

Some Fixed Point Theorems in Pseudo Multiplicative Metric Spaces with Convex Structure

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Abstract

In the framework of a multiplicative metric space. We have discuss some Properties of convex structures in pseudo multiplicative metric space and have applied these properties to obtain a fixed point results in complete pseudo It would be interesting to prove some further results in such metric spaces with completeness property.

Keywords: contractive mapping; Convex Pseudo multiplicative metric space; fixed point theorems

1 Introduction and Preliminaries

Fixed point theory is one of the most dynamic research area and has many applications in differential and integral equations. The most impressive result in this area was given by Banach in 1922. This result is called Banach contraction principle. Every contraction has a unique fixed point in a complete metric space. Due to elementary technique used in the proof by Banach and its usefulness in solving various problems in many areas, efforts have been made for generalizing this celebrated result in many abstract spaces and for various operators.

In 1906, French mathematician M. Frechet carried axiomatic development of the notion of a metric space. The metric spaces play an important role in the growth of Functional Analysis. Inspired by the impact of the notion of a metric space in Mathematics, several researchers attempted various generalizations of this notion such as rectangular metric spaces, semi-metric spaces, quasi-metric spaces,

Moreover Eniola Funmilayo kazeem [9] the concept of a quasi-pseudo-metric space was introduced and some fixed point theorem was proven. The concept of a b-metric space has been studied Czerwick in [4,5] and many others obtained many fixed point results for single and multi-value mappings.

In 1970, Takahashi [15-16] introduced convexity in metric spaces and studied certain fixed-point theorems in such metric spaces for non-expansive maps.

The convex structure on convex spaces was studied by several authors after that. Ding[8] dealt with the iteration scheme of Ishikawa in 1988 to construct fixed points of quasi-contractive, generalized quasi-contractive and quasi-non expansive maps in convex metric spaces. In 1989, Sahab and Khan [13] extended some known results of best approximation in spaces with convex structure. In 1998, Beg [1] proved the existence of fixed points in uniformly convex complete metric spaces.

In 2003, Change and Kim et al. [2] demonstrated some convergence theorems of the iterative sequences of Ishikawas type with errors metric spaces for nonlinear generalized quasi-contractive maps. In 2008, Ciric et al. [3] showed the strong convergence of Mann's iterative type process for pseudo-contractive maps in Hilbert spaces; in the same year, Wang and Liu used the iterative type process of Ishikawa with errors to approximate the fixed point of two uniformly quasi-Lipschitzain maps in convex metric spaces.

Fixed Point theorems in pseudo multiplicative metric space with convex structure are being investigated now days. The purpose of this paper to find some new fixed point theorems in these metric spaces.

Definition 1.[Pseudo Metric Space]

Let X is a non- empty set. A function $D: X \times X \rightarrow \mathbb{R}^+$ is called a Pseudo multiplicative metric. If for any $x, y, z \in X$ is satisfied with the following conditions:

$$(N_1) \quad D(x, x) = 0,$$

$$(N_2) \quad D(x, y) = D(y, x),$$

$$(N_3) \quad D(x, z) \leq D(x, y) + D(y, z). \quad \text{for all } x, y, z \in X$$

Definition 2.[Pseudo Multiplicative Metric Space]

Let X is a set. A function $D: X \times X \rightarrow \mathbb{R}^+$ is called a Pseudo multiplicative metric if for any $x, y, z \in X$, the following condition hold:

$$(t_1) \quad D(x, x) = 1,$$

$$(t_2) \quad D(x, y) = D(y, x),$$

$$(t_3) \quad D(x, z) \leq D(x, y) \cdot D(y, z).$$

Then it is said that function D is a Pseudo Multiplicative metric on X .

Definition 3. Let (X, D) be a pseudo multiplicative metric space

Assume that $\{x_n\}$ is a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ in X is said to be convergent x , if

$$\lim_{n \rightarrow \infty} d(x, x) = 1$$

(ii) The sequence $\{x_n\}$ in X is said to be a Cauchy Sequence If for any $\epsilon > 1$, there exists a positive integer n_0 such that $d(x_n, x_k) < \epsilon$ for all $n, k > n_0$

(iii) The Pseudo multiplicative metric space is said to be complete if every Cauchy sequence converges to a point in it.

(iv) Let $T: X \rightarrow X$ be a mapping. The mapping T is said to be sequentially continuous if $x_n \rightarrow x$ implies $T(x_n) \rightarrow T(x)$, as $n \rightarrow \infty$.

Definition 4.[Multiplicative Contraction Mapping]

Let (X, D) be a multiplicative pseudo metric space. A map $T: X \rightarrow X$ is said to be a contraction if there a exists a number λ with $0 < \lambda \leq 1$ such that

$$D(Tx, Ty) \leq [D(x, y)]^\lambda$$

Definition 5.

- (i) A point $x \in X$, if $Tx = x$, is a fixed point of T
(ii) It is said that a point $x \in X$ is a common fixed point of T and S :
If $Tx=Sx= x$ when $T,S:X \rightarrow X$ are mappings.

Definition 6.[Self Mapping]

Let's make T a set.. A self - mapping on T maps itself from T .

Definition 7.

Suppose (X, D) is a multiplicative pseudo metric space.

A mapping $W: X \times X \times [0,1] \rightarrow X$ is said to be convex on X , if for each

$(x,y, \lambda) \in X \times X \times [0,1]$ and $u \in X$

$$D(u, w(x,y,\lambda)) \leq D(u, x)^\lambda \cdot D(u, x)^{1-\lambda} .$$

A Pseudo-

Multiplicative Space X equipped with Convex structure W is called Convex Pseudo-Multiplicative Space. Denoted (X,D,W) .

Definition 8.

A sequence $\{x_n\}$ is multiplicative Cauchy, if for each $\epsilon > 1$, there exists $N_0 \in \mathbb{N}$ such that $D(x_n, x_m) < \epsilon$ for each $m, n \geq N_0$

Definition 9.

A mapping $T: X \rightarrow X$ it is said to be continuous multiplicative at the point $x \in X$ iff $T(x_n) \rightarrow T(x)$ for every sequence $\{x_n\}$ with $x_n \rightarrow x$.

Definition 10.

Let E be a non- empty subset of Multiplicative Space X . A mapping $T:E \rightarrow E$ is said to be a k -Lipschitzian if there exists a $k \in [0, \infty)$ such that $D(Tx, Ty) \leq D(x, y)^k, \forall x, y \in E$.

Theorem . [11] Let (X, D_p) be a metric space of multiplication. Define

$$D_p: X \times X \rightarrow [0, +\infty] \text{ by}$$

$$D_p(x, y) = \ln(p(x, y))$$

Then (X, D_p) is a metric space.

Theorem :[12] Let (X, D) be a metric space of multiplication and let $f: X \rightarrow X$ be a multiple contraction. If (X, D) is complete, f has a fixed point that is unique.

Lemma2.1 :-Let (X, D, W) be a convex Pseudo multiplicative metric space for all $x, y \in X, \lambda \in [0, 1)$ the following property are satisfied:

a) $D(x, y) = D(x, w(x, y, \lambda)) \cdot D(w(x, y, \lambda), y)$

b) $D(x, w(x, y, \lambda)) = D(x, y)^{1-\lambda}$

c) $D(y, w(x, y, \lambda)) = D(x, y)^\lambda$

Proof:-

a) Let $x, y \in X$. Then

by definition of convex structure

$$D(x, w(x, y, \lambda)) \cdot D(w(x, y, \lambda), y) \leq (D(x, x)^\lambda \cdot D(x, y)^{1-\lambda}) (D(x, y)^\lambda \cdot D(y, y)^{1-\lambda})$$

because (X, D) is pseudo-metric space multiplicative.

Since $D(x, x) = D(y, y) = 1$, so

$$D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y) \leq (1) D(x, y)^{1-\lambda} D(x, y)^\lambda (1) = D(x, y) \\ D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y) \leq D(x, y) \quad \text{_____ (i)}$$

By t_3 we have

$$D(x, y) \leq D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y) \quad \text{_____ (ii)}$$

From (i) and (ii), we get

$$D(x, y) = D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y)$$

b) For every $x, y \in X$ Further.

$$D(x, w(x, y, \lambda)) = D(x, y)^{1-\lambda}$$

By definition of Convex Structure we have

$$\text{This implies } D(x, w(x, y, \lambda)) \leq D(x, x)^\lambda . D(x, y)^{1-\lambda} \\ \leq (1) D(x, y)^{1-\lambda}$$

$$D(x, w(x, y, \lambda)) \leq D(x, y)^{1-\lambda} \quad \text{_____ (iii)}$$

and using (a) we have

$$D(x, y)^{1-\lambda} = D(x, y). D(x, y)^{-\lambda} \\ = [D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y)]. D(x, y)^{-\lambda}$$

But

$$D(w(x, y, \lambda), y) \leq D(x, y)^\lambda D(y, y)^{1-\lambda} \text{ This implies}$$

$$D(w(x, y, \lambda), y) \leq D(x, y)^\lambda$$

Therefore,

$$D(x, y)^{1-\lambda} \leq [D(x, w(x, y, \lambda)). D(x, y)^\lambda] d(x, y)^{-\lambda}, \text{ which gives}$$

$$D(x, y)^{1-\lambda} \leq D(x, w(x, y, \lambda)) \quad \text{_____ (iv)}$$

From (iii) and (iv)

$$\text{Hence } D(x, w(x, y, \lambda)) = D(x, y)^{1-\lambda}$$

C) By definition of convex structure

$$D(w(x, y, \lambda), y) \leq D(x, y)^\lambda D(y, y)^{1-\lambda} \\ \leq D(x, y)^\lambda (1). \text{ Thus}$$

$$D(w(x, y, \lambda), y) \leq D(x, y)^\lambda \quad \text{_____ (v)}$$

By t_3

$$D(x, y)^\lambda \leq [D(x, w(x, y, \lambda)). D(w(x, y, \lambda), y)]^\lambda \\ \leq \left[(D(x, x)^\lambda . D(x, y)^{1-\lambda}) . (D(x, y)^\lambda . D(y, y)^{1-\lambda}) \right]^\lambda \\ \leq [D(x, y)^{1-\lambda} (1). D(x, y)^\lambda (1)]^\lambda$$

$$D(x, y)^\lambda \leq [D(x, y)]^\lambda = D(w(x, y, \lambda), y)$$

$$D(x, y)^\lambda \leq D(w(x, y, \lambda), y) \quad \text{_____ (vi)}$$

From (V) and (vi), we get

$$D(w(x, y, \lambda), y) = D(x, y)^\lambda$$

Lemma 2.2:- Let (M, D) be a Pseudo multiplicative metric Space $K \subseteq M$ be closed subset of M . Let $T: K \rightarrow K$ be a sequentially continuous mapping then $D(x, T(K)) = 1$ implies $x \in T(K)$.

Proof:- Since T is a self mapping and K is closed subset of M . So, $T(K)$ is a closed subset of K . Due to definition of distance of a point of (M, D) from a subset of (M, D) . We have

$$D(x, T(k)) = \inf \{D(x, y) : y \in T(k)\}$$

and since it is given that $D(x, T(k)) = 1$, therefore

$$D(x, T(k)) = \inf \{D(x, y) : y \in T(k) = 1$$

Sequentially continuity of T implies that there exists at least one distance between x, y such that $D(x, y) = 1$,

which in turn implies that x is a limit point of $T(k)$. Since $T(k)$ is a closed subset of M , therefore $x \in T(k)$.

Lemma 2.3:- Let (X, D) be a convex multiplicative metric space, then the following hold :

i) $D(x, y) = D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y)$,

ii) $D(x, w(x, y, \frac{1}{2})) = D(y, w(x, y, \frac{1}{2})) = D(x, y)^{\frac{1}{2}}$ For all $x, y \in X$

Proof:- (i) Let $x, y \in X$, By definition of convexness.

$$D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y) \leq (D(x, x)^{\frac{1}{2}} \cdot D(x, y)^{1-\frac{1}{2}}) \cdot (D(x, y)^{\frac{1}{2}} \cdot D(y, y)^{1-\frac{1}{2}})$$

Since (X, D) is a multiplicative metric space, so

$$D(x, x) = D(y, y) = 1$$

$$\leq (1) \cdot D(x, y)^{1-\frac{1}{2}} \cdot (D(x, y)^{\frac{1}{2}} \cdot (1)). \text{ Thus}$$

$$D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y) \leq D(x, y)^{1-\frac{1}{2}} \cdot D(x, y)^{\frac{1}{2}} \text{ . so}$$

$$D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y) \leq D(x, y) \quad \text{_____ (i)}$$

By definition of Convexness

$$D(x, y) \leq D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y) \quad \text{_____ (ii)}$$

From (i) and (ii) we get

$$D(x, y) = D(x, w(x, y, \frac{1}{2})) \cdot D(w(x, y, \frac{1}{2}), y) \quad \text{for all } x, y \in X$$

(ii) Let $x, y \in X$, By definition of convexness.

$$D(x, w(x, y, \frac{1}{2})) \leq D(x, x)^{1-\frac{1}{2}} \cdot D(x, y)^{\frac{1}{2}}$$

$$D(x, w(x, y, \frac{1}{2})) \leq D(x, y)^{\frac{1}{2}} \quad \text{_____ (A)}$$

and on the other hand

$$D(x, y)^{1-\frac{1}{2}} = D(x, y) \cdot D(x, y)^{-\frac{1}{2}}$$

By(i) we have

$$D(x, y)^{\frac{1}{2}} = [D(x, w(x, y, \frac{1}{2})) \cdot D(y, w(x, y, \frac{1}{2}))] \cdot D(x, y)^{-\frac{1}{2}}$$

$$\text{From (A), } D(y, w(x, y, \frac{1}{2})) \leq D(x, y)^{\frac{1}{2}}$$

$$\leq [D(x, w(x, y, \frac{1}{2})) \cdot D(x, y)^{\frac{1}{2}}] \cdot D(x, y)^{-\frac{1}{2}}$$

$$D(x, y)^{\frac{1}{2}} \leq D(x, w(x, y, \frac{1}{2})) \quad \text{_____ (B)}$$

From (A) and (B) we get

$$D(x, w(x, y, \frac{1}{2})) = D(x, y)^{\frac{1}{2}}$$

Similarly, $D(y, w(x, y, \frac{1}{2})) = D(x, y)^{\frac{1}{2}}$

Hence

$$D(x, w(x, y, \frac{1}{2})) = D(y, W(x, y, \frac{1}{2})) = D(x, y)^{\frac{1}{2}}$$

2 Main Results

In this section we discuss some fixed point theorems in Pseudo Multiplicative metric spaces with convex structure.

Theorem 1:- Let Y be a non-empty closed convex subset of a complete Convex Pseudo multiplicative metric space (X, D, W) and f be a sequentially continuous self mapping of Y . if there exists

$$k \in [0, \frac{1}{2}) \text{ such that} \\ D(x, f(y)) \cdot D(f(x), f(y)) \leq D(y, f(x))^k \quad \text{_____ (1)}$$

for all $x, y \in Y$. Then f has at least one point fixed.

Proof :- Using Pseudo multiplicative convex structure definition we have

$$D(x, W(x, y, \lambda)) \leq D(x, x)^\lambda \cdot D(x, y)^{1-\lambda} \quad \text{_____ (2)}$$

and

$$D(y, W(x, y, \lambda)) \leq D(y, x)^\lambda \cdot D(y, y)^{1-\lambda} \quad \text{_____ (3)}$$

Since $D(x, x) = D(y, y) = 1$ Consequently (2) and (3)

$$D(x, W(x, y, \lambda)) \cdot D(y, w(x, y, \lambda)) \leq D(x, y) \quad \text{_____ (4)}$$

Now let $x_0 \in Y$. Define $\{x_n\}$ sequence as follows.

$$x_n = W(x_{n-1}, f x_{n-1}, \lambda) \quad ; n=1, 2, 3, \dots \quad \text{_____ (5)}$$

Since Y is convex. so $x_n \in Y$ and using (2), (4) and (5), we have

$$D(x_n, x_{n+1}) \leq D(x_n, x_n)^\lambda \cdot D(x_n, f x_n)^{1-\lambda} \text{ and} \quad \text{_____ (6)}$$

$$D(f x_n, f x_{n+1}) \leq D(x_n, f x_n)^\lambda \cdot D(f x_n, f x_n)^{1-\lambda} \quad \text{_____ (7)}$$

Multiplying (6) and (7)

$$D(x_n, x_{n+1}) \cdot D(f x_n, f x_{n+1}) \leq D(x_n, f x_n) \quad \text{_____ (8)}$$

Now using t_3 , we obtain

$$D(x_n, f x_n) \leq D(x_n, f x_{n-1}) \cdot D(f x_{n-1}, f x_n) \quad \text{_____ (9)}$$

From (8) and (9), we get

$$D(x_n, x_{n+1}) \cdot D(f x_n, f x_{n+1}) \leq D(x_n, f x_{n-1}) \cdot D(f x_{n-1}, f x_n)$$

This implies

$$\frac{D(x_n, x_{n+1}) \cdot D(f x_n, f x_{n+1})}{D(x_n, f x_{n-1})} \leq D(f x_{n-1}, f x_n) \quad \text{_____ (10)}$$

By substituting x for x_n and y for x_{n-1} in (1), we have

$$D(x_n, x_{n+1}) \cdot D(fx_n, x_{n+1}) \leq D(x_n, fx_{n-1})$$

This implies that

$$D(fx_n, fx_{n-1}) \leq \frac{D(x_{n-1}, x_n)^k}{D(fx_n, fx_{n-1})} \quad \text{_____ (11)}$$

Using (10) and (11)

$$D(x_n, x_{n+1}) \cdot D(x_{n+1}, fx_n) \leq D(x_{n-1}, fx_n)^k \quad \text{_____ (12)}$$

Again using t_3 , we get

$$D(x_{n-1}, fx_n) \leq D(x_{n-1}, x_n) \cdot D(x_n, fx_n) \quad \text{_____ (13)}$$

Combining (12) and (13), we have

$$D(x_n, x_{n+1}) \cdot D(x_{n+1}, fx_n) \leq [D(x_{n-1}, x_n) \cdot D(x_n, fx_n)]^k$$

By t_3 , we have

$$D(x_n, fx_n) \leq D(x_{n-1}, x_n)^k \cdot D(x_n, fx_n)^k$$

$$D(x_n, fx_n)^{1-k} \leq D(x_{n-1}, x_n)^k$$

Using pseudo multiplicative convex structure, we have

$$D(x_n, x_{n+1}) \leq D(x_n, fx_n)^{1-\lambda} \leq D(fx_n, x_n)$$

That is,

$$D(x_n, x_{n+1}) \leq D(x_n, fx_n)$$

Thus

$$D(x_n, x_{n+1})^{1-k} \leq D(x_{n-1}, x_n)^k$$

$$D(x_n, x_{n+1}) \leq D(x_{n-1}, x_n)^{\frac{k}{1-k}}, \text{ which implies}$$

$$D(x_n, x_{n+1}) \leq D(x_{n-1}, x_n)^\theta ; \quad \text{where } \theta = \frac{k}{1-k}$$

so,

$$D(x_n, x_{n+1}) \leq D(x_0, x_1)^\theta^n$$

Similarly, we have

$$D(x_{n+1}, x_{n+2}) \leq D(x_0, x_1)^\theta^{n+1}$$

Continuing in similar way we obtain

$$D(x_{n+k-1}, x_{n+k}) \leq D(x_0, x_1)^\theta^{n+k-1}$$

Using t_3 , we obtain

$$D(x_n, x_{n+k}) \leq D(x_n, x_{n+1}) \cdot D(x_{n+1}, x_{n+2}) \cdot \dots \cdot D(x_{n+k-1}, x_{n+k})$$

$$\leq D(x_0, x_1)^{(\theta^n + \theta^{n+1} + \theta^{n+2} + \dots + \theta^{n+k-1})}$$

$$\leq D(x_0, x_1)^{\frac{\theta^n}{1-\theta}}$$

Since $0 \leq \theta < 1$ therefore we get $D(x_0, x_1) = 1$ as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+k}) = 1$$

This implies that $\{x_n\}$ is a Sequence Cauchy in Y. Since Y is closed subset of complete space (X,D).so Y is also complete subspace.

Consequently $\{x_n\}_{n=1}^{\infty}$ Converges to a point z in Y.

$$\lim_{n \rightarrow \infty} x_n = z$$

Now we prove that z is a fixed point of f. Because f is continuous sequentially multiplicative this implies

$$x_n \rightarrow z \Rightarrow fx_n \rightarrow fz \quad \text{as } n \rightarrow \infty$$

By t_3 , we have

$$D(x_n, fx_n) \leq D(fx_{n-1}, fx_{n-1})^\lambda \cdot D(x_{n-1}, fx_{n-1})^{1-\lambda},$$

Which implies

$$D(x_n, fx_n) \leq D(x_{n-1}, fx_{n-1})^{1-\lambda}$$

Letting $n \rightarrow \infty$ We're getting

$$D(z, fz) \leq D(z, Tz)^{1-\lambda} \leq D(z, fz)$$

Hence $D(z, fz) = 1$ and we can deduce this from Lemma 2.2 $z=f(z)$.

Which completes the proof.

Theorem 2:- Let Y be a non-empty closed convex subset of a complete Convex Pseudo multiplicative metric space (X,D,W) and f be a sequentially continuous self mapping of Y. if there exists $k \in [2,4)$ such that

$$D(x,fx).D(y,fy) \leq D(x,y)^k \quad \text{_____ (14)}$$

for all $x,y \in Y$. Then f has at least one fixed point.

Proof :- Let $x_0 \in Y$. Define a sequence $\{x_n\}$ by

$$x_n = W(x_{n-1}, fx_{n-1}, \lambda) \quad ; n=1,2,3,\dots$$

As Y is convex, so $x_n \in Y$ and from the Lemma 2.1, We get the following equation

$$\text{for } \lambda = \frac{1}{2}$$

$$D(x_n, x_{n+1}) = D(x_n, fx_n)^{\frac{1}{2}} \quad \text{_____ (15)}$$

and

$$D(x_n, x_{n-1}) = D(x_{n-1}, fx_{n-1})^{\frac{1}{2}} \quad \text{_____ (16)}$$

By replacing x with x_n and y with x_{n-1} in equation (14) we get

$$D(x_n, fx_n) \cdot D(x_{n-1}, fx_{n-1}) \leq D(x_n, x_{n-1})^k \quad \text{_____ (17)}$$

Using equations (15), (16) and (17), we have

$$D(x_n, x_{n+1})^2 \cdot D(x_n, x_{n-1})^2 \leq D(x_n, x_{n-1})^k$$

$$D(x_n, x_{n+1})^2 \leq D(x_n, x_{n-1})^{k-2}$$

$$D(x_n, x_{n+1}) \leq D(x_n, x_{n-1})^{\frac{k-2}{2}}$$

$$D(x_n, x_{n+1}) \leq D(x_n, x_{n-1})^\lambda \quad \text{where } \lambda = \frac{k-2}{2}, \text{ so}$$

$$D(x_n, x_{n+1}) \leq D(x_0, x_1) \lambda^n$$

Similarly,

$$D(x_{n+1}, x_{n+2}) \leq D(x_0, x_1) \lambda^{n+1}$$

Continuing in a similar way we get

$$D(x_{n+k-1}, x_{n+k}) \leq D(x_0, x_1) \lambda^{n+k-1}$$

Using t_3 , we have

$$D(x_n, x_{n+k}) \leq D(x_n, x_{n+1}) \cdot D(x_{n+1}, x_{n+2}) \cdot \dots \cdot D(x_{n+k-1}, x_{n+k})$$

$$\leq D(x_0, x_1) (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+k-1}) \quad \text{Thus}$$

$$D(x_n, x_{n+k}) \leq D(x_0, x_1) \frac{\lambda^n}{1 - \lambda}$$

Since $0 \leq \lambda < 1$, therefore we get

$$\lim_{n \rightarrow \infty} D(x_n, x_{n+k}) = 1$$

This implies that $\{x_n\}$ is a Cauchy Sequence in Y . Since Y is a complete space subset closed (X, D) . so Y is also a complete subspace consequently $\{x_n\}_{n=1}^{\infty}$ Converges to a point z in Y and

$$\lim_{n \rightarrow \infty} x_n = z$$

Now we prove that z is a fixed point of f . Because f is sequentially continuous multiplicative this implies

$$x_n \rightarrow z \Rightarrow fx_n \rightarrow fz \quad \text{as } n \rightarrow \infty$$

Also due to t_3

$$D(x_n, fx_n) \leq D(fx_{n-1}, fx_{n-1}) \lambda \cdot D(x_{n-1}, fx_{n-1})^{1-\lambda}, \text{ which implies}$$

$$D(x_n, fx_n) \leq D(x_{n-1}, fx_{n-1})^{1-\lambda}$$

Letting $n \rightarrow \infty$ we have

$$D(z, fz) \leq D(z, Tz)^{1-\lambda} \leq D(z, fz)$$

Therefore $D(z, fz) = 1$ and Lemma 2.2, $Z = f(z)$ can be deduced. Which complete the proof.

Theorem 3:- Let Y be a non-empty closed convex subset of a complete Convex Pseudo multiplicative metric space (X, D, W) and f be a self-mapping of Y . If there is a, b, c, k that exists

$$\frac{2b}{c} \leq b \leq \frac{2(a+b+c)}{c} \quad \text{_____ (a)}$$

$$D(x, f(x))^a \cdot D(y, f(y))^b \cdot D(f(x), f(y))^c \leq D(x, y)^k \quad \text{_____ (18)}$$

for all $x, y \in Y$. Then f has at least one point fixed.

Proof :- Let $x_0 \in Y$ is arbitrary. A sequence is defined $\{x_n\}_{n=1}^{\infty}$ T a follows:

$$x_n = W(x_{n-1}, fx_{n-1}, \frac{1}{2}) \quad ; \quad n=1, 2, 3, \dots \quad \text{_____ (19)}$$

Since Y is convex, so $x_n \in Y$ for all $n \in \mathbb{N}$. By Lemma 2.1 and (19), we have

$$D(x_n, f(x_n)) = D(x_n, x_{n+1})^2 \quad \text{_____ (20)}$$

and

$$D(x_n, f(x_{n-1})) = D(x_n, x_{n-1}) \quad \text{_____ (21)}$$

For all $n \in \mathbb{N}$, Now, by replacement x with x_n and y with x_{n-1} in (18) we get

$$D(x_n, f(x_n))^a \cdot D(x_{n-1}, f(x_{n-1}))^b \cdot D(f(x_n), f(x_{n-1}))^c \leq D(x_n, x_{n-1})^k$$

For all $n \in \mathbb{N}$. Therefore, from (20) and (21) it follow that

$$D(x_n, x_{n+1})^{2a} \cdot D(x_n, x_{n-1})^{2b} \cdot D(f(x_n), f(x_{n-1}))^c \leq D(x_n, x_{n-1})^k \quad \text{_____ (22)}$$

for all $n \in \mathbb{N}$. Let Y be non-negative number. Using the inequality Triangle, (20) and (21), We have

$$\frac{D(x_n, x_{n+1})^{2c}}{D(x_n, x_{n-1})^c} \leq D(f(x_n), f(x_{n-1}))^c$$

For all $n \in \mathbb{N}$. Likewise, for the case $Y < 0$ we have

$$D(x_n, x_{n+1})^{2c} \cdot D(x_n, x_{n-1})^c \leq D(f(x_n), f(x_{n-1}))^c$$

for all $n \in \mathbb{N}$. That is why we have for each case

$$\frac{D(x_n, x_{n+1})^{2c}}{D(x_n, x_{n-1})^c} \leq D(f(x_n), f(x_{n-1}))^c \quad \text{_____ (23)}$$

For each $n \in \mathbb{N}$. Now form (22) and (23) it follows that

$$\frac{D(x_n, x_{n+1})^{2a} \cdot D(x_n, x_{n-1})^{2b} \cdot D(x_n, x_{n+1})^{2c}}{D(x_n, x_{n-1})^c} \leq D(x_n, x_{n-1})^k$$

For all $n \in \mathbb{N}$. That implies

$$D(x_n, x_{n+1}) \leq D(x_n, x_{n-1})^{\frac{k-2b+|c|}{2(a+c)}}$$

for all $n \in \mathbb{N}$. From (a) $\frac{k-2b+|c|}{2(a+c)} \in [0, 1)$ and therefore, $\{x_n\}_{n=1}^{\infty}$ is construction Sequence in Y . So it's Cauchy Sequence. Since Y is a complete space.

there exist $v \in Y$ such that $\lim_{n \rightarrow \infty} x_n = v$. Now, by applying x to v and y with x_n in

(18). We get

$$D(v, f(v))^a \cdot D(x_n, f(x_n))^b \cdot D(f(v), f(x_n))^c \leq D(v, x_n)^k$$

for all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ It follow from the above inequality

$$D(v, f(v))^{(a+c)} \leq 0$$

Since $(a+c)$ is positive. it follows that $D(v, f(v)) = 1$. Hence $f(v) = v$ and complete the theorem.

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