharp with the function given by (11).

Proof. Let

$$A_6=\sum_{k=2}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k} - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big],$$

and

$$B_6=\sum_{k=2}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

It is easy to see that

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\frac{f_n(z)}{f(z)}-\frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}\Big]=\frac{1+A_6}{1+B_6}.$$

Details of the proof is omitted and is simillar to the Theorem 2. □

If $f(z)$ of the form (1) with $b_1=0$ satisfies condition (2),

(i) If $n(n+2+\alpha)+2\alpha\geq m(m+2-\alpha)$ or for all $k\geq 2$ we have $b_k=0$ then

$$Re\Big\{\frac{f(z)}{f_{m,n}(z)}\Big\}\geq\frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)},\quad z\in\mathbb{D}.$$

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(ii) if $n(n+2+\alpha)+2\alpha \le m(m+2-\alpha)$ or for all $k \ge 2$ we have $a_k = 0$ then

$$Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\} \ge \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}, \quad z \in \mathbb{D}.$$

Proof. (i) Let

$$A_7 = +\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\left[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\right],$$

and

$$B_7 = \sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(m+1)(m+1-\alpha)}{1-\alpha}\left[\frac{f(z)}{f_{m,n}(z)} - \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}\right] = \frac{1+A_7}{1+B_7}.$$

Then

$$\omega(z) = \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\left[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\right]}{2+2A_7}.$$

Therefore

$$|\omega(z)| \le \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right) \le 1. \tag{14}$$

Since the left hand side of (14) is bounded above by

$$\sum_{k=1}^{\infty}\left(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\right),$$

the following inequality holds

$$\sum_{k=2}^{m}\left(\frac{k(k-\alpha)}{1-\alpha}-1\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{k(k-\alpha)}{1-\alpha} - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\right)|a_k| + \sum_{k=2}^{n}\left(\frac{k(k+\alpha)}{1-\alpha}-1\right)|b_k| + \sum_{k=n+1}^{\infty}\left(\frac{k(k+\alpha)}{1-\alpha} - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\right)|b_k| \ge 0.$$

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To see $f(z) = z + \dfrac{1-\alpha}{(m+1)(m+1-\alpha)} z^{m+1}$ gives the sharp result, let $z = re^{i\frac{\pi}{m}}$. We have

$$\frac{f(z)}{f_{m,n}(z)} = 1 + \frac{1-\alpha}{(m+1)(m+1-\alpha)} z^m$$
$$\Longrightarrow 1 - \frac{1-\alpha}{(m+1)(m+1-\alpha)} = \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)},$$

when $r \to 1^-$.

(ii) Let

$$C_7 = \sum_{k=2}^{m} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=2}^{n} r^{k-1} e^{-i(k+1)\theta} \overline{b_k} + \frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big[\sum_{k=m+1}^{\infty} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=n+1}^{\infty} r^{k-1} e^{-i(k+1)\theta} \overline{b_k} \Big],$$

and

$$D_7 = \sum_{k=2}^{m} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=2}^{n} r^{k-1} e^{-i(k+1)\theta} \overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big[\frac{f(z)}{f_{m,n}(z)} - \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)} \Big] = \frac{1+C_7}{1+D_7}.$$

Then

$$\omega(z) = \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big[\sum_{k=m+1}^{\infty} r^{k-1} e^{i(k-1)\theta} a_k + \sum_{k=n+1}^{\infty} r^{k-1} e^{-i(k+1)\theta} \overline{b_k} \Big]}{2 + 2C_7}.$$

Consequently

$$|\omega(z)| \le \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big[\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k| \Big]}{2 - 2\Big(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k| \Big) - \frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k| \Big)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k| + \frac{(n+1)(n+1+\alpha)}{1-\alpha} \Big(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k| \Big) \le 1. \tag{15}$$

It suffices to show that the left hand side of (15) is bounded above by

$$\sum_{k=1}^{\infty} \Big(\frac{k(k-\alpha)}{1-\alpha} |a_k| + \frac{k(k+\alpha)}{1-\alpha} |b_k| \Big),$$

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which is equivalent to

$$\sum_{k=2}^{m}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k|+\sum_{k=m+1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}-\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|a_k|$$
$$+\sum_{k=2}^{n}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k|+\sum_{k=n+1}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha}-\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|b_k|\geq 0.$$

To show that $f(z)=z+\frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}$ gives the sharp result, let $z=re^{i\frac{\pi}{n+2}}$. We have

$$\frac{f(z)}{f_{m,n}(z)}=1+\frac{1-\alpha}{(n+1)(n+1+\alpha)}r^ne^{-\frac{i\pi}{n+2}(n+2)}$$
$$\Longrightarrow 1-\frac{1-\alpha}{(n+1)(n+1+\alpha)}=\frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)},$$

when $r\to 1^-$. The result follows.
□

If $f(z)$ of the form (1) with $b_1=0$ satisfies condition (2),

(i) if $n(n+2+\alpha)+2\alpha\geq m(m+2-\alpha)$ or for all $k\geq 2$ we have $b_k=0$ then

$$Re\Big\{\frac{f_{m,n}(z)}{f(z)}\Big\}\geq\frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)},\quad z\in\mathbb{D}.$$

(ii) if $n(n+2+\alpha)+2\alpha\leq m(m+2-\alpha)$ or for all $k\geq 2$ we have $a_k=0$ then

$$Re\Big\{\frac{f_{m,n}(z)}{f(z)}\Big\}\geq\frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2},\quad z\in\mathbb{D}.$$

Proof. (i) Let

$$A_8=\sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}$$
$$-\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big],$$

and

$$B_8=\sum_{k=2}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\frac{f_{m,n}(z)}{f(z)}-\frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}\Big]=\frac{1+A_8}{1+B_8}.$$

Then

$$|\omega(z)|\leq\frac{\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|\Big)-\frac{m(m+2-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\Big)}\leq 1.$$

Therefore

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{m(m+2-\alpha)+(1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\Big) \leq 1. \tag{16}$$

Since the left hand side of (16) is bounded above by

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big),$$

the proof is complete.

(ii) Let

$$\begin{aligned} C_8 = &\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} \\ &- \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big], \end{aligned}$$

and

$$D_8 = \sum_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\frac{f_{m,n}(z)}{f(z)} - \frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}\Big] = \frac{1+C_8}{1+D_8}.$$

Then

$$|\omega(z)| \leq \frac{\frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{n(n+2+\alpha)+2\alpha}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big)} \leq 1.$$

Therefore

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{n(n+2+\alpha)+(1+\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\Big) \leq 1. \tag{17}$$

Since the left hand side of (17) is bounded above by

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big),$$

which completes the proof.

□

If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\Big\{\frac{f'(z)}{f'_{m,n}(z)}\Big\} \geq \frac{m}{m+1-\alpha}, \quad \text{for } n > m, z \in \mathbb{D}. \tag{18}$$

The result (18) is sharp with the function given by (4).

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Proof. Let

$$A_9 = \sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{n} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}$$
$$+ \frac{m+1-\alpha}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=n+1}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big],$$

and

$$B_9 = \sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{n} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}.$$

We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+1-\alpha}{1-\alpha}\Big[\frac{f'(z)}{f'_{m,n}(z)} - \frac{m}{m+1-\alpha}\Big] = \frac{1+A_9}{1+B_9}.$$

Then

$$\omega(z) = \frac{\frac{m+1-\alpha}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=n+1}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{2+2A_9},$$

and

$$|\omega(z)| \le \frac{\frac{m+1-\alpha}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} k|a_k| - \sum_{k=n+1}^{\infty} k|b_k|\Big]}{2-2\Big(\sum_{k=2}^{m} k|a_k| + \sum_{k=2}^{n} k|b_k|\Big) - \frac{m+1-\alpha}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} k|a_k| + \sum_{k=n+1}^{\infty} k|b_k|\Big)} \le 1.$$

Therefore

$$\sum_{k=2}^{m} k|a_k| + \sum_{k=2}^{n} k|b_k| + \frac{m+1-\alpha}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} k|a_k| + \sum_{k=n+1}^{\infty} k|b_k|\Big) \le 1. \tag{19}$$

Since the left hand side of (19) is bounded above by

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big),$$

the proof is complete. □

In a similar way to the previews theorem, we may prove the next corollary. If f of the form (1) with $b_1 = 0$ satisfies condition (2), then

$$Re\Big\{\frac{f'_{m,n}(z)}{f'(z)}\Big\} \ge \frac{m+1-\alpha}{m+2(1-\alpha)}, \quad z \in \mathbb{D}. \tag{20}$$

The result (20) is sharp with the function $f(z) = z + \frac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}$.

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