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Article

Four-Dimensional Almost Einstein Manifolds with Skew-Circulant Structures

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Abstract: We consider a four-dimensional Riemannian manifold M equipped with an additional tensor structure S, whose fourth power is minus identity and the second power is an almost complex structure. In a local coordinate system the components of the metric g and the structure S form skew-circulant matrices. Both structures S and g are compatible, such that an isometry is induced in every tangent space of M. By a special identity for the curvature tensor, generated by the Riemannian connection of g, we determine classes of an Einstein manifolds and an almost Einstein manifolds. For such manifolds we obtain propositions for the sectional curvatures of some special 2-planes in a tangent space of M. We consider an almost Hermitian manifold associated with the studied manifold and find conditions for g, under which it is a Kähler manifold. We construct some examples of the considered manifolds on Lie groups.

Keywords: Riemannian manifold; Einstein manifold; sectional curvatures; Ricci curvature; Lie group

MSC: 53B20; 53C15; 53C25; 53C55; 22E60

1. Introduction

The right circulant matrices and the right skew-circulant matrices are Toeplitz matrices, which are thoroughly studied in [1] and [3]. The set of invertible circulant (skew-circulant) matrices form a group with respect to the matrix multiplication. Such matrices have application to geometry, linear codes, graph theory, vibration analysis (for example [2,7,9,11–13]).

A. Gray, L. Hervella and L. Vanhecke used curvature identities to classify and to study the almost Hermitian manifolds (for instance in [4–6,15]). The Hermitian manifolds form a class of manifolds with an integrable complex structure J. The class of the Kähler manifolds is their subclass and such manifolds have a parallel structure J. According to A. Gray, the Kähler manifolds have an especially rich geometric structure, due to the Kähler curvature identity $R(\cdot,\cdot,J\cdot,J\cdot) = R(\cdot,\cdot,\cdot,\cdot)$. Some of the recent investigations on the curvature properties of the almost Hermitian manifolds are made in [8,10,14,16].

In the present work we study a four-dimensional differentiable manifold M with a Riemannian metric g. The manifold M is equipped with an additional tensor structure S of type $(1,1)$, which satisfies $S^4 = -\mathrm{id}$. Moreover, the component matrix of S is a special skew-circulant matrix. The structure S is compatible with g, such that an isometry is induced in every tangent space of M. Such

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a manifold (M, g, S) is associated with an almost Hermitian manifold (M, g, J), where $J = S^2$ is an almost complex structure.

The paper is organized as follows. In Sect. 2, we introduce the manifold (M, g, S). In Sect. 3, we find conditions under which an orthogonal basis of the type $\{x, Sx, S^2x, S^3x\}$ exists in every tangent space of (M, g, S). In Sect. 4, we consider a class of almost Einstein manifolds (M, g, S). Also, we obtain conditions for (M, g, S) to be an Einstein manifold. In Sect. 5, we find some curvature properties of these manifolds. In Sect. 6, we obtain a necessary and sufficient condition for S to be parallel with respect to the Riemannian connection of g. Also, we get conditions for (M, g, J) to be a Kähler manifold. In Sect. 7, we construct examples of the considered manifolds on Lie groups and find some of their geometric characteristics.

2. Preliminaries

Let M be a 4-dimensional Riemannian manifold equipped with a tensor structure S in every tangent space T_pM at a point p on M. Let S have a skew-circulant matrix, with respect to some basis $\{e_i\}$, as follows

$$(S_j^k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Then S has the property

$$S^4 = -\mathrm{id}. \tag{2}$$

Let the metric g and the structure S satisfy

$$g(Sx, Sy) = g(x, y). \tag{3}$$

Here and anywhere in this work, x, y, z, u will stand for arbitrary elements of the algebra on smooth vector fields on M or vectors in T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

The conditions (1) and (3) imply that the matrix of g has the form

$$(g_{ij}) = \begin{pmatrix} A & B & 0 & -B \\ B & A & B & 0 \\ 0 & B & A & B \\ -B & 0 & B & A \end{pmatrix}, \tag{4}$$

i.e. it is skew-circulant. Here $A = A(p)$ and $B = B(p)$ are smooth functions of an arbitrary point $p(X^1, X^2, X^3, X^4)$ on M. The determinant of g has a value $\det(g_{ij}) = (A^2 - 2B^2)^2$. It is supposed that

$$A(p) > \sqrt{2}B(p) > 0 \tag{5}$$

in order g to be positive definite. A manifold M introduced in this way we denote by (M, g, S).

Now, we consider an associated metric $\tilde{g}$ with g, determined by

$$\tilde{g}(x, y) = g(x, Sy) + g(Sx, y). \tag{6}$$

Using (1), (4) and (6) we get that the matrix of its components is

$$(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A & 0 & -A \\ A & 2B & A & 0 \\ 0 & A & 2B & A \\ -A & 0 & A & 2B \end{pmatrix}. \tag{7}$$

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Since (5) is valid, it is easy to see that $\tilde{g}$ is an indefinite metric.

The inverse matrices of (g_{ij}) and $(\tilde{g}_{ij})$ are as follows:

$$(g^{ij}) = \frac{1}{D}\begin{pmatrix} A & -B & 0 & B \\ -B & A & -B & 0 \\ 0 & -B & A & -B \\ B & 0 & -B & A \end{pmatrix}, \tag{8}$$

$$(\tilde{g}^{ij}) = \frac{1}{2D}\begin{pmatrix} -2B & A & 0 & -A \\ A & -2B & A & 0 \\ 0 & A & -2B & A \\ -A & 0 & A & -2B \end{pmatrix}, \tag{9}$$

where $D = A^2 - 2B^2$.

3. Orthogonal S-basis of T_pM

If x is a nonzero vector on (M, g, S), then according to (1) we have $Sx \neq \pm x$. Therefore the angle φ between x and Sx belongs to the interval $(0, \pi)$. Evidently, the vectors x, Sx, S^2x and S^3x determine six angles, which belong to $(0, \pi)$. For these angles we establish the next statement.

Theorem 1. *Let x be a nonzero vector on (M, g, S). Then*

$$\angle(x, Sx) = \angle(Sx, S^2x) = \angle(S^2x, S^3x) = \varphi, \quad \angle(x, S^3x) = \pi - \varphi, \quad \angle(x, S^2x) = \angle(Sx, S^3x) = \frac{\pi}{2}, \tag{10}$$

where $\varphi \in (0, \pi)$.

Proof. Let $x = (x^1, x^2, x^3, x^4)$ be a nonzero vector on (M, g, S). By using (1), we get

$$Sx = (x^2, x^3, x^4, -x^1), \quad S^2x = (x^3, x^4, -x^1, -x^2), \quad S^3x = (x^4, -x^1, -x^2, -x^3). \tag{11}$$

From (2) and (3) it follows

$$g(x, Sx) = -g(x, S^3x), \qquad g(x, S^2x) = 0. \tag{12}$$

Having in mind (4) and (11), we calculate

$$\begin{aligned} g(x, x) &= A\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big) + 2B(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4), \\ g(x, Sx) &= A(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4) + B\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big). \end{aligned} \tag{13}$$

Now, due to (3) and (5), we can determine the angle between x and Sx and the angle between x and S^2x as follows:

$$\cos\varphi = \frac{g(x, Sx)}{g(x, x)}, \qquad \cos\phi = \frac{g(x, S^2x)}{g(x, x)}. \tag{14}$$

We apply (12) and (13) in (14) and find

$$\cos\varphi = \frac{A(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4) + B\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big)}{A\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big) + 2B(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4)},$$

$$\cos\phi = 0.$$

Then, bearing in mind (3) and (12), we get (10). $\square$

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52 **Definition 1 .** *A basis of type $\{x, Sx, S^2x, S^3x\}$ of T_pM is called a S-basis. In this case we say that the vector x*
53 *induces a S-basis of T_pM.*

54 The following statements hold.

Theorem 2. *Every nonzero vector $x = (x^1, x^2, x^3, x^4)$, which satisfies*

$$4x^2x^4\left((x^1)^2 - (x^3)^2\right) + 4x^1x^3\left((x^4)^2 - (x^2)^2\right) + \left((x^1)^2 + (x^3)^2\right)^2 + \left((x^2)^2 + (x^4)^2\right)^2 \neq 0, \tag{15}$$

55 *induces a S-basis of T_pM.*

Proof. If a nonzero vector $x \in T_pM$ has coordinates (x^1, x^2, x^3, x^4), then using (11) we get the determinant formed by the coordinates of the vectors x, Sx, S^2x and S^3x. It is

$$\triangle = 4x^2x^4\left((x^1)^2 - (x^3)^2\right) + 4x^1x^3\left((x^4)^2 - (x^2)^2\right) + \left((x^1)^2 + (x^3)^2\right)^2 + \left((x^2)^2 + (x^4)^2\right)^2.$$

56 In case that (15) is valid, we have $\triangle \neq 0$, i.e. x, Sx, S^2x and S^3x form a basis. $\square$

Lemma 1. *Let a vector x induce a S-basis and let φ be the angle between x and Sx. The following inequalities are valid:*

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \tag{16}$$

Proof. We suppose without loss of generality that $g(x, x) = 1$. Then, from (3), (12) and (14), we find

$$g(x, Sx) = g(Sx, S^2x) = g(S^2x, S^3x) = -g(x, S^3x) = \cos\varphi, \quad g(x, S^2x) = g(Sx, S^3x) = 0. \tag{17}$$

We consider a nonzero vector y, such that

$$y = -\cos\varphi x + Sx - \cos\varphi S^2x. \tag{18}$$

Since g is a Riemannian metric we have $g(y, y) > 0$. Substituting (18) into the latter inequality, and using (17), we get

$$1 - 2\cos^2\varphi > 0.$$

57 Then, taking into account $0 < \varphi < \pi$, we obtain (16). $\square$

58 Bearing in mind Theorem 1, Theorem 2 and Lemma 1, we arrive at the following

59 **Theorem 3.** *For every manifold (M, g, S) an orthogonal S-basis of T_pM exists.*

60 4. Almost Einstein manifolds

Let ∇ be the Riemannian connection of g. The curvature tensor R of ∇ is determined by

$$R(x, y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z. \tag{19}$$

The tensor of type $(0, 4)$ associated with R is defined by

$$R(x, y, z, u) = g(R(x, y)z, u). \tag{20}$$

The Ricci tensor ρ with respect to g is given by the well-known formula

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j). \tag{21}$$

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The scalar curvature τ with respect to g and its associated quantity are determined by

$$\tau = g^{ij}\rho(e_i, e_j), \qquad \tau^* = \tilde{g}^{ij}\rho(e_i, e_j). \tag{22}$$

Now, we consider a manifold (M, g, S) with the condition

$$\nabla S = 0. \tag{23}$$

i.e., S is a parallel structure with respect to ∇.

Proposition 1. *Every manifold (M, g, S) with a parallel structure S satisfies the curvature identity*

$$R(x, y, Sz, Su) = R(x, y, z, u). \tag{24}$$

Proof. The well-known formula $(\nabla_x S)y = \nabla_x Sy - S\nabla_x y$, together with (23), yields

$$\nabla_x Sy = S\nabla_x y. \tag{25}$$

On the other hand (19) implies $R(x, y, Sz, Su) = g\big(R(x, y)Sz, Su\big)$. Because of the latter identity, using (3), (19) and (25), we have successively

$$\begin{aligned} R(x, y, Sz, Su) &= g\big(\nabla_x\nabla_y Sz - \nabla_y\nabla_x Sz - \nabla_{[x,y]}Sz, Su\big) \\ &= g\big(\nabla_x S(\nabla_y z) - \nabla_y S(\nabla_x z) - S(\nabla_{[x,y]}z), Su\big) \\ &= g\big(S(\nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z), Su\big) = g\big(R(x, y)z, u\big), \end{aligned}$$

which completes the proof. □

We will note that the identity (24) defines a more general class of manifolds (M, g, S) than the class with the condition (23). Farther in this paper, we investigate the properties of manifolds in these two classes.

Let R_{ijkh} be the components of the curvature tensor R of type $(0, 4)$. The local form of (24) is $R_{ijlm}S^l_k S^m_h = R_{ijkh}$. Then, using (1), we find the equalities

$$\begin{aligned} R_{1313} =& R_{2424} = R_{1324}, \\ R_{1212} =& R_{1414} = R_{2323} = R_{3434} = R_{1223} = R_{1214} = R_{1434} = R_{1234} = R_{2334} = R_{2314}, \\ R_{1213} =& R_{1224} = R_{1413} = R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434}. \end{aligned}$$

By applying the Bianchi identity to the above components of R, we obtain

$$\begin{aligned} R_{1313} = R_{2424} = R_{1324} =& 2R_{1212} = 2R_{1414} = 2R_{2323} = 2R_{3434} = 2R_{1223} = 2R_{1214} \\ =& 2R_{1434} = 2R_{1234} = 2R_{2334} = 2R_{2314}, \\ R_{1213} = R_{1224} = R_{1413} =& R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434}. \end{aligned} \tag{26}$$

Vice versa, from (1) and (26) it follows (24).

Hence we arrive at the following

Proposition 2. *The property (24) of the curvature tensor R of the manifold (M, g, S) is equivalent to the conditions (26).*

Proposition 3. *If a manifold (M, g, S) has the property (24), then the components of the Ricci tensor ρ satisfy*

$$\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44}, \quad \rho_{12} = \rho_{23} = \rho_{34} = -\rho_{14}, \quad \rho_{13} = \rho_{24} = 0. \tag{27}$$

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Proof. Due to Proposition 2 the components of the curvature tensor R satisfy (26). For brevity, we denote

$$R_1 = R_{1313}, \quad R_2 = R_{1213}. \tag{28}$$

Thus, having in mind (8), (21), (26) and (28), we get the components of ρ, as follows:

$$\begin{aligned} \rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} &= \frac{2}{D}\big(-AR_1 + 2BR_2\big), \\ \rho_{12} = \rho_{23} = \rho_{34} = -\rho_{14} &= \frac{2}{D}\big(BR_1 - AR_2\big), \\ \rho_{13} = \rho_{24} &= 0. \end{aligned} \tag{29}$$

i.e. the equalities (27) are valid. □

A Riemannian manifold is said to be Einstein if its Ricci tensor ρ is a constant multiple of the metric tensor g, i.e.

$$\rho(x, y) = \alpha g(x, y). \tag{30}$$

In [17], for locally decomposable Riemannian manifolds is defined a class of almost Einstein manifolds. For the considered in our paper manifolds, we give the following

Definition 2. *A Riemannian manifold (M, g, S) is called almost Einstein if the metrics g and $\tilde{g}$ satisfy*

$$\rho(x, y) = \alpha g(x, y) + \beta \tilde{g}(x, y), \tag{31}$$

where α and β are smooth functions on M.

Theorem 4. *If a manifold (M, g, S) has the property (24), then it is almost Einstein.*

Proof. Due to Proposition 3, for (M, g, S) the equalities (27) are valid. Consequently, from (22), using (8), (9) and (27), we get the values of the scalar curvatures τ and τ^*, as follows:

$$\tau = \frac{4}{D}\big(A\rho_{11} - 2B\rho_{12}\big), \quad \tau^* = \frac{4}{D}\big(-B\rho_{11} + A\rho_{12}\big).$$

Immediately from the latter equalities we have

$$\rho_{11} = \frac{\tau}{4}A + \frac{2\tau^*}{4}B, \quad \rho_{12} = \frac{\tau}{4}B + \frac{\tau^*}{4}A, \tag{32}$$

and bearing in mind (4) and (7) we get

$$\rho_{11} = \frac{\tau}{4}g_{11} + \frac{\tau^*}{4}\tilde{g}_{11}, \quad \rho_{12} = \frac{\tau}{4}g_{12} + \frac{\tau^*}{4}\tilde{g}_{12}.$$

Then, taking into account (4), (7), (27) and (32), we obtain

$$\rho_{ij} = \frac{\tau}{4}g_{ij} + \frac{\tau^*}{4}\tilde{g}_{ij}, \tag{33}$$

i.e.

$$\rho(x, y) = \frac{\tau}{4}g(x, y) + \frac{\tau^*}{4}\tilde{g}(x, y). \tag{34}$$

Therefore, comparing (34) with (31), we state that (M, g, S) is an almost Einstein manifold. □

Corollary 1. *The manifold (M, g, S) with (24) is Einstein if and only if the scalar curvature τ^* vanishes.*

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Proof. If (M, g, S) has the scalar curvature which satisfies

$$\tau^* = 0, \tag{35}$$

then the equality (34) implies $\rho(x, y) = \frac{\tau}{4} g(x, y)$, i.e. (M, g, S) is an Einstein manifold.

Conversely. Since (M, g, S) is an Einstein manifold its Ricci tensor ρ has the form (30). Thus (34) implies (35). □

In the next theorem, we explicitly express the curvature tensor R of an almost Einstein manifold (M, g, S) by both structures g and S.

Theorem 5. *Let (M, g, S) have the property (24). Then the curvature tensor R has an expression*

$$R = \frac{\tau}{16}(2\pi_1 + \pi_3) + \frac{\tau^*}{8}\pi_2, \tag{36}$$

where

$$\begin{aligned} \pi_1(x, y, z, u) &= g(y, z)g(x, u) - g(x, z)g(y, u), \\ \pi_2(x, y, z, u) &= g(y, z)\tilde{g}(x, u) + g(x, u)\tilde{g}(y, z) - g(x, z)\tilde{g}(y, u) - g(y, u)\tilde{g}(x, z), \\ \pi_3(x, y, z, u) &= \tilde{g}(y, z)\tilde{g}(x, u) - \tilde{g}(x, z)\tilde{g}(y, u). \end{aligned} \tag{37}$$

Proof. Due to Proposition 3, the components of the Ricci tensor ρ of (M, g, S) are given by (29). Therefore, by straightforward computation, we get

$$R_1 = -\frac{1}{2}(A\rho_{11} + 2B\rho_{12}) \qquad R_2 = -\frac{1}{2}(B\rho_{11} + A\rho_{12}).$$

We substitute (32) into the above equalities and obtain

$$R_1 = -\frac{1}{8}((A^2 + 2B^2)\tau + 4AB\tau^*), \qquad R_2 = -\frac{1}{8}(2AB\tau + (2B^2 + A^2)\tau^*). \tag{38}$$

From (4), (7), (28) and (38) it follows

$$\begin{aligned} R_{1313} &= \frac{\tau}{16}\Big(2(g_{13}g_{31} - g_{11}g_{33}) + \tilde{g}_{13}\tilde{g}_{31} - \tilde{g}_{11}\tilde{g}_{33}\Big) + \frac{\tau^*}{8}(g_{13}\tilde{g}_{31} + \tilde{g}_{13}g_{31} - \tilde{g}_{11}g_{33} - g_{11}\tilde{g}_{33}), \\ R_{1213} &= \frac{\tau}{16}\Big(2(g_{13}g_{21} - g_{11}g_{23}) + \tilde{g}_{13}\tilde{g}_{21} - \tilde{g}_{11}\tilde{g}_{23}\Big) + \frac{\tau^*}{8}(g_{13}\tilde{g}_{21} + \tilde{g}_{13}g_{21} - \tilde{g}_{11}g_{23} - g_{11}\tilde{g}_{23}), \end{aligned}$$

Consequently, using (4), (7), (26), (28) and (38), we have

$$R_{ijkh} = \frac{\tau}{16}\Big(2(g_{ih}g_{jk} - g_{ik}g_{jh}) + \tilde{g}_{ih}\tilde{g}_{jk} - \tilde{g}_{ik}\tilde{g}_{jh}\Big) + \frac{\tau^*}{8}(g_{ih}\tilde{g}_{jk} + \tilde{g}_{ih}g_{jk} - \tilde{g}_{ik}g_{jh} - g_{ik}\tilde{g}_{jh}),$$

which is equivalent to (36) with (37). □

5. Curvature properties of (M, g, S)

The sectional curvature of a non-degenerate 2-plane $\{x, y\}$ spanned by the vectors $x, y \in T_pM$ is the value

$$k(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}. \tag{39}$$

Let x induce a S-basis of T_pM for (M, g, S) and let $\sigma = \{x, Sx\}$ be a 2-plane. Evidently, if $y \in \sigma$ and $y \neq x$, then $Sy \notin \sigma$. Consequently, σ has only two S-bases: $\{x, Sx\}$ and $\{-x, -Sx\}$. Thus the sectional curvature $k(x, Sx)$ depends only on $\varphi = \angle(x, Sx)$.

Theorem 6. *Let (M, g, S) have the property (24) and let a vector x induce a S-basis. Then the sectional curvatures, determined by the S-basis, are*

$$\begin{aligned} k(x, Sx) = k(Sx, S^2x) = k(x, S^3x) = k(S^2x, S^3x)& \\ = \frac{1}{16(\cos^2\varphi - 1)}\Big(\tau(1 + 2\cos^2\varphi) + 4\tau^*\cos\varphi\Big)&, \\ k(x, S^2x) = k(Sx, S^3x) = -\frac{1}{8}\Big(\tau(1 + 2\cos^2\varphi) + 4\tau^*\cos\varphi\Big)&, \end{aligned} \tag{40}$$

where $\varphi = \angle(x, Sx)$.

Proof. Let a vector x induce a S-basis. The equalities (3), (12) and (14) imply

$$\begin{aligned} &g(x, Sx) = g(Sx, S^2x) = g(S^2x, S^3x) = -g(x, S^3x) = g(x, x)\cos\varphi, \\ &g(x, S^2x) = g(Sx, S^3x) = 0. \end{aligned} \tag{41}$$

Due to Lemma 1, the angle $\varphi = \angle(x, Sx)$ satisfies (16).

Now, from (2), (3), (6) and (41) we find

$$\tilde{g}(x, x) = 2g(x, x)\cos\varphi, \quad \tilde{g}(x, Sx) = g(x, x), \quad \tilde{g}(x, S^2x) = 0, \quad \tilde{g}(x, S^3x) = -g(x, x). \tag{42}$$

Applying (36), (37), (41) and (42) in (39), we obtain (40). □

Corollary 2. *Let a vector x induce an orthonormal S-basis. Then*

$$\begin{aligned} k(x, Sx) =& k(Sx, S^2x) = k(x, S^3x) = k(S^2x, S^3x) = -\frac{\tau}{16}, \\ k(x, S^2x) =& k(Sx, S^3x) = -\frac{\tau}{8}. \end{aligned}$$

Proof. The proof follows directly from (40), when $\varphi = \frac{\pi}{2}$. □

Due to Theorem 6 and Corollary 1 we establish the following

Proposition 4. *If (M, g, S) with (24) is an Einstein manifold, then the sectional curvatures, determined by an S-basis, are*

$$\begin{aligned} k(x, Sx) =& k(Sx, S^2x) = k(x, S^3x) = k(S^2x, S^3x) = \frac{\tau(1 + 2\cos^2\varphi)}{16(\cos^2\varphi - 1)}, \\ k(x, S^2x) =& k(Sx, S^3x) = -\frac{\tau}{8}(1 + 2\cos^2\varphi). \end{aligned}$$

Now, we recall that the Ricci curvature in the direction of a non-zero vector x is the value

$$r(x) = \frac{\rho(x, x)}{g(x, x)}. \tag{43}$$

Theorem 7. *Let (M, g, S) have the property (24) and let a vector x induce a S-basis. Then the Ricci curvatures are*

$$r(x) = r(Sx) = r(S^2x) = r(S^3x) = \frac{\tau}{4} + \frac{\tau^*}{2}\cos\varphi, \tag{44}$$

where $\varphi = \angle(x, Sx)$.

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Proof. According to Theorem 4, the Ricci tensor ρ is given by (34). Then, using (3), we find

$$\rho(x,x) = \rho(Sx,Sx) = \rho(S^2x,S^2x) = \rho(S^3x,S^3x) = \frac{\tau}{4}g(x,x) + \frac{\tau^*}{4}\tilde{g}(x,x). \tag{45}$$

Let a vector x induce a S-basis. From (3), (42), (43) and (45) it follows (44). □

Further, Theorem 7 and Corollary 1 imply the next statement.

Proposition 5. *Let (M,g,S) with (24) be an Einstein manifold. Then the Ricci curvatures are*

$$r(x) = r(Sx) = r(S^2x) = r(S^3x) = \frac{\tau}{4}.$$

Proof. These equalities follow directly by substituting $\tau^* = 0$ into (44). □

6. Manifolds with parallel structures

In this section we study a manifold (M,g,S), whose structure S satisfies (23). Also, we consider an associated manifold (M,g,J) with a structure $J = S^2$. Bearing in mind (2) and (3), we get that the manifold (M,g,J) is almost Hermitian and the structure J is almost complex. In case that J is parallel (M,g,J) is a Kähler manifold. The characteristic condition of a Kähler manifold is

$$\nabla J = 0. \tag{46}$$

We note that equalities (23) and $J = S^2$ imply (46).

Theorem 8. *Let (M,g,S) have the property (23). Then the scalar curvatures τ and τ^* satisfy*

$$3\tau_1 = \tau_2^* - \tau_4^*, \quad 3\tau_2 = \tau_1^* + \tau_3^*, \quad 3\tau_3 = \tau_2^* + \tau_4^*, \quad 3\tau_4 = -\tau_1^* + \tau_3^*, \tag{47}$$

where $\tau_i = \frac{\partial \tau}{\partial X^i}$, $\tau_i^* = \frac{\partial \tau^*}{\partial X^i}$.

Proof. It is known that in a Riemannian manifold for the scalar curvature τ and the Ricci tensor ρ it is valid

$$\nabla_i \rho_k^i = \frac{1}{2}\nabla_k \tau, \tag{48}$$

where $\rho_k^i = \rho_{ak}g^{ai}$.

On the other hand, if (M,g,S) satisfies (23), then it satisfies (24). Therefore, the Ricci tensor has the expression (33). Hence, from (1), (4), (7), (8) and (33), we get

$$\rho_k^i = \frac{\tau}{4}\delta_k^i + \frac{\tau^*}{4}(S_k^i - (S_k^i)^3),$$

where δ_k^i are the Kronecker symbols. Using the above equalities, (23) and (48) we obtain

$$\tau_k = \frac{\tau_i}{4}\delta_k^i + \frac{\tau_i^*}{4}(S_k^i - (S_k^i)^3).$$

Then, from (1) it follows (47). □

According to Theorem 8 and Corollary 1 we establish the following

Proposition 6. *If (M,g,S) with (23) is an Einstein manifold, then the scalar curvature τ is a constant.*

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6.1. Conditions for parallel structures

Theorem 9. *The manifold (M, g, S) satisfies (23) if and only if*

$$A_1 = B_2 - B_4, \qquad A_2 = B_1 + B_3, \qquad A_3 = B_2 + B_4, \qquad A_4 = B_4 = B_3 - B_1, \tag{49}$$

where $A_i = \dfrac{\partial A}{\partial X^i}$, $B_i = \dfrac{\partial B}{\partial X^i}$.

Proof. If Γ_{ij}^s are the Christoffel symbols of ∇, then

$$\nabla_i S_j^t = \partial_i S_j^t + \Gamma_{ik}^t S_j^k - \Gamma_{ij}^k S_k^t. \tag{50}$$

Together with (23), (50) yields

$$\Gamma_{ik}^t S_j^k = \Gamma_{ij}^k S_k^t. \tag{51}$$

From (1) and (51) we get

$$\begin{aligned}
\Gamma_{11}^1 =&\Gamma_{12}^2 = \Gamma_{13}^3 = \Gamma_{14}^4 = \Gamma_{22}^3 = \Gamma_{23}^4 = -\Gamma_{24}^1 = -\Gamma_{33}^1 = -\Gamma_{34}^2 = -\Gamma_{44}^3,\\
\Gamma_{11}^2 =&\Gamma_{12}^3 = \Gamma_{13}^4 = -\Gamma_{14}^1 = \Gamma_{22}^4 = -\Gamma_{23}^1 = -\Gamma_{24}^2 = -\Gamma_{33}^2 = -\Gamma_{34}^3 = -\Gamma_{44}^4,\\
\Gamma_{11}^3 =&\Gamma_{12}^4 = -\Gamma_{13}^1 = -\Gamma_{14}^2 = -\Gamma_{22}^1 = -\Gamma_{23}^2 = -\Gamma_{24}^3 = -\Gamma_{33}^3 = -\Gamma_{34}^4 = \Gamma_{44}^1,\\
\Gamma_{11}^4 =&-\Gamma_{12}^1 = -\Gamma_{13}^2 = -\Gamma_{14}^3 = -\Gamma_{22}^2 = -\Gamma_{23}^3 = -\Gamma_{24}^4 = -\Gamma_{33}^4 = \Gamma_{34}^1 = \Gamma_{44}^2.
\end{aligned} \tag{52}$$

Now, using (1), (4), (8) and the well known identities

$$2\Gamma_{ij}^s = g^{as}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}),$$

we calculate

$$\begin{array}{ll}
\Gamma_{11}^1 = \frac{1}{2D}\big(AA_1 - B(4B_1 - A_2 + A_4)\big), & \Gamma_{11}^2 = \frac{1}{2D}\big(A(2B_1 - A_2) + B(A_3 - A_1)\big),\\
\Gamma_{11}^3 = \frac{1}{2D}\big(B(A_2 + A_4) - AA_3\big), & \Gamma_{11}^4 = \frac{1}{2D}\big(B(A_1 + A_3) - A(2B_1 + A_4)\big),\\
\Gamma_{12}^1 = \frac{1}{2D}\big(AA_2 - B(A_1 + B_2 + B_4)\big), & \Gamma_{12}^2 = \frac{1}{2D}\big(AA_1 - B(A_2 + B_1 - B_3)\big),\\
\Gamma_{12}^3 = \frac{1}{2D}\big(A(B_1 - B_3) - B(A_1 - B_2 - B_4)\big), & \Gamma_{12}^4 = \frac{1}{2D}\big(B(A_2 + B_3 - B_1) - A(B_2 + B_4)\big),\\
\Gamma_{13}^1 = \frac{1}{2D}\big(AA_3 - 2BB_3\big), & \Gamma_{13}^2 = \frac{1}{2D}\big(A(B_1 + B_3) - B(A_1 + A_3)\big),\\
\Gamma_{13}^3 = \frac{1}{2D}\big(AA_1 - 2BB_1\big), & \Gamma_{13}^4 = \frac{1}{2D}\big(A(B_1 - B_3) + B(A_3 - A_1)\big),\\
\Gamma_{14}^1 = \frac{1}{2D}\big(AA_4 + B(A_1 - B_2 - B_4)\big), & \Gamma_{14}^2 = \frac{1}{2D}\big(A(B_2 + B_4) - B(A_4 + B_1 + B_3)\big),\\
\Gamma_{14}^3 = \frac{1}{2D}\big(A(B_1 + B_3) - B(A_1 + B_2 + B_4)\big), & \Gamma_{14}^4 = \frac{1}{2D}\big(AA_1 + B(A_4 - B_1 - B_3)\big),\\
\Gamma_{22}^1 = \frac{1}{2D}\big(A(2B_2 - A_1) - B(A_2 + A_4)\big), & \Gamma_{22}^2 = \frac{1}{2D}\big(AA_2 - B(4B_2 - A_1 - A_3)\big),\\
\Gamma_{22}^3 = \frac{1}{2D}\big(A(2B_2 - A_3) + B(A_4 - A_2)\big), & \Gamma_{22}^4 = \frac{1}{2D}\big(B(A_3 - A_1) - AA_4\big),\\
\Gamma_{23}^1 = \frac{1}{2D}\big(A(B_3 - B_1) - B(A_3 - B_2 + B_4)\big), & \Gamma_{23}^2 = \frac{1}{2D}\big(AA_3 + B(B_1 - B_3 - A_2)\big),\\
\Gamma_{23}^3 = \frac{1}{2D}\big(AA_2 - B(B_2 - B_4 + A_3)\big), & \Gamma_{23}^4 = \frac{1}{2D}\big(A(B_2 - B_4) - B(A_2 + B_1 - B_3)\big),\\
\Gamma_{24}^1 = \frac{1}{2D}\big(A(B_4 - B_2) - B(A_4 - A_2)\big), & \Gamma_{24}^2 = \frac{1}{2D}\big(AA_4 - 2BB_4\big),\\
\Gamma_{24}^3 = \frac{1}{2D}\big(A(B_2 + B_4) - B(A_2 + A_4)\big), & \Gamma_{24}^4 = \frac{1}{2D}\big(AA_2 - 2BB_2\big),\\
\Gamma_{33}^1 = \frac{1}{2D}\big(B(A_2 - A_4) - AA_1\big), & \Gamma_{33}^2 = \frac{1}{2D}\big(A(2B_3 - A_2) + B(A_1 - A_3)\big),\\
\Gamma_{33}^3 = \frac{1}{2D}\big(AA_3 - B(4B_3 - A_2 - A_4)\big), & \Gamma_{33}^4 = \frac{1}{2D}\big(A(2B_3 - A_4) - B(A_1 + A_3)\big),\\
\Gamma_{34}^1 = \frac{1}{2D}\big(B(A_3 + B_2 - B_4) - A(B_1 + B_3)\big), & \Gamma_{34}^2 = \frac{1}{2D}\big(A(B_4 - B_2) + B(B_3 + B_1 - B_4)\big),\\
\Gamma_{34}^3 = \frac{1}{2D}\big(AB_4 - B(A_3 - B_2 + B_4)\big), & \Gamma_{34}^4 = \frac{1}{2D}\big(AA_3 - B(B_1 + B_3 + B_4)\big),\\
\Gamma_{44}^1 = \frac{1}{2D}\big(B(A_2 + A_4) - A(2B_4 + A_1)\big), & \Gamma_{44}^2 = \frac{1}{2D}\big(B(A_1 + A_3) - AA_2\big),\\
\Gamma_{44}^3 = \frac{1}{2D}\big(A(2B_4 - A_3) + B(A_2 - A_4)\big), & \Gamma_{44}^4 = \frac{1}{2D}\big(AA_4 - B(4B_4 + A_1 - A_3)\big).
\end{array} \tag{53}$$

We apply (53) in (52) and obtain the conditions (49).

Vice versa. Let (49) hold true. We put equalities (49) into (53) and find (52). Hence (1) and (52) imply (51). Consequently, from (1), (50) and (51) we get (23). □

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Theorem 10. *The manifold (M, g, J) is Kähler if and only if the equalities (49) are valid.*

Proof. Having in mind (1), we get that the components of the structure $J = S^2$ on (M, g, J) are given by the skew-circulant matrix

$$(J_j^k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{54}$$

Therefore, from (46), (54) and

$$\nabla_i J_j^t = \partial_i J_j^t + \Gamma_{ik}^t J_j^k - \Gamma_{ij}^k J_k^t$$

it follows

$$\Gamma_{ik}^t J_j^k = \Gamma_{ij}^k J_k^t. \tag{55}$$

Together with (54), (55) yields

$$\begin{array}{lll} \Gamma_{11}^1 = \Gamma_{13}^3 = -\Gamma_{33}^1, & \Gamma_{14}^4 = \Gamma_{23}^4 = \Gamma_{12}^2 = -\Gamma_{34}^2, & \Gamma_{22}^3 = -\Gamma_{24}^1 = -\Gamma_{44}^3, \\ \Gamma_{11}^2 = \Gamma_{13}^4 = -\Gamma_{33}^2, & \Gamma_{14}^1 = \Gamma_{23}^1 = -\Gamma_{12}^3 = \Gamma_{34}^3, & \Gamma_{22}^4 = -\Gamma_{24}^2 = -\Gamma_{44}^4, \\ \Gamma_{11}^3 = -\Gamma_{13}^1 = -\Gamma_{33}^3, & \Gamma_{14}^2 = \Gamma_{23}^2 = -\Gamma_{12}^4 = \Gamma_{34}^4, & \Gamma_{22}^1 = \Gamma_{24}^3 = -\Gamma_{44}^1, \\ \Gamma_{11}^4 = -\Gamma_{13}^2 = -\Gamma_{33}^4, & \Gamma_{14}^3 = \Gamma_{23}^3 = \Gamma_{12}^1 = -\Gamma_{34}^1, & \Gamma_{22}^2 = \Gamma_{24}^4 = -\Gamma_{44}^2. \end{array} \tag{56}$$

We apply (53) in (56) and obtain conditions (49).

Vice versa. From (49) it follows (23). Obviously (23) implies (46). □

Bearing in mind Theorem 9 and Theorem 10 we state the following

Corollary 3. *The structure S of (M, g, S) is parallel with respect to ∇ if and only if the structure J of (M, g, J) is parallel with respect to ∇.*

7. Lie groups as 4-dimensional Riemannian manifolds with skew-circulant structures

Let G be a 4-dimensional real connected Lie group and $\mathbf{g}$ be its Lie algebra with a basis $\{x_1, x_2, x_3, x_4\}$. We introduce a structure S and left invariant metric g as follows

$$Sx_1 = x_2,\ Sx_2 = x_3,\ Sx_3 = x_4,\ Sx_4 = -x_1, \tag{57}$$

$$g(x_i, x_j) = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases} \tag{58}$$

Obviously (2) and (3) are valid. Therefore (G, g, S) is a Riemannian manifold of the considered type.

For the manifold (G, g, S) we suppose that S is an Abelian structure, i.e.

$$[x_i, x_j] = [Sx_i, Sx_j]. \tag{59}$$

According to (57), (59) and the Jacobi identity for the commutators $[x_i, x_j]$ we obtain

$$\begin{aligned} [x_1, x_2] = [x_1, x_4] =& [x_2, x_3] = [x_3, x_4] = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, \\ [x_1, x_3] = [x_2, x_4] =& (\lambda_2 - \lambda_4)x_1 + (\lambda_1 + \lambda_3)x_2 + (\lambda_2 + \lambda_4)x_3 + (\lambda_3 - \lambda_1)x_4, \end{aligned} \tag{60}$$

where $\lambda_i \in \mathbb{R}$.

It is easy to see that a manifold (G, g, S) with a Lie algebra $\mathbf{g}$, determined by (60), has an Abelian structure S.

Theorem 11. *Let (G, g, S) be a manifold with a Lie algebra $\mathbf{g}$ determined by (60). Then (G, g, S) satisfies the identity (23).*

Proof. The well-known Koszul formula implies

$$2g(\nabla_{x_i}x_j, x_k) = g([x_i, x_j], x_k) + g([x_k, x_i], x_j) + g([x_k, x_j], x_i),$$

and having in mind (58) and (60), we get

$$\begin{array}{ll}
\nabla_{x_1}x_1 = -\lambda_1(x_2 + x_4) + (\lambda_4 - \lambda_2)x_3, & \nabla_{x_1}x_2 = \lambda_1(x_1 - x_3) + (\lambda_4 - \lambda_2)x_4, \\
\nabla_{x_1}x_3 = \lambda_1(x_2 - x_4) + (\lambda_2 - \lambda_4)x_1, & \nabla_{x_1}x_4 = \lambda_1(x_1 + x_3) + (\lambda_2 - \lambda_4)x_2, \\
\nabla_{x_2}x_1 = -\lambda_2(x_2 + x_4) - (\lambda_1 + \lambda_3)x_3, & \nabla_{x_2}x_2 = \lambda_2(x_1 - x_3) - (\lambda_1 + \lambda_3)x_4, \\
\nabla_{x_2}x_3 = \lambda_2(x_2 - x_4) + (\lambda_1 + \lambda_3)x_1, & \nabla_{x_2}x_4 = \lambda_2(x_1 + x_3) + (\lambda_1 + \lambda_3)x_2, \\
\nabla_{x_3}x_1 = -\lambda_3(x_2 + x_4) - (\lambda_2 + \lambda_4)x_3, & \nabla_{x_3}x_2 = \lambda_3(x_1 - x_3) - (\lambda_2 + \lambda_4)x_4, \\
\nabla_{x_3}x_3 = \lambda_3(x_2 - x_4) + (\lambda_2 + \lambda_4)x_1, & \nabla_{x_3}x_4 = \lambda_3(x_1 + x_3) + (\lambda_2 + \lambda_4)x_2, \\
\nabla_{x_4}x_1 = -\lambda_4(x_2 + x_4) + (\lambda_1 - \lambda_3)x_3, & \nabla_{x_4}x_2 = \lambda_4(x_1 - x_3) + (\lambda_1 - \lambda_3)x_4, \\
\nabla_{x_4}x_3 = \lambda_4(x_2 - x_4) + (\lambda_3 - \lambda_1)x_1, & \nabla_{x_4}x_4 = \lambda_4(x_1 + x_3) + (\lambda_3 - \lambda_1)x_2.
\end{array} \tag{61}$$

From (57), (61) and the formula $(\nabla_{x_i}S)x_j = \nabla_{x_i}Sx_j - S\nabla_{x_i}x_j$ we get $(\nabla_{x_i}S)x_j = 0$, i.e. (23) is valid. □

Further, using (19), (20), (58), (60) and (61) we calculate the following components of the curvature tensor R:

$$\begin{aligned}
R_{1313} = R_{2424} = R_{1324} &= 2R_{1212} = 2R_{1414} = 2R_{2323} = 2R_{3434} = 2R_{1223} = 2R_{1214} \\
&= 2R_{1434} = 2R_{1234} = 2R_{2334} = 2R_{2314} = 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\
R_{1213} = R_{1224} = R_{1413} &= R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434} \\
&= 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 - \lambda_1\lambda_4).
\end{aligned} \tag{62}$$

The rest of the nonzero components are obtained from the properties

$$R_{ijks} = R_{ksij}, \; R_{ijks} = -R_{jiks} = -R_{ijsk}.$$

From (58), (62) and the formula (21) we get the components of the Ricci tensor ρ:

$$\begin{aligned}
\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} &= -4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\
\rho_{12} = \rho_{23} = \rho_{34} &= -4(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 - \lambda_1\lambda_4), \\
\rho_{13} = \rho_{24} &= 0, \quad \rho_{14} = -\rho_{12}.
\end{aligned} \tag{63}$$

Now, using (6) and (58), we find the components of $\tilde{g}$ and the components of its inverse. They are as follows:

$$\begin{array}{lll}
\tilde{g}_{11} = \tilde{g}_{22} = \tilde{g}_{33} = \tilde{g}_{44} = 0, & \tilde{g}_{12} = \tilde{g}_{23} = \tilde{g}_{34} = -\tilde{g}_{14} = 1, & \tilde{g}_{13} = \tilde{g}_{24} = 0, \\
\tilde{g}^{11} = \tilde{g}^{22} = \tilde{g}^{33} = \tilde{g}^{44} = 0, & \tilde{g}^{12} = \tilde{g}^{23} = \tilde{g}^{34} = -\tilde{g}^{14} = \frac{1}{2}, & \tilde{g}^{13} = \tilde{g}^{24} = 0.
\end{array}$$

Then, applying (58), (63) in (22), we get the values of the scalar curvatures τ and τ^* as follows:

$$\tau = -16(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \quad \tau^* = -16(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 - \lambda_1\lambda_4). \tag{64}$$

Consequently, the equalities (58), (63) and (64) imply (33), i.e. (G, g, S) is an almost Einstein manifold.

Further, using (39), (58) and (62), for the sectional curvatures of the basic 2-planes we find

$$\begin{aligned}
k(x_2, x_4) = k(x_1, x_3) &= 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\
k(x_1, x_2) = k(x_1, x_4) = k(x_2, x_3) = k(x_3, x_4) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2.
\end{aligned} \tag{65}$$

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Therefore, we arrive at the following

Theorem 12. *Let (G, g, S) be a manifold with a Lie algebra* $\mathbf{g}$ *determined by* (60). *Then*

(i) *The nonzero components of the curvature tensor R are* (62);
(ii) *The components of the Ricci tensor ρ are* (63);
(iii) *The scalar curvatures τ and τ^* are* (64). *The manifold is almost Einstein;*
(iv) *The sectional curvatures of the basic 2-planes are* (65).

Bearing in mind Corollary 1 and the second equality of (64) we construct two examples of Einstein manifolds (G, g, S).

Proposition 7. *Let (G, g, S) be a manifold with a Lie algebra* $\mathbf{g}$ *determined by*

$$\begin{aligned}
[x_1, x_2] = [x_1, x_4] = &[x_2, x_3] = [x_3, x_4] = \lambda_1 x_1 + \lambda_1 x_3 + \lambda_4 x_4,\\
[x_1, x_3] = [x_2, x_4] = &-\lambda_4 x_1 + 2\lambda_1 x_2 + \lambda_4 x_3.
\end{aligned}$$

Then

(i) *The nonzero components of the Ricci tensor ρ are*

$$\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = -4(2\lambda_1^2 + \lambda_4^2);$$

(ii) *The manifold is Einstein and the scalar curvatures τ and τ^* are*

$$\tau = -16(2\lambda_1^2 + \lambda_4^2), \quad \tau^* = 0;$$

(iii) *The sectional curvatures of the basic 2-planes are*

$$k(x_2, x_4) = k(x_1, x_3) = 2(2\lambda_1^2 + \lambda_4^2), \quad k(x_1, x_2) = k(x_1, x_4) = k(x_2, x_3) = k(x_3, x_4) = 2\lambda_1^2 + \lambda_4^2.$$

Proof. This example we get by substituting $\lambda_3 = \lambda_1$ and $\lambda_2 = 0$ into each of the equalities (60), (63), (64) and (65). □

Proposition 8. *Let (G, g, S) be a manifold with a Lie algebra* $\mathbf{g}$ *determined by*

$$\begin{aligned}
[x_1, x_2] = [x_1, x_4] = &[x_2, x_3] = [x_3, x_4] = (\lambda_2 + \lambda_4)x_1 + \lambda_2 x_2 + (\lambda_4 - \lambda_2)x_3 + \lambda_4 x_4,\\
[x_1, x_3] = [x_2, x_4] = &(\lambda_2 - \lambda_4)x_1 + 2\lambda_4 x_2 + (\lambda_2 + \lambda_4)x_3 - 2\lambda_2 x_4.
\end{aligned}$$

Then

(i) *The nonzero components of the Ricci tensor ρ are*

$$\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = -12(\lambda_2^2 + \lambda_4^2);$$

(ii) *The manifold is Einstein and the scalar curvatures τ and τ^* are*

$$\tau = -48(\lambda_2^2 + \lambda_4^2), \quad \tau^* = 0;$$

(iii) *The sectional curvatures of the basic 2-planes are*

$$k(x_2, x_4) = k(x_1, x_3) = 6(\lambda_2^2 + \lambda_4^2), \quad k(x_1, x_2) = k(x_1, x_4) = k(x_2, x_3) = k(x_3, x_4) = 3(\lambda_2^2 + \lambda_4^2).$$

Proof. We put $\lambda_1 = \lambda_2 + \lambda_4$ and $\lambda_3 = \lambda_4 - \lambda_2$ into each of the equalities (60), (63), (64) and (65). □

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