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On Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers

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Abstract. In this paper, we present Binet's formulas, generating functions, and the summation formulas for generalized Pentanacci numbers, and as special cases, we investigate Pentanacci and Pentanacci-Lucas numbers with their properties. Also, we define Gaussian generalized Pentanacci numbers and as special cases, we investigate Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers with their properties. Moreover, we give some identities for these numbers. Furthermore, we present matrix formulations of generalized Pentanacci numbers and Gaussian generalized Pentanacci numbers.

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1. Introduction and Preliminaries

In this work, we investigate generalized Pentanacci numbers and give properties of Pentanacci and Pentanacci-Lucas numbers as special cases. We also define Gaussian generalized Pentanacci numbers and give properties of Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers as special cases. First, in this section, we present some background about generalized Pentanacci numbers.

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Pentanacci and Pentanacci-Lucas which are special case of generalized Pentanacci numbers. A generalized Pentanacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}, \quad (1.1)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} + V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pentanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Pentanacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$-c_0 - c_1 - c_2 - c_3 + c_4$
2	c_2	$2c_3 - c_4$
3	c_3	$2c_2 - c_3$
4	c_4	$2c_1 - c_2$
5	$c_0 + c_1 + c_2 + c_3 + c_4$	$2c_0 - c_1$
6	$c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4$	$-3c_0 - 2c_1 - 2c_2 - 2c_3 + 2c_4$
7	$2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4$	$c_0 + c_1 + c_2 + 5c_3 - 3c_4$
8	$4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4$	$4c_2 - 4c_3 + c_4$
9	$8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4$	$4c_1 - 4c_2 + c_3$
10	$16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4$	$4c_0 - 4c_1 + c_2$

We consider two special cases of V_n : $V_n(0, 1, 1, 2, 4) = P_n$ is the sequence of Pentanacci numbers (sequence $A001591$ in [20]) and $V_n(5, 1, 3, 7, 15) = Q_n$ is the sequence of Pentanacci-Lucas numbers ($A074048$ in [20]). In other words, Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined by the fifth-order recurrence relations

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \qquad P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4 \tag{1.2}$$

and

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5}, \qquad Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15 \tag{1.3}$$

respectively. Pentanacci sequence has been studied by many authors, see for example [14], [15], [19].

Next, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Pentanacci and Pentanacci-Lucas Numbers

n	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
P_n	−3	2	0	0	0	−1	1	0	0	0	0	1	1	2	4	8	16	31	61	120	236
Q_n	19	−1	−1	−1	−7	9	−1	−1	−1	−1	5	1	3	7	15	31	57	113	223	439	863

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For all integers n, usual Pentanacci and Pentanacci-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

(see Theorem 2.2) or

$$P_n=\frac{\alpha-1}{6\alpha-10}\alpha^{n-1}+\frac{\beta-1}{6\beta-10}\beta^{n-1}+\frac{\gamma-1}{6\gamma-10}\gamma^{n-1}+\frac{\delta-1}{6\delta-10}\delta^{n-1}+\frac{\lambda-1}{6\lambda-10}\lambda^{n-1} \tag{1.4}$$

(see for example [5])

and

$$Q_n=\alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n$$

respectively, where $\alpha,\beta,\gamma,\delta$ and λ are the roots of the equation

$$x^5-x^4-x^3-x^2-x-1=0. \tag{1.5}$$

Moreover, the approximate value of $\alpha,\beta,\gamma,\delta$ and λ are given by

$$\begin{aligned} \alpha &= 1.9659 \\ \beta &= -0.67835+0.45854i \\ \gamma &= -0.67835-0.45854i \\ \delta &= 0.19538+0.84885i \\ \lambda &= 0.19538-0.84885i. \end{aligned}$$

In fact, there are no solutions of the characteristic equation (1.5) in terms of radicals, see [27].

Note that we have the following identities:

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\lambda &= 1, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1 \\ \alpha\beta\gamma\delta\lambda &= 1. \end{aligned}$$

2. Properties of Generalized Pentanacci Numbers

In this section, we present Binet's formulas, generating functions, and the summation formulas for generalized Pentanacci numbers.

First, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized Pentanacci sequence* $\{V_n\}_{n\geq 0}$. *Then* $f_{V_n}(x)$ *is given by*

(2.1)
$$f_{V_n}(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 + (V_4 - V_3 - V_2 - V_1 - V_0)x^4}{1 - x - x^2 - x^3 - x^4 - x^5}.$$

Proof. Using (1.1) and some calculation, we obtain

$$\begin{aligned} & f_{V_n}(x) - x f_{V_n}(x) - x^2 f_{V_n}(x) - x^3 f_{V_n}(x) - x^4 f_{V_n}(x) - x^5 f_{V_n}(x) \\ = \; & V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 + (V_4 - V_3 - V_2 - V_1 - V_0)x^4 \end{aligned}$$

which gives (2.1).

The previous Lemma gives the following results as particular examples: generating function of the Pentanacci sequence P_n is

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4 - x^5} \tag{2.2}$$

and generating function of the Pentanacci-Lucas sequence Q_n is

$$f_{Q_n}(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{5 - 4x - 3x^2 - 2x^3 - x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{2.3}$$

We next find Binet formula of Pentanacci numbers by the use of generating function for P_n.

Theorem 2.2. *(Binet formula of Pentanacci numbers)*

$$\begin{aligned} P_n \; = \; & \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ & + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ & + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}. \end{aligned} \tag{2.4}$$

Proof. Let

$$h(x) = 1 - x - x^2 - x^3 - x^4 - x^5.$$

Then for some $\alpha, \beta, \gamma, \delta$ and λ, we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)(1 - \lambda x)$$

i.e.,

$$1 - x - x^2 - x^3 - x^4 - x^5 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)(1 - \lambda x) \tag{2.5}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ ve $\frac{1}{\lambda}$ are the roots of $h(x)$. This gives $\alpha, \beta, \gamma, \delta$ and λ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5} = 0.$$

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This implies $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$. Now, by (2.2) and (2.5), it follows that

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)}.$$

Then we write

$$\frac{x}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)} = \frac{A}{(1-\alpha x)} + \frac{B}{(1-\beta x)} + \frac{C}{(1-\gamma x)} + \frac{D}{(1-\delta x)} + \frac{E}{(1-\lambda x)}. \tag{2.6}$$

So

$$\begin{aligned} x &= A(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) + B(1-\alpha x)(1-\gamma x)(1-\delta x)(1-\lambda x) \\ &\quad + C(1-\alpha x)(1-\beta x)(1-\delta x)(1-\lambda x) + D(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x) \\ &\quad + E(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $\frac{1}{\alpha} = A(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})(1-\frac{\lambda}{\alpha})$. This gives $A = \frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}$. Similarly, we obtain

$$\begin{aligned} B &= \frac{\beta^3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}, C = \frac{\gamma^3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}, \\ D &= \frac{\delta^3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}, E = \frac{\lambda^3}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}. \end{aligned}$$

Thus (2.6) can be written as

$$f_{P_n}(x) = A(1-\alpha x)^{-1} + B(1-\beta x)^{-1} + C(1-\gamma x)^{-1} + D(1-\delta x)^{-1} + E(1-\lambda x)^{-1}.$$

This gives

$$\begin{aligned} f_{P_n}(x) &= A\sum_{n=0}^{\infty}\alpha^n x^n + B\sum_{n=0}^{\infty}\beta^n x^n + C\sum_{n=0}^{\infty}\gamma^n x^n + D\sum_{n=0}^{\infty}\delta^n x^n + E\sum_{n=0}^{\infty}\lambda^n x^n \\ &= \sum_{n=0}^{\infty}(A\alpha^n + B\beta^n + C\gamma^n + D\delta^n + E\lambda^n)x^n. \end{aligned}$$

Using the values of A, B, C, D and E we get

$$\begin{aligned} f_{P_n}(x) &= \sum_{n=0}^{\infty} P_n x^n = \sum_{n=0}^{\infty}(\frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} \\ &\quad + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{\delta^3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^3}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})x^n \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we get (2.4).

Next, we give an identity related with generalized Pentanacci numbers and Pentanacci numbers.

THEOREM 2.3. *For $n \geq 0$ and $m \geq 0$, the following identity holds:*

(2.7)

$$V_{m+n} = P_{m-4}V_n + (P_{m-4}+P_{m-5})V_{n+1} + (P_{m-4}+P_{m-5}+P_{m-6})V_{n+2} + (P_{m-4}+P_{m-5}+P_{m-6}+P_{m-7})V_{n+3} + P_{m-3}V_{n+4}$$

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Proof. We prove the identity by induction on m. If $m=0$ then

$$V_n = P_{-4}V_n + (P_{-4}+P_{-5})V_{n+1} + (P_{-4}+P_{-5}+P_{-6})V_{n+2} + (P_{-4}+P_{-5}+P_{-6}+P_{-7})V_{n+3} + P_{-3}V_{n+4}$$

which is true because $P_{-3}=0$, $P_{-4}=1$, $P_{-5}=-1$, $P_{-6}=0$, $P_{-7}=0$. Assume that the equaliy holds for all $m \leq k$. For $m=k+1$, we have

$$\begin{aligned}
V_{(k+1)+n} &= V_{n+k}+V_{n+k-1}+V_{n+k-2}+V_{n+k-3}+V_{n+k-4} \\
&= (P_{k-4}V_n+(P_{k-4}+P_{k-5})V_{n+1}+(P_{k-4}+P_{k-5}+P_{k-6})V_{n+2} \\
&\quad +(P_{k-4}+P_{k-5}+P_{k-6}+P_{k-7})V_{n+3}+P_{k-3}V_{n+4}) \\
&\quad +(P_{k-5}V_n+(P_{k-5}+P_{k-6})V_{n+1}+(P_{k-5}+P_{k-6}+P_{k-7})V_{n+2} \\
&\quad +(P_{k-5}+P_{k-6}+P_{k-7}+P_{k-8})V_{n+3}+P_{k-4}V_{n+4}) \\
&\quad +(P_{k-6}V_n+(P_{k-6}+P_{k-7})V_{n+1}+(P_{k-6}+P_{k-7}+P_{k-8})V_{n+2} \\
&\quad +(P_{k-6}+P_{k-7}+P_{k-8}+P_{k-9})V_{n+3}+P_{k-5}V_{n+4}) \\
&\quad +(P_{k-7}V_n+(P_{k-7}+P_{k-8})V_{n+1}+(P_{k-7}+P_{k-8}+P_{k-9})V_{n+2} \\
&\quad +(P_{k-7}+P_{k-8}+P_{k-9}+P_{k-10})V_{n+3}+P_{k-6}V_{n+4}) \\
&\quad +(P_{k-8}V_n+(P_{k-8}+P_{k-9})V_{n+1}+(P_{k-8}+P_{k-9}+P_{k-10})V_{n+2} \\
&\quad +(P_{k-8}+P_{k-9}+P_{k-10}+P_{k-11})V_{n+3}+P_{k-7}V_{n+4}) \\
&= (P_{k-3}V_n+(P_{k-3}+P_{k-4})V_{n+1}+(P_{k-3}+P_{k-4}+P_{k-5})V_{n+2} \\
&\quad +(P_{k-3}+P_{k-4}+P_{k-5}+P_{k-6})V_{n+3}+P_{k-2}V_{n+4}) \\
&= P_{(k+1)-4}V_n+(P_{(k+1)-4}+P_{(k+1)-5})V_{n+1}+(P_{(k+1)-4}+P_{(k+1)-5}+P_{(k+1)-6})V_{n+2} \\
&\quad +(P_{(k+1)-4}+P_{(k+1)-5}+P_{(k+1)-6}+P_{(k+1)-7})V_{n+3}+P_{(k+1)-3}V_{n+4}
\end{aligned}$$

By induction on m, this proves (2.7).

The previous Theorem gives the following results as particular examples: For $n \geq 0$ and $m \geq 0$, we have (taking $V_n = P_n$)

$$P_{m+n} = P_{m-4}P_n+(P_{m-4}+P_{m-5})P_{n+1}+(P_{m-4}+P_{m-5}+P_{m-6})P_{n+2}+(P_{m-4}+P_{m-5}+P_{m-6}+P_{m-7})P_{n+3}+P_{m-3}P_{n+4}$$

and (taking $V_n = Q_n$)

$$Q_{m+n} = P_{m-4}Q_n+(P_{m-4}+P_{m-5})Q_{n+1}+(P_{m-4}+P_{m-5}+P_{m-6})Q_{n+2}+(P_{m-4}+P_{m-5}+P_{m-6}+P_{m-7})Q_{n+3}+P_{m-3}Q_{n+4}.$$

Next we present the Binet's formula of the generalized Pentanacci sequence.

LEMMA 2.4. *The Binet's formula of the generalized Pentanacci sequence $\{V_n\}$ is given as*

$$V_n = P_{n-4}V_0+(P_{n-4}+P_{n-5})V_1+(P_{n-4}+P_{n-5}+P_{n-6})V_2+(P_{n-4}+P_{n-5}+P_{n-6}+P_{n-7})V_3+P_{n-3}V_4.$$

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Proof. Take $n = 0$ and then replace n with m in Theorem 2.3.

For another proof of the Lemma 2.4, see [19]. This Lemma is also a special case of a work on the nth k-generalized Fibonacci number (which is also called k-step Fibonacci number) in [2, Theorem 2.2].

COROLLARY 2.5. *The Binet's formula of the generalized Pentanacci sequence $\{V_n\}$ is given as*

$$V_n = A_1\alpha^{n-8} + A_2\beta^{n-8} + A_3\gamma^{n-8} + A_4\delta^{n-8} + A_5\lambda^{n-8}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha-1}{6\alpha-10}(V_4\alpha^4 + (V_0+V_1+V_2+V_3)\alpha^3 + (V_1+V_2+V_3)\alpha^2 + (V_2+V_3)\alpha + V_3),\\
A_2 &= \frac{\beta-1}{6\beta-10}(V_4\beta^4 + (V_0+V_1+V_2+V_3)\beta^3 + (V_1+V_2+V_3)\beta^2 + (V_2+V_3)\beta + V_3),\\
A_3 &= \frac{\gamma-1}{6\gamma-10}(V_4\gamma^4 + (V_0+V_1+V_2+V_3)\gamma^3 + (V_1+V_2+V_3)\gamma^2 + (V_2+V_3)\gamma + V_3),\\
A_4 &= \frac{\delta-1}{6\delta-10}(V_4\delta^4 + (V_0+V_1+V_2+V_3)\delta^3 + (V_1+V_2+V_3)\delta^2 + (V_2+V_3)\delta + V_3),\\
A_5 &= \frac{\lambda-1}{6\lambda-10}(V_4\lambda^4 + (V_0+V_1+V_2+V_3)\lambda^3 + (V_1+V_2+V_3)\lambda^2 + (V_2+V_3)\lambda + V_3).
\end{aligned}$$

Proof. The proof follows from Lemma 2.4 and (1.4).

In fact, Corollary 2.5 is a special case of a result in [2, Remark 2.3].

The following Theorem present some summation formulas of generalized Pentanacci numbers.

THEOREM 2.6. *For $n \geq 1$ we have the following summing formulas:*

(a): *(Sum of the generalized Pentanacci numbers)*

$$\sum_{k=1}^{n} V_k = \frac{1}{4}(V_{n+4} - V_{n+2} - 2V_{n+1} + V_n - V_4 + V_2 + 2V_1 - V_0)$$

(b): $\sum_{k=1}^{n} V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} - 3V_4 + 4V_3 - V_2 - 2V_1 + V_0)$

(c): $\sum_{k=1}^{n} V_{2k} = \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 - 3V_0)$.

Proof.

(a): Using the recurrence relation

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$V_{n-5} = V_n - V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4}$$

we obtain

$$\begin{aligned}
V_0 &= V_5 - V_4 - V_3 - V_2 - V_1 \\
V_1 &= V_6 - V_5 - V_4 - V_3 - V_2 \\
V_2 &= V_7 - V_6 - V_5 - V_4 - V_3 \\
V_3 &= V_8 - V_7 - V_6 - V_5 - V_4 \\
V_4 &= V_9 - V_8 - V_7 - V_6 - V_5 \\
&\vdots \\
V_{n-5} &= V_n - V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4} \\
V_{n-4} &= V_{n+1} - V_n - V_{n-1} - V_{n-2} - V_{n-3} \\
V_{n-3} &= V_{n+2} - V_{n+1} - V_n - V_{n-1} - V_{n-2} \\
V_{n-2} &= V_{n+3} - V_{n+2} - V_{n+1} - V_n - V_{n-1} \\
V_{n-1} &= V_{n+4} - V_{n+3} - V_{n+2} - V_{n+1} - V_n \\
V_n &= V_{n+5} - V_{n+4} - V_{n+3} - V_{n+2} - V_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
V_0 + \sum_{k=1}^{n} V_k &= \left(-V_4 - V_3 - V_2 - V_1 + \sum_{k=1}^{n} V_k\right) + \left(V_3 + V_2 + V_1 - \sum_{k=1}^{n} V_k\right) \\
&\quad + \left(V_2 + V_1 - \sum_{k=1}^{n} V_k\right) + \left(V_1 - \sum_{k=1}^{n} V_k\right) + \left(-\sum_{k=1}^{n} V_k\right) \\
&\quad -3V_{n+1} - 2V_{n+2} - V_{n+3} + V_{n+5} \\
&= -3V_{n+1} - 2V_{n+2} - V_{n+3} + V_{n+5} + (-V_4 + V_2 + 2V_1) - 3\sum_{k=1}^{n} V_k
\end{aligned}$$

and then

$$4\sum_{k=1}^{n} V_k = V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 - V_0.$$

It follows that

$$\begin{aligned}
\sum_{k=1}^{n} V_k &= \frac{1}{4}(V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 - V_0) \\
&= \frac{1}{4}((V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n) - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 - V_0) \\
&= \frac{1}{4}(V_{n+4} - V_{n+2} - 2V_{n+1} + V_n - V_4 + V_2 + 2V_1 - V_0).
\end{aligned}$$

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(b): When we use (3.1), we obtain the following equalities:

$$
\begin{aligned}
V_k &= V_{k-1} + V_{k-2} + V_{k-3} + V_{k-4} + V_{k-5} \\
V_4 &= V_3 + V_2 + V_1 + V_0 + V_{-1} \\
V_6 &= V_5 + V_4 + V_3 + V_2 + V_1 \\
V_8 &= V_7 + V_6 + V_5 + V_4 + V_3 \\
V_{10} &= V_9 + V_8 + V_7 + V_6 + V_5 \\
&\vdots \\
V_{2n} &= V_{2n-1} + V_{2n-2} + V_{2n-3} + V_{2n-4} + V_{2n-5} \\
V_{2n+2} &= V_{2n+1} + V_{2n} + V_{2n-1} + V_{2n-2} + V_{2n-3}.
\end{aligned}
$$

If we rearrange the above equalities, we obtain

$$
\begin{aligned}
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 \\
V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} \\
&\vdots \\
V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\
V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3}.
\end{aligned}
$$

Now, if we add the above equations by side by, we get (as a I. Method)

$$
\begin{aligned}
\sum_{k=1}^{n} V_{2k+1} &= -V_{-1} - V_0 - V_2 + (-V_1 - V_3 - V_5 - V_7 - \ldots - V_{2n-1}) - \left(\sum_{k=1}^{2n-2} V_k\right) + V_{2n+2} \\
&= -V_{-1} - V_0 - V_2 + \left(V_{2n+1} - V_1 - \sum_{k=1}^{n} V_{2k+1}\right) - \left(\sum_{k=1}^{2n-2} V_k\right) + V_{2n+2}
\end{aligned}
$$

and then, using (a), we obtain

$$
\sum_{k=1}^{n} V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} - 3V_4 + 4V_3 - V_2 - 2V_1 + V_0).
$$

Note that, as an alternative method (II. Method), the following equality can be used:

$$\sum_{k=1}^{n} V_{2k+1} = \left(-V_2 + V_{2n+2} + \sum_{k=1}^{n} V_{2k}\right) + \left(-\sum_{k=1}^{n} V_{2k}\right) + \left(-V_1 + V_{2n+1} - \sum_{k=1}^{n} V_{2k+1}\right)$$
$$+ \left(-V_0 + V_{2n} - \sum_{k=1}^{n} V_{2k}\right) + \left(-V_{-1} - V_1 + V_{2n-1} + V_{2n+1} - \sum_{k=1}^{n} V_{2k+1}\right).$$

(c): (c) follows from (a), (b) and the equality

$$\sum_{k=1}^{n} V_{2k} = \sum_{k=1}^{2n+1} V_k - \sum_{k=1}^{n} V_{2k+1} - V_1.$$

This completes the proof.

All the listed identities in Theorem 2.6 may be proved by induction, but that method of proof gives no clue about their discovery.

As special cases of the above Theorem, we have the following two Corollaries. First one present some summing formulas of Pentanacci numbers.

Corollary 2.7. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the Pentanacci numbers)*

$$\sum_{k=1}^{n} P_k = \frac{1}{4}(P_{n+4} - P_{n+2} - 2P_{n+1} + P_n - 1)$$

(b): $\sum_{k=1}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2} + 4P_{2n+1} + P_{2n} + 2P_{2n-1} - P_{2n-2} - 7)$

(c): $\sum_{k=1}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2} + 4P_{2n+1} + 5P_{2n} + 2P_{2n-1} + 3P_{2n-2} - 3)$.

Second Corollary gives some summing formulas of Pentanacci-Lucas numbers.

Corollary 2.8. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the Pentanacci-Lucas numbers)*

$$\sum_{k=1}^{n} Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+2} - 2Q_{n+1} + Q_n - 15)$$

(b): $\sum_{k=1}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + 4Q_{2n+1} + Q_{2n} + 2Q_{2n-1} - Q_{2n-2} - 17)$

(c): $\sum_{k=1}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 4Q_{2n+1} + 5Q_{2n} + 2Q_{2n-1} + 3Q_{2n-2} - 21)$.

Sometimes, we need to start summing from zero, such as when dealing with Pentanacci quaternions, sedenions, etc. We can state Theorem 2.6 in the following form.

Theorem 2.9. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} V_k = \frac{1}{4}(V_{n+4} - V_{n+2} - 2V_{n+1} + V_n - V_4 + V_2 + 2V_1 + 3V_0)$,

(b): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} - 3V_4 + 4V_3 - V_2 + 6V_1 + V_0)$

(c): $\sum_{k=0}^{n} V_{2k} = \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 + 5V_0)$.

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As special cases of above Theorem, we have the following two Corollaries. First one present some summation formulas of Pentanacci numbers.

Corollary 2.10. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+4} - P_{n+2} - 2P_{n+1} + P_n - 1)$

(b): $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2} + 4P_{2n+1} + P_{2n} + 2P_{2n-1} - P_{2n-2} + 1)$

(c): $\sum_{k=0}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2} + 4P_{2n+1} + 5P_{2n} + 2P_{2n-1} + 3P_{2n-2} - 3)$.

Next Corollary gives some summation formulas of Pentanacci-Lucas numbers.

Corollary 2.11. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+2} - 2Q_{n+1} + Q_n + 5)$

(b): $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + 4Q_{2n+1} + Q_{2n} + 2Q_{2n-1} - Q_{2n-2} - 9)$

(c): $\sum_{k=0}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 4Q_{2n+1} + 5Q_{2n} + 2Q_{2n-1} + 3Q_{2n-2} + 19)$.

3. Gaussian Generalized Pentanacci Numbers

In this section, we introduce Gaussian generalized Pentanacci numbers and present Binet's formulas, generating functions, and the summation formulas for Gaussian generalized Pentanacci numbers.

First we recall Gaussian integers. A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. These numbers is denoted by $\mathbb{Z}[i]$. For more information about this kind of integers, see the work of Fraleigh [6].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

In 1963, Horadam [11] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [17] defined the complex Tribonacci numbers at Gaussian integers, see also [7]. There are other several studies dedicated to these sequences of Gaussian numbers such as the works in [1], [3], [4], [8], [9], [10], [12], [13], [16], [21], [22], [24], [25], [26], among others.

Gaussian generalized Pentanacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4)\}_{n\geq 0}$ are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i, GV_1 = c_1 + c_0 i, GV_2 = c_2 + c_1 i, \\ GV_3 &= c_3 + c_2 i, GV_4 = c_4 + c_3 i \end{aligned}$$

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not all being zero. The sequences $\{GV_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} + GV_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (3.1) hold for all integer n. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1} \tag{3.2}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}$$

The first few generalized Gaussian Pentanacci numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4:

Table 3. A few Gaussian generalized Pentanacci numbers with positive subscript

n	GV_n
0	$c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i$
1	$c_1 + c_0 i$
2	$c_2 + c_1 i$
3	$c_3 + c_2 i$
4	$c_4 + c_3 i$
5	$(c_0 + c_1 + c_2 + c_3 + c_4) + c_4 i$
6	$(c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4) + (c_0 + c_1 + c_2 + c_3 + c_4)i$
7	$(2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4) + (c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4)i$
8	$(4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4) + (2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4)i$
9	$(8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4) + (4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4)i$
10	$(16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4) + (8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4)i$

Table 4. A few Gaussian generalized Pentanacci numbers with negative subscript

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n	GV_{-n}
0	$c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i$
1	$(-c_0 - c_1 - c_2 - c_3 + c_4) + (2c_3 - c_4)i$
2	$(2c_3 - c_4) + (2c_2 - c_3)i$
3	$(2c_2 - c_3) + (2c_1 - c_2)i$
4	$(2c_1 - c_2) + (2c_0 - c_1)i$
5	$(2c_0 - c_1) + (-3c_0 - 2c_1 - 2c_2 - 2c_3 + 2c_4)i$
6	$(-3c_0 - 2c_1 - 2c_2 - 2c_3 + 2c_4) + (c_0 + c_1 + c_2 + 5c_3 - 3c_4)i$
7	$(c_0 + c_1 + c_2 + 5c_3 - 3c_4) + (4c_2 - 4c_3 + c_4)i$
8	$(4c_2 - 4c_3 + c_4) + (4c_1 - 4c_2 + c_3)i$
9	$(4c_1 - 4c_2 + c_3) + (4c_0 - 4c_1 + c_2)i$
10	$(4c_0 - 4c_1 + c_2) + (4c_4 - 3c_1 - 4c_2 - 4c_3 - 8c_0)i$

We consider two special cases of $GV_n : GV_n(0, 1, 1 + i, 2 + i, 4 + 2i) = GP_n$ is the sequence of Gaussian Pentanacci numbers and $GV_n(5-i, 1+5i, 3+i, 7+3i, 15+7i) = GQ_n$ is the sequence of Gaussian Pentanacci-Lucas numbers. We formally define them as follows:

Gaussian Pentanacci numbers are defined by

$$GP_n = GP_{n-1} + GP_{n-2} + GP_{n-3} + GP_{n-4} + GP_{n-5}, \tag{3.3}$$

with the initial conditions

$$GP_0 = 0, GP_1 = 1, GP_2 = 1 + i, GP_3 = 2 + i, GP_4 = 4 + 2i$$

and Gaussian Pentanacci-Lucas numbers are defined by

$$GQ_n = GQ_{n-1} + GQ_{n-2} + GQ_{n-3} + GQ_{n-4} + GQ_{n-5} \tag{3.4}$$

with the initial conditions

$$GQ_0 = 5 - i, GQ_1 = 1 + 5i, GQ_2 = 3 + i, GQ_3 = 7 + 3i, GQ_4 = 15 + 7i.$$

Note that for $n \geq 0$

$$GP_n = M_n + iM_{n-1},\ GQ_n = R_n + iR_{n-1}$$

and

$$GP_{-n} = M_{-n} + iM_{-n-1}, GQ_{-n} = R_{-n} + iR_{-n-1}.$$

Next, we present the first few values of the Gaussian Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 4:

Table 4. A few Gaussian Pentanacci and Pentanacci-Lucas Numbers

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n	0	1	2	3	4	5	6	7	8	9
GP_n	0	1	$1+i$	$2+i$	$4+2i$	$8+4i$	$16+8i$	$31+16i$	$61+31i$	$120+61i$
GP_{-n}	0	0	0	i	$1-i$	-1	0	0	$2i$	$2-3i$
GQ_n	$5-i$	$1+5i$	$3+i$	$7+3i$	$15+7i$	$31+15i$	$57+31i$	$113+57i$	$223+113i$	$439+223i$
GQ_{-n}	$5-i$	$-1-i$	$-1-i$	$-1-i$	$-1+9i$	$9-7i$	$-7-i$	$-1-i$	$-1-i$	$-1+19i$

The following Theorem presents the generating function $f_{GV_n}(x) = \sum_{n=0}^{\infty} GVx^n$ of Gaussian generalized Pentanacci numbers GV_n.

THEOREM 3.1. *The generating function of Gaussian generalized Pentanacci numbers is given as*

$$f_{GV_n}(x) = \frac{\begin{array}{c} GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3 \\ +(GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^4 \end{array}}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{3.5}$$

Proof. Using (3.1) and some calculation, we obtain

$$\begin{aligned} & f_{GV_n}(x) - xf_{GV_n}(x) - x^2 f_{GV_n}(x) - x^3 f_{GV_n}(x) - x^4 f_{GV_n}(x) - x^5 f_{GV_n}(x) \\ = \; & GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3 \\ & +(GV_4 - GV_3 - GV_2 - GV_1 - GV_0)x^4 \end{aligned}$$

which gives (3.5).

The previous Theorem gives the following results as particular examples: the generating function of Gaussian Pentanacci numbers is

$$f_{GP_n}(x) = \frac{x + ix^2}{1 - x - x^2 - x^3 - x^4 - x^5} \tag{3.6}$$

and the generating function of Gaussian Pentanacci-Lucas numbers is

$$f_{GQ_n}(x) = \frac{5 - i - (4 - 6i)\,x - (3 + 3i)\,x^2 - (2 + 2i)\,x^3 - (1 + i)\,x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{3.7}$$

We now present the Binet formula for the Gaussian generalized Pentanacci numbers.

THEOREM 3.2. *The Binet formula for the Gaussian generalized Pentanacci numbers is*

$$\begin{aligned} GV_n &= \left(A_1\alpha^{n-8} + A_2\beta^{n-8} + A_3\gamma^{n-8} + A_4\delta^{n-8} + A_5\lambda^{n-8}\right) \\ &\quad +i\left(A_1\alpha^{n-9} + A_2\beta^{n-9} + A_3\gamma^{n-9} + A_4\delta^{n-9} + A_5\lambda^{n-9}\right) \end{aligned}$$

where A_1, A_2, A_3, A_4 and A_5 are as in Corollary 2.5.

Proof. The proof follows from Corollary 2.5 and $GV_n = V_n + iV_{n-1}$.

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The previous Theorem gives the following results as particular examples: the Binet formula for the Gaussian Pentanacci numbers is

$$\begin{aligned} GP_n &= (\frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &+ \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}) \\ &+ i(\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &+ \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+2}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}) \end{aligned}$$

or

$$\begin{aligned} GP_n &= \left(\frac{\alpha-1}{6\alpha-10}\alpha^{n-1} + \frac{\beta-1}{6\beta-10}\beta^{n-1} + \frac{\gamma-1}{6\gamma-10}\gamma^{n-1} + \frac{\delta-1}{6\delta-10}\delta^{n-1} + \frac{\lambda-1}{6\lambda-10}\lambda^{n-1}\right) \\ &+ i\left(\frac{\alpha-1}{6\alpha-10}\alpha^{n-2} + \frac{\beta-1}{6\beta-10}\beta^{n-2} + \frac{\gamma-1}{6\gamma-10}\gamma^{n-2} + \frac{\delta-1}{6\delta-10}\delta^{n-2} + \frac{\lambda-1}{6\lambda-10}\lambda^{n-2}\right) \end{aligned}$$

and the Binet formula for the Gaussian Pentanacci-Lucas numbers is

$$GQ_n = (\alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n) + i\left(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1} + \lambda^{n-1}\right).$$

The following Theorem present some summation formulas of Gaussian generalized Pentanacci numbers.

THEOREM 3.3. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the Gaussian generalized Pentanacci numbers)*

$$\sum_{k=1}^{n} GV_k = \frac{1}{4}(GV_{n+4} - GV_{n+2} - 2GV_{n+1} + GV_n - GV_4 + GV_2 + 2GV_1 - GV_0)$$

(b): $\sum_{k=1}^{n} GV_{2k+1} = \frac{1}{8}(3GV_{2n+2} + 4GV_{2n+1} + GV_{2n} + 2GV_{2n-1} - GV_{2n-2} - 3GV_4 + 4GV_3 - GV_2 - 2GV_1 + GV_0)$

(c): $\sum_{k=1}^{n} GV_{2k} = \frac{1}{8}(-GV_{2n+2} + 4GV_{2n+1} + 5GV_{2n} + 2GV_{2n-1} + 3GV_{2n-2} + GV_4 - 4GV_3 + 3GV_2 - 2GV_1 - 3GV_0)$.

Proof. (a), (b) and (c) can be proved exactly as in the proof of Theorem 2.6.

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Pentanacci numbers.

COROLLARY 3.4. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the Gaussian Pentanacci numbers)*

$$\sum_{k=1}^{n} GP_k = \frac{1}{4}(GP_{n+4} - GP_{n+2} - 2GP_{n+1} + GP_n - 1 - i)$$

(b): $\sum_{k=1}^{n} GP_{2k+1} = \frac{1}{8}(3GP_{2n+2} + 4GP_{2n+1} + GP_{2n} + 2GP_{2n-1} - GP_{2n-2} - 7 - 3i)$

(c): $\sum_{k=1}^{n} GP_{2k} = \frac{1}{8}(-GP_{2n+2} + 4GP_{2n+1} + 5GP_{2n} + 2GP_{2n-1} + 3GP_{2n-2} - 3 + i)$.

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Second Corollary gives summation formulas of Gaussian Pentanacci-Lucas numbers.

Corollary 3.5. *For $n \geq 1$ we have the following formulas:*

(a): *(Sum of the Gaussian Pentanacci-Lucas numbers)*

$$\sum_{k=1}^{n} GQ_k = \frac{1}{4}(GQ_{n+4} - GQ_{n+2} - 2GQ_{n+1} + GQ_n - 15 + 5i)$$

(b): $\sum_{k=1}^{n} GQ_{2k+1} = \frac{1}{8}(3GQ_{2n+2} + 4GQ_{2n+1} + GQ_{2n} + 2GQ_{2n-1} - GQ_{2n-2} - 17 - 21i)$

(c): $\sum_{k=1}^{n} GQ_{2k} = \frac{1}{8}(-GQ_{2n+2} + 4GQ_{2n+1} + 5GQ_{2n} + 2GQ_{2n-1} + 3GQ_{2n-2} - 21 - 9i)$.

4. Basic Relations and Simson Formulas

In this section, we obtain some identities of Pentanacci numbers and Pentanacci-Lucas numbers and some identities of Gaussian Pentanacci numbers and Gaussian Pentanacci-Lucas numbers. Moreover, we present Simson formulas of these numbers.

First, we can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

Theorem 4.1. *The following equalities are true:*

$$Q_n = -P_{n+3} + 7P_{n+1} - 2P_n - P_{n-1}, \tag{4.1}$$

$$Q_n = -P_{n+2} + 6P_{n+1} - 3P_n - 2P_{n-1} - P_{n-2}, \tag{4.2}$$

$$Q_n = 3P_{n+4} - 7P_{n+2} - 9P_n - 8P_{n-1} - 4P_{n-2}, \tag{4.3}$$

$$Q_{2n+1} = 287P_{n+2} - 286P_{n+1} - 281P_n - 264P_{n-1} - 213P_{n-2}, \tag{4.4}$$

and

$$1198P_n = 27Q_{n+3} + 18Q_{n+2} + 3Q_{n+1} - 22Q_n + 136Q_{n-1}, \tag{4.5}$$

$$1198P_n = 45Q_{n+3} - 15Q_{n+1} - 40Q_n + 118Q_{n-1} - 18Q_{n-2}. \tag{4.6}$$

Proof. Note that the last six identities hold for all integers n. For example, to show (4.1), writing

$$Q_n = aP_{n+3} + bP_{n+2} + cP_{n+1} + dP_n + eP_{n-1}$$

and solving the system of equations

$$\begin{aligned}
Q_0 &= aP_3 + bP_2 + cP_1 + dP_0 + eP_{-1} \\
Q_1 &= aP_4 + bP_3 + cP_2 + dP_1 + eP_0 \\
Q_2 &= aP_5 + bP_4 + cP_3 + dP_2 + eP_1 \\
Q_3 &= aP_6 + bP_5 + cP_4 + dP_3 + eP_2 \\
Q_4 &= aP_7 + bP_6 + cP_5 + dP_4 + eP_3
\end{aligned}$$

we find that $a = -1, b = 0, c = 7, d = -2, e = -1$. The other equalities can be proved similarly.

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We present a few basic relations between $\{GP_n\}$ and $\{GQ_n\}$.

THEOREM 4.2. *The following equalities are true:*

$$GQ_n = -GP_{n+3} + 7GP_{n+1} - 2GP_n - GP_{n-1}, \tag{4.7}$$

$$GQ_n = -GP_{n+2} + 6GP_{n+1} - 3GP_n - 2GP_{n-1} - GP_{n-2}, \tag{4.8}$$

$$GQ_n = 3GP_{n+4} - 7GP_{n+2} - 9GP_n - 8GP_{n-1} - 4GP_{n-2}, \tag{4.9}$$

$$GQ_{2n+1} = 287GP_{n+2} - 286GP_{n+1} - 281GP_n - 264GP_{n-1} - 213GP_{n-2}, \tag{4.10}$$

and

$$1198GP_n = 27GQ_{n+3} + 18GQ_{n+2} + 3GQ_{n+1} - 22GQ_n + 136GQ_{n-1}, \tag{4.11}$$

$$1198GP_n = 45GQ_{n+3} - 15GQ_{n+1} - 40GQ_n + 118GQ_{n-1} - 18GQ_{n-2}. \tag{4.12}$$

Proof. Note that the last six identities hold for all integers n. For example, to show (4.7), writing

$$GQ_n = aGP_{n+3} + bGP_{n+2} + cGP_{n+1} + dGP_n + eGP_{n-1}$$

and solving the system of equations

$$\begin{aligned} GQ_0 &= aGP_3 + bGP_2 + cGP_1 + dGP_0 + eGP_{-1} \\ GQ_1 &= aGP_4 + bGP_3 + cGP_2 + dGP_1 + eGP_0 \\ GQ_2 &= aGP_5 + bGP_4 + cGP_3 + dGP_2 + eGP_1 \\ GQ_3 &= aGP_6 + bGP_5 + cGP_4 + dGP_3 + eGP_2 \\ GQ_4 &= aGP_7 + bGP_6 + cGP_5 + dGP_4 + eGP_3 \end{aligned}$$

we find that $a = -1, b = 0, c = 7, d = -2, e = -1$. Or using the relations $GP_n = P_n + iP_{n-1}$, $GQ_n = Q_n + iQ_{n-1}$ and identity $Q_n = -P_{n+3} + 7P_{n+1} - 2P_n - P_{n-1}$ (see Theorem 4.1) we obtain the identity (4.7). In fact, note that

$$\begin{aligned} GQ_n &= Q_n + iQ_{n-1} \\ &= (-P_{n+3} + 7P_{n+1} - 2P_n - P_{n-1}) + i(-P_{n+2} + 7P_n - 2P_{n-1} - P_{n-2}) \\ &= -(P_{n+3} + iP_{n+2}) + 7(P_{n+1} + iP_n) - 2(P_n + iP_{n-1}) - (P_{n-1} + iP_{n-2}) \\ &= -GP_{n+3} + 7GP_{n+1} - 2GP_n - GP_{n-1}. \end{aligned}$$

The other equalities can be proved similarly.

We can also give a few basic relations between $\{GQ_n\}$ and $\{P_n\}$.

$$GQ_n = (5-i)\,P_{n+1} - (4-6i)\,P_n - (3+3i)\,P_{n-1} - (2+2i)\,P_{n-2} - (1+i)\,P_{n-3}, \tag{4.13}$$

$$GQ_{n+4} = (15+7i)\,P_{n+1} + (16+8i)\,P_n + (11+9i)\,P_{n-1} + (10+4i)\,P_{n-2} + (7+3i)\,P_{n-3}. \tag{4.14}$$

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We present an identity related with Gaussian generalized Pentanacci numbers and Pentanacci numbers.

THEOREM 4.3. *For $n \geq 0$ and $m \geq 0$, the following identity holds:*

$$GV_{m+n} = P_{m-4}GV_n + (P_{m-4} + P_{m-5})GV_{n+1} + (P_{m-4} + P_{m-5} + P_{m-6})GV_{n+2} \tag{4.15}$$

$$+(P_{m-4} + P_{m-5} + P_{m-6} + P_{m-7})GV_{n+3} + P_{m-3}GV_{n+4} \tag{4.16}$$

Proof. The identity (4.15) can be proved by induction on m as in Theorem 2.3.

The previous Theorem gives the following results as particular examples: For $n \geq 0$ and $m \geq 0$, we have (taking $GV_n = GP_n$)

$$\begin{aligned} GP_{m+n} &= P_{m-4}GP_n + (P_{m-4} + P_{m-5})GP_{n+1} + (P_{m-4} + P_{m-5} + P_{m-6})GP_{n+2} \\ &\quad +(P_{m-4} + P_{m-5} + P_{m-6} + P_{m-7})GP_{n+3} + P_{m-3}GP_{n+4} \end{aligned}$$

and (taking $GV_n = GQ_n$)

$$\begin{aligned} GQ_{m+n} &= P_{m-4}GQ_n + (P_{m-4} + P_{m-5})GQ_{n+1} + (P_{m-4} + P_{m-5} + P_{m-6})GQ_{n+2} \\ &\quad +(P_{m-4} + P_{m-5} + P_{m-6} + P_{m-7})GQ_{n+3} + P_{m-3}GQ_{n+4}. \end{aligned}$$

One of the oldest and best known identities for the Fibonacci sequence $\{F_n\}$ is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 [18]. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n$$

and called as Simson or Cassini formula (Identity). The following Theorem gives generalization of this result to generalized Pentanacci numbers.

THEOREM 4.4. *(Simson's formula of generalized Pentanacci numbers) For all integers n, we have*

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \tag{4.17}$$

Proof. (4.17) is given in Soykan [23].

COROLLARY 4.5. *For all integers n, we have*

(a): *(Simson's formula of Pentanacci numbers)*

$$\begin{vmatrix} P_{n+4} & P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+3} & P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} & P_{n-3} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} & P_{n-4} \end{vmatrix} = 1,$$

(b): *(Simpson's formula of Pentanacci-Lucas numbers)*

$$\begin{vmatrix} Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \end{vmatrix} = 9584.$$

Note that Simson's formula of Gaussian Pentanacci numbers is

$$\begin{vmatrix} GP_{n+4} & GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} & GP_{n-4} \end{vmatrix} = 1+i,$$

and Simson's formula of Gaussian Pentanacci-Lucas numbers is

$$\begin{vmatrix} GQ_{n+4} & GQ_{n+3} & GQ_{n+2} & GQ_{n+1} & GQ_n \\ GQ_{n+3} & GQ_{n+2} & GQ_{n+1} & GQ_n & GQ_{n-1} \\ GQ_{n+2} & GQ_{n+1} & GQ_n & GQ_{n-1} & GQ_{n-2} \\ GQ_{n+1} & GQ_n & GQ_{n-1} & GQ_{n-2} & GQ_{n-3} \\ GQ_n & GQ_{n-1} & GQ_{n-2} & GQ_{n-3} & GQ_{n-4} \end{vmatrix} = 9584 + 9584i.$$

5. Matrix Formulations of V_n and GV_n

In this section, we present some matrix formulation of generalized Pentanacci numbers and Gaussian generalized Pentanacci numbers.

We define the square matrix A of order 5 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Induction proof may be used to establish

$$A^n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} + P_{n-2} + P_{n-3} & P_n + P_{n-1} + P_{n-2} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} & P_{n-1} + P_{n-2} + P_{n-3} & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} & P_{n-2} + P_{n-3} + P_{n-4} & P_{n-2} + P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-3} + P_{n-4} + P_{n-5} + P_{n-6} & P_{n-3} + P_{n-4} + P_{n-5} & P_{n-3} + P_{n-4} & P_{n-3} \\ P_{n-3} & P_{n-4} + P_{n-5} + P_{n-6} + P_{n-7} & P_{n-4} + P_{n-5} + P_{n-6} & P_{n-4} + P_{n-5} & P_{n-4} \end{pmatrix}$$

Matrix formulation of P_n and Q_n can be given as

$$\begin{pmatrix} P_{n+4} \\ P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} P_4 \\ P_3 \\ P_2 \\ P_1 \\ P_0 \end{pmatrix} \tag{5.1}$$

and

$$\begin{pmatrix} Q_{n+4} \\ Q_{n+3} \\ Q_{n+2} \\ Q_{n+1} \\ Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Q_4 \\ Q_3 \\ Q_2 \\ Q_1 \\ Q_0 \end{pmatrix}. \tag{5.2}$$

Induction proofs may be used to establish the matrix formulations P_n and Q_n. Similarly, matrix formulation of V_n can be given as

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

Consider the matrices N_P, E_P defined by as follows:

$$N_P = \begin{pmatrix} 4+2i & 2+i & 1+i & 1 & 0 \\ 2+i & 1+i & 1 & 0 & 0 \\ 1+i & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 1-i \end{pmatrix},$$

$$E_P = \begin{pmatrix} GP_{n+4} & GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} & GP_{n-4} \end{pmatrix}.$$

Next Theorem presents the relations between A^n, N_P and E_P.

THEOREM 5.1. *For $n \geq 4$, we have*

$$A^n N_P = E_P.$$

Proof. Using the relations

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GP_{n+4} &= P_{n+4} + iP_{n+3} = (4+2i)\,P_{n+1} + (4+2i)\,P_n + (4+2i)\,P_{n-1} + (3+2i)\,P_{n-2} + (2+i)\,P_{n-3}, \\ GP_{n+3} &= (2+i)\,P_{n+1} + (2+i)\,P_n + (2+i)\,P_{n-1} + (2+i)\,P_{n-2} + (1+i)\,P_{n-3}, \end{aligned}$$

we get $A^n N_P = E_P$.

Above Theorem can be proved by mathematical induction as well.

Consider the matrices N_Q, E_Q defined by as follows:

$$N_Q = \begin{pmatrix} 15+7i & 7+3i & 3+i & 1+5i & 5-i \\ 7+3i & 3+i & 1+5i & 5-i & -1-i \\ 3+i & 1+5i & 5-i & -1-i & -1-i \\ 1+5i & 5-i & -1-i & -1-i & -1-i \\ 5-i & -1-i & -1-i & -1-i & -1+9i \end{pmatrix}$$

$$E_Q = \begin{pmatrix} GQ_{n+4} & GQ_{n+3} & GQ_{n+2} & GQ_{n+1} & GQ_n \\ GQ_{n+3} & GQ_{n+2} & GQ_{n+1} & GQ_n & GQ_{n-1} \\ GQ_{n+2} & GQ_{n+1} & GQ_n & GQ_{n-1} & GQ_{n-2} \\ GQ_{n+1} & GQ_n & GQ_{n-1} & GQ_{n-2} & GQ_{n-3} \\ GQ_n & GQ_{n-1} & GQ_{n-2} & GQ_{n-3} & GQ_{n-4} \end{pmatrix}.$$

The following Theorem presents the relations between A^n, N_Q and E_Q.

Theorem 5.2. *We have*

$$A^n N_Q = E_Q.$$

Proof. The proof requires some lengthy calculation, so we omit it.

The previous Theorem, also, can be proved by mathematical induction.

6. Conclusions

- In the section 1, we present some background about generalized Pentanacci numbers.
- In the section 2, we present Binet's formulas, generating functions, and the summation formulas for generalized Pentanacci numbers.
- In the section 3, first we recall Gaussian integers and then we define Gaussian generalized Pentanacci numbers and as special cases, we investigate Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers, with their properties such as the generating functions, Binet's formulas and sums formulas of these Gaussian numbers.
- In the section 4, we obtain some identities of Pentanacci numbers and Pentanacci-Lucas numbers and some identities of Gaussian Pentanacci numbers and Gaussian Pentanacci-Lucas numbers. Furthermore, we present Simson formulas of those numbers.
- In the section 5, we give some matrix formulation of generalized Pentanacci numbers and Gaussian generalized Pentanacci numbers.

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