

Effect of the Hair on Deflection Angle by Asymptotically Flat Black Holes in Einstein-Maxwell-Dilaton Theory

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In this paper, we are interested in a model of exact asymptotically flat charged hairy black holes in the background of dilaton potential. We study the weak gravitational lensing in the spacetime of hairy black hole in Einstein-Maxwell theory with a non-minimally coupled dilaton and its non-trivial potential. In doing so, we use the optical geometry of the flat charged hairy black hole for some range of parameter γ . For this purpose, by using Gauss-Bonnet theorem, we obtain the deflection angle of photon in a spherically symmetric and asymptotically flat spacetime. Moreover, we also investigate the impact of plasma medium on weak gravitational lensing by asymptotically flat charged hairy black hole with a dilaton potential. Our analytically analyses show the effect of the hair on the deflection angle in weak field limits.

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I. INTRODUCTION

At the darkest points in the universe, their boundaries perilous and invisible, space warps. The singularity constitutes the centre of a black hole and is hidden by the object's surface, the event horizon. A black hole is a location in space that possesses so much gravity, nothing can escape its pull, even light. It is said that fact is sometimes stranger than fiction, and nowhere is this more true than in the case of black holes. Since the first image of a black hole by Event Horizon Telescope [1], physicist now try to take even sharper images so that Einstein's theory of general relativity can be tested and also see the properties of the black holes [2–10] because there are many theoretically obtained black hole solutions with different properties.

In this paper, we try to understand the effect of the hair on deflection angle by asymptotically flat black holes in Einstein-Maxwell-dilaton (EMD) theory, where is derived from a string theory at low energy limits [11]. The EMD black holes have a scalar hair, in addition to mass, rotation and charge so that they offers an fascinating theoretical black hole models to check experimentally in black hole experiments, and investigate possible differences between Einstein gravity and modified gravity theories [11–13].

Further, let us now briefly review the GBT which connects the topologically surfaces. First, using Euler characteristic χ and a Riemannian metric g , one can choose the subset oriented surface domain as (D, χ, g) to find the Gaussian curvature K . Then the Gauss-Bonnet theorem is defined as follow [14]

$$\iint_D K dS + \oint_{\partial D} \kappa dt + \sum_i \theta_i = 2\pi\chi(D) \quad (1)$$

where κ is the geodesic curvature for $\partial D : \{t\} \rightarrow D$ and θ_i is the exterior angle with i^{th} vertex. Following this approach, global symmetric lenses are considered to be Riemannian metric manifolds, which are geodesic spatial light rays. In optical geometry, we calculate the Gaussian optical curvature K to find the asymptotic bending angle which can be calculated as follows [14]:

$$\hat{\alpha} = - \int \int_{D_\infty} K dS. \quad (2)$$

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Note that this equation is an exact result for the deflection angle. In this equation, we integrate over an infinite region of the surface D_∞ which is bounded by the light ray. By assumption, one can use the above relation only for asymptotically Euclidian optical metrics. Therefore it will be interesting to see the form of the deflection angle in the case of non-asymptotically Euclidian metrics. This method has been applied in various papers for different types of spacetimes [15–43].

This paper is organized as follows: In section 2, we review some basic concepts about asymptotically flat hairy BH. In section 3, we compute the Gaussian optical curvature for deflection angle and calculate the deflection angle by using GBT for $\gamma = 1$ also find the deflection angle in a plasma medium. In section 4, we calculate the deflection angle for $\gamma = \sqrt{3}$ also find the deflection angle in a plasma medium. The last section comprises of concluding remarks and results obtained from graphical analysis.

II. ASYMPTOTICALLY FLAT BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

In this section, we briefly review the asymptotically flat hairy BH in EMD theory in the background of dilaton potential. We consider the following action of Einstein-Maxwell-dilaton theory [11, 12]:

$$I[g_{\mu\nu}, A_\mu, \phi] = \frac{1}{2k} \int_{\mathcal{M}} d^4x \sqrt{-g} [R - e^{\gamma\phi} F^2 - \frac{1}{2}(\partial\phi)^2 - V(\phi)], \quad (3)$$

where

$$F^2 = F_{\mu\nu} F^{\mu\nu}, (\partial\phi)^2 = \partial_\mu \phi \partial^\mu \phi, \quad (4)$$

$V(\phi)$ is the dilaton potential, and $c = G_N = 4\pi\epsilon_0 = 1$ by using the convention $k = 8\pi$. The equations of motion for gauge field, dilaton and metric are following:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^\phi + T_{\mu\nu}^{EM} \quad (5)$$

$$\partial_\mu (\sqrt{-g} e^{\gamma\phi} F^{\mu\nu}) = 0 \quad (6)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = \frac{dV(\phi)}{d\phi} + \gamma e^{\gamma\phi} F^2 \quad (7)$$

where the stress tensors of matter fields are defined as

$$T_{\mu\nu}^\phi \equiv \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} [\frac{1}{2}(\partial\phi)^2 + V(\phi)], \quad (8)$$

$$T_{\mu\nu}^{EM} \equiv 2e^{\gamma\phi} (F_{\mu\alpha} F_{\nu}^\alpha - \frac{1}{4} g_{\mu\nu} F^2). \quad (9)$$

For exact regular hairy BH solutions, for a general scalar potential there is a new method in a number of papers [13], by using a specific ansatz. For flat spacetime, we should apply the similar special ansatz for the metric and dilaton. For the sake of simplicity, we are going to focus on two particular cases for which the exponent coefficient of the dilaton coupling with the gauge field takes the values $\gamma = 1$ and $\gamma = \sqrt{3}$, but for more general solution there exist a literature [12].

III. WEAK DEFLECTION ANGLE OF CALCULATION OF PHOTON LENSING FOR $\gamma = 1$ BY GAUSS-BONNET THEOREM

For this solution we consider the following scalar field potential:

$$V(\phi) = 2\alpha(2\phi + \phi \cosh \phi - 3 \sinh \phi), \quad (10)$$

here α is an arbitrary parameter. For $\gamma = 1$ the static hairy BH metric and gauge field, yields that:

$$ds^2 = \Omega(x) \left[-f(x) dt^2 + \frac{\eta^2 dx^2}{x^2 f(x)} + d\Sigma^2 \right] \quad (11)$$

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{qe^{-\phi}}{x} dt \wedge dx$$

where η and q are defined as independent parameters of the solution and correspond to the mass and charge of this BH. $d\Sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$ is spherical line element, and the coordinate x is restricted to be positive, $x \in [0, \infty)$. We can suppose that $\eta > 0$. One can use the conformal factor as follows:

$$\Omega(x) = \frac{x}{\eta^2(x-1)^2} \quad (12)$$

and then check that the equations of motion are satisfied for the following spacetime metric function:

$$f(x) = \alpha \left[\frac{x^2-1}{2x} - \ln(x) \right] + \frac{\eta^2(x-1)^2}{x} \left[1 - \frac{2q^2(x-1)}{x} \right]. \quad (13)$$

It is appropriate to first find the black hole optical metric by imposing the null condition $ds^2 = 0$, and solving the space-time metric for dt and also set the metric into equatorial plane with $\theta = \frac{\pi}{2}$, which yield as:

$$dt^2 = \frac{\eta^2}{x^2 f(x)^2} dx^2 + \frac{1}{f(x)} d\phi^2. \quad (14)$$

The optical geometry is in two dimensions, and is obtained for thermodynamically stable asymptotically flat hairy BH with a dilaton potential as follows [14]. By using Gauss-Bonnet theorem, initially we find the Gaussian curvature \mathcal{K} of the optical spacetime, as

$$\mathcal{K} = \frac{R_{\text{icciScalar}}}{2} \quad (15)$$

$$\begin{aligned} \mathcal{K} \approx & -\eta^2 x - \frac{\eta^2}{x} + 1/4 \frac{\eta^2}{x^2} + 3/2 \eta^2 + 1/4 x^2 \eta^2 - 1/2 x \alpha \ln(x) - 1/2 \frac{\alpha \ln(x)}{x} + 1/4 \alpha x^2 - 1/4 \frac{\alpha}{x^2} - x^2 \eta^2 q^2 \\ & + 5 \eta^2 q^2 x - 11 \frac{\eta^2 q^2}{x^2} + 16 \frac{\eta^2 q^2}{x} + 3 \frac{\eta^2 q^2}{x^3} - 12 \eta^2 q^2 + 5 \frac{\alpha q^2}{x} - 3/2 \frac{\alpha q^2}{x^3} \\ & - 4 \frac{\alpha q^2 \ln(x)}{x^2} + 3 \frac{\alpha q^2 \ln(x)}{x} + x \alpha \ln(x) q^2 - 3 \alpha q^2 - 1/2 x^2 \alpha q^2 + 1/2 x \alpha q^2 - 1/2 \frac{\alpha q^2}{x^2} \end{aligned} \quad (16)$$

For multiple images, we use the global theory (Gauss-Bonnet theorem) to relate with the local feature of the space-time such that Gaussian optical curvature.

The above equation will be apply to calculate the deflection angle by taking a non singular domain S_R outside of the light ray (along with boundaries $\partial S_R = \gamma_g \cup C_R$) with Euler characteristic $\chi(S_R)$, Gaussian curvature \mathcal{K} , geodesic curvature k and exterior jump angles $\alpha_i = (\alpha_O, \alpha_S)$ at vertices.

$$\int \int_{S_R} \mathcal{K} dS + \oint_{\partial S_R} k dt + \sum_j \alpha_j = 2\pi \chi(S_R), \quad (17)$$

at weak limit approximation ($\rho \rightarrow \infty$), $\alpha_O + \alpha_S \rightarrow \pi$. Then GBT becomes

$$\int \int_{S_R} \mathcal{K} dS + \oint_{C_r} k dt \stackrel{\rho \rightarrow \infty}{=} \int \int_{S_\infty} k dS + \int_0^{\pi+\Theta} d\varphi = \pi. \quad (18)$$

Now, by geodesic property, the geodesic curvature vanishes $k(\gamma_g) = 0$, and we get

$$k(C_R) = |\nabla_{\dot{C}_r} \dot{C}_r|, \quad (19)$$

with $C_r := \rho(\varphi) = r = \text{constant}$. Then GBT reduces

$$\lim_{R \rightarrow \infty} \int_0^{\pi+\Theta} \left[k_g \frac{d\sigma}{d\varphi} \right] |_{C_R} d\varphi = \pi - \lim_{R \rightarrow \infty} \int \int_{S_R} \mathcal{K} dS. \quad (20)$$

Now, for radial distance

$$k(C_R) dt = d\varphi. \quad (21)$$

Therefore,

$$\lim_{R \rightarrow \infty} k_g \frac{d\sigma}{d\varphi} \Big|_{C_R} = 1. \quad (22)$$

In the weak field regions, the light ray follows a straight line approximation, so that we can use the condition of $r = b / \sin \varphi$ at zero order.

$$\Theta = - \lim_{R \rightarrow \infty} \int_0^\pi \int_{b/\sin \varphi}^R \mathcal{K} dS, \quad (23)$$

where

$$KdS \approx 1/4 \frac{\eta^2}{x^2} + 1/4 \frac{\alpha}{x^2} - \frac{\eta^2 q^2}{x^2} - 1/2 \frac{\alpha q^2}{x^2}. \quad (24)$$

After simplification, we find the deflection angle of photon for asymptotically flat hairy BH in leading order terms as

$$\tilde{\alpha} \simeq -1/2 \frac{\eta^2}{b} - 1/2 \frac{\alpha}{b} + 2 \frac{\eta^2 q^2}{b} + \frac{\alpha q^2}{b} \quad (25)$$

Therefore, we can say that GBT provides a globally as well as topologically effect, this method is very useful for quantitative tool and can be apply in any asymptotically flat metrics.

A. Photon lensing in a plasma medium

In this section, we analyze the effect of plasma medium on the photon lensing by asymptotically hairy black hole. The refractive index for hairy black hole is as follows [15],

$$n(x) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2} \left(\frac{xf(x)}{\eta^2(x-1)^2} \right)}, \quad (26)$$

then, the corresponding optical metric yields that

$$d\tilde{\sigma}^2 = g_{jk}^{opt} dx^j dx^k = \frac{n^2(x)}{f(x)} \left(\frac{\eta^2}{x^2 f(x)} dx^2 + d\varphi^2 \right). \quad (27)$$

The determinant of above optical metric is:

$$\det g_{x\varphi}^{opt} = \frac{xf(x)\omega_e^4 - 2\eta^2\omega_\infty^2\omega_e^2(x-1)^2}{\eta^2\omega_\infty^4 xf(x)^2(x-1)^4} \quad (28)$$

where the metric function is given as

$$f(x) = \alpha \left[\frac{x^2-1}{2x} - \ln(x) \right] + \frac{\eta^2(x-1)^2}{x} \left[1 - \frac{2q^2(x-1)}{x} \right] \quad (29)$$

Now, we have

$$\frac{d\tilde{\sigma}}{d\varphi} = n(x) \left(\frac{\alpha^2 x^2}{f(x)} \right)^{1/2}, \quad (30)$$

hence we get differently which goes to α :

$$\lim_{x \rightarrow \infty} k_g \frac{d\tilde{\sigma}}{d\varphi} \Big|_{C_R} = 1. \quad (31)$$

We use straight line approximation $r = b / \sin \varphi$, for the limit $x \rightarrow \infty$, then GBT stated as

$$\lim_{x \rightarrow \infty} \int_0^{\pi+\Theta} \left[k_g \frac{d\tilde{\sigma}}{d\varphi} \right] |_{C_R} d\varphi = \pi - \lim_{x \rightarrow \infty} \int_0^\pi \int_{b/\sin \varphi}^x \mathcal{K} dS, \quad (32)$$

where

$$\mathcal{K} dS = -3/2 \frac{\omega_e^2 \eta^2 q^2}{x^2 \omega_\infty^2} - 3/2 \frac{\omega_e^2 \alpha q^2}{x^2 \omega_\infty^2} + 1/4 \frac{\omega_e^2 \eta^2}{x^2 \omega_\infty^2} + 3/8 \frac{\omega_\infty^2 \alpha}{x^2 \omega_\infty^2} + 1/4 \frac{\alpha}{x^2} + 1/4 \frac{\eta^2}{x^2} - 1/2 \frac{\alpha q^2}{x^2} - \frac{\eta^2 q^2}{x^2}. \quad (33)$$

Note that we only consider the first order terms. After simplification, we obtain the deflection angle in weak field limits as follows:

$$\tilde{\alpha} \simeq 3 \frac{\omega_e^2 \eta^2 q^2}{b \omega_\infty^2} + 2 \frac{\eta^2 q^2}{b} + 3 \frac{\omega_e^2 \alpha q^2}{b \omega_\infty^2} + \frac{\alpha q^2}{b} - 1/2 \frac{\omega_e^2 \eta^2}{b \omega_\infty^2} - 1/2 \frac{\eta^2}{b} - 3/4 \frac{\omega_e^2 \alpha}{b \omega_\infty^2} - 1/2 \frac{\alpha}{b} \quad (34)$$

The above results shows that the photon rays are moving in a medium of homogeneous plasma.

IV. NEW ASYMPTOTICALLY FLAT BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

In this section, we analyzed the exact asymptotically flat charged hairy BH's in the background of dilaton potential. There exist a literature related to this BH recently discussed by Astefanesei et al. [12]. We are interested in a action of Einstein-Maxwell-dilaton theory ($\kappa = 8\pi G_N$):

$$I[g_{\mu\nu}, A_\mu, \phi] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} e^{\gamma\phi} F^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \quad (35)$$

where the gauge coupling and potential are functions of the dilaton. The corresponding equations of motion are

$$\nabla_\mu (e^{\gamma\phi} F^{\mu\nu}) = 0 \quad (36)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - \frac{\partial V}{\partial \phi} - \frac{1}{4} \gamma e^{\gamma\phi} F^2 = 0 \quad (37)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} [T_{\mu\nu}^\phi + T_{\mu\nu}^{EM}] \quad (38)$$

where $T_{\mu\nu}^\phi$ and $T_{\mu\nu}^{EM}$ are the stress tensors of the matter fields and represented as follows

$$T_{\mu\nu}^\phi = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} (\partial\phi)^2 + V(\phi) \right], \quad T_{\mu\nu}^{EM} = e^{\gamma\phi} \left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F^2 \right) \quad (39)$$

Now, we consider only asymptotically flat solutions. To implement this condition, we require

$$\lim_{x \rightarrow 1} \Omega(x) f(x) = 1 \quad (40)$$

on this account, we fix f_0 to obtain [12]

$$\begin{aligned} f(x) = & \frac{\eta^2}{\nu} \left(x^2 + \frac{2x^{2-\nu}}{\nu-2} - \frac{\nu}{\nu-2} \right) + f_1 \left(\frac{x^{\nu+2}}{\nu+2} - x^2 + \frac{x^{2-\nu}}{2-\nu} + \frac{\nu^2}{\nu^2-4} \right) \\ & + \frac{Q^2 \eta^2}{(1-p)\nu^2} \left(\frac{x^{3-p+\nu}}{3-p+\nu} + \frac{x^{3-p-\nu}}{3-p-\nu} - 2 \frac{x^{3-p}}{3-p} - \frac{2\nu^2}{(3-p)(3-p+\nu)(3-p-\nu)} \right) \\ & + \frac{P^2 \eta^4}{(1+p)\nu^2} \left(\frac{x^{3+p+\nu}}{3+p+\nu} + \frac{x^{3+p-\nu}}{3+p-\nu} - 2 \frac{x^{3+p}}{3+p} - \frac{2\nu^2}{(3+p)(3+p+\nu)(3+p-\nu)} \right) \end{aligned} \quad (41)$$

A. Solutions with a non-trivial dilaton potential ($\gamma = 1$)

For this solution we consider the following scalar field potential [12]:

$$\Omega(x) = \frac{x}{\eta^2 (x-1)^2} \quad , \quad \phi(x) = \ln(x) \quad (42)$$

$$ds^2 = \Omega(x) \left[-f(x)dt^2 + \frac{\eta^2 dx^2}{x^2 f(x)} + d\theta^2 + \sin^2 \theta d\varphi^2 \right] \quad (43)$$

Here we study only the $\gamma = 1$ case, which is smoothly connected with a solution of $\mathcal{N} = 4$ supergravity which gives the solutions of metric function as follows:

$$A = \frac{Q}{x} dt + P \cos \theta d\varphi \quad (44)$$

$$V(\phi) = \alpha [2\phi + \phi \cosh(\phi) - 3 \sinh(\phi)] \quad (45)$$

$$f(x) = \frac{\eta^2 (x-1)^2}{x} + \left[\frac{x}{4} - \frac{1}{4x} - \frac{1}{2} \ln(x) \right] \alpha + \frac{\eta^2 (x-1)^3}{2x} (\eta^2 P^2 - x^{-1} Q^2) \quad (46)$$

Note that the dilaton field is vanishing at the boundary, $x = 1$ and also the dilaton potential is vanishing when $\alpha = 0$. At the boundary $x = 1$, one can find a asymptotically flat spacetime. After make a change of coordinates using:

$$\Omega(x) = r^2 + O(r^{-4}), \quad (47)$$

which is given for, the $x < 1$ black holes, by

$$x = 1 - \frac{1}{\eta r} + \frac{1}{2\eta^2 r^2} - \frac{1}{8\eta^3 r^3} + \frac{1}{2^7 \eta^5 r^5} \quad (48)$$

The spacetime metric becomes asymptotically flat [12]

$$g_{tt} = \Omega(x)f(x) = 1 - \frac{\alpha + 6\eta^2 (\eta^2 P^2 - Q^2)}{12\eta^3 r} + O(r^{-2}) \quad (49)$$

$$g_{rr}^{-1} = \frac{x^2 f(x)}{\eta^2 \Omega(x)} \left(\frac{dr}{dx} \right)^2 = 1 - \frac{\alpha + 6\eta^2 (\eta^2 P^2 - Q^2)}{12\eta^3 r} + O(r^{-2}). \quad (50)$$

It is noted that the scalar field potential is regular everywhere, except at the spacetime singularities.

V. DEFLECTION ANGLE OF PHOTONS BY ASYMPTOTICALLY FLAT HAIRY BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

Initially we find the Gaussian curvature \mathcal{K} of the optical spacetime, as

$$\mathcal{K} = \frac{R_{\text{icciScalar}}}{2}, \quad (51)$$

and

$$\mathcal{K} = \frac{(-6\eta^4 p^2 + 6Q^2 \eta^2 - \alpha)(-6\eta^4 p^2 + 6Q^2 \eta^2 + 16\eta^3 r - \alpha)}{192\eta^6 r^4}. \quad (52)$$

In weak field limits,

$$\mathcal{K} = -\frac{\alpha}{12\eta^3 r^3} + \frac{\alpha^2}{192\eta^6 r^4} + \frac{(-8\eta^3 r + \alpha) p^2}{16\eta^2 r^4} - \frac{(-8\eta^3 r + \alpha) Q^2}{16\eta^4 r^4} + \mathcal{O}(Q^3, p^3). \quad (53)$$

For multiple images, we use the global theory (Gauss-Bonnet theorem) to relate with the local feature of the space-time such that Gaussian optical curvature.

In the weak field regions, the light ray follows a straight line approximation, so that we can use the condition of $r = b/\sin\phi$ at zero order.

$$\tilde{\alpha} = -\lim_{R \rightarrow \infty} \int_0^\pi \int_{b/\sin\phi}^R \mathcal{K} dS. \quad (54)$$

Now by using Eq. (18), the deflection angle of photon by exact asymptotically flat charged hairy black hole with a dilaton potential in weak field limit is found as:

$$\tilde{\alpha} = \frac{3Q^2 p^2 \pi}{32b^2} + \frac{\eta p^2}{b} - \frac{Q^2}{b\eta} + \frac{\pi Q^2 \alpha}{64b^2 \eta^4} + \frac{\alpha}{6b\eta^3} + \mathcal{O}(Q^3, p^3). \quad (55)$$

A. Deflection angle of photons in plasma medium by asymptotically flat hairy black holes in Einstein-Maxwell-dilaton theory

In this section, we analyze the effect of plasma medium on the photon lensing by asymptotically hairy black hole. The refractive index for hairy black hole is as follows [15],

$$n(x) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2} \left(\frac{xf(x)}{\eta^2(x-1)^2} \right)}, \quad (56)$$

then, the corresponding optical metric yields that

$$d\tilde{\sigma}^2 = g_{jk}^{opt} dx^j dx^k = \frac{n^2(x)}{f(x)} \left(\frac{\eta^2}{x^2 f(x)} dx^2 + d\varphi^2 \right). \quad (57)$$

The determinant of above optical metric is:

$$\det g_{x\varphi}^{opt} = \frac{xf(x)\omega_e^4 - 2\eta^2\omega_\infty^2\omega_e^2(x-1)^2}{\eta^2\omega_\infty^4 xf(x)^2(x-1)^4}. \quad (58)$$

Now, by using Eq. 17 and 29, in weak field limit the Gaussian optical curvature stated as follows:

$$\begin{aligned} \mathcal{K} = & -\frac{\omega_\infty^2(\omega_e^2 - 2\omega_\infty^2)Q^2}{4r^3\eta(\omega_e^2 - \omega_\infty^2)^2} + \frac{\omega_\infty^2(3Q^2\omega_e^2\omega_\infty^2 + 3Q^2\omega_\infty^4 + 2\eta r\omega_e^4 - 6\eta r\omega_e^2\omega_\infty^2 + 4\eta r\omega_\infty^4)p^2}{8r^4(\omega_e^2 - \omega_\infty^2)^3} \\ & + \frac{\alpha(\Psi)\omega_\infty^2}{48\eta^4 r^5(\omega_e^2 - \omega_\infty^2)^4}, \end{aligned} \quad (59)$$

where

$$\begin{aligned} \Psi = & 9Q^2\eta p^2\omega_e^4\omega_\infty^2 + 9Q^2\eta p^2\omega_e^2\omega_\infty^4 - 3\eta^2 p^2 r\omega_e^4\omega_\infty^2 + 3\eta^2 p^2 r\omega_\infty^6 + 3Q^2 r\omega_e^4\omega_\infty^2 - 3Q^2 r\omega_\infty^6 + 2\eta r^2\omega_e^6 \\ & - 8\eta r^2\omega_e^4\omega_\infty^2 + 10\eta r^2\omega_e^2\omega_\infty^4 - 4\eta r^2\omega_\infty^6 \end{aligned} \quad (60)$$

Now, we have

$$\frac{d\tilde{\sigma}}{d\varphi} = n(x) \left(\frac{\alpha^2 x^2}{f(x)} \right)^{1/2}, \quad (61)$$

hence we get differently which goes to α :

$$\lim_{x \rightarrow \infty} k_g \frac{d\tilde{\sigma}}{d\varphi} \Big|_{C_R} = \alpha. \quad (62)$$

We use straight line approximation $r = b / \sin \varphi$, for the limit $x \rightarrow \infty$, then GBT stated as

$$\lim_{x \rightarrow \infty} \int_0^{\pi + \tilde{\alpha}} \left[k_g \frac{d\tilde{\sigma}}{d\varphi} \right] |_{C_R} d\varphi = \pi - \lim_{x \rightarrow \infty} \int_0^\pi \int_{b/\sin \varphi}^x \mathcal{K} dS. \quad (63)$$

After simplification, we obtain

$$\begin{aligned} \tilde{\alpha} = & \frac{21 \pi Q^2 p^2 \omega_e^2}{64 \omega_\infty^2 b^2} + \frac{3 Q^2 p^2 \pi}{32 b^2} + \frac{\eta p^2 \omega_e^2}{\omega_\infty^2 b} + \frac{\eta p^2}{b} - \frac{7 \pi \alpha p^2 \omega_e^2}{128 \omega_\infty^2 b^2 \eta^2} - \frac{\pi \alpha p^2}{64 b^2 \eta^2} - \frac{Q^2 \omega_e^2}{\omega_\infty^2 b \eta} \\ & - \frac{Q^2}{b \eta} + \frac{7 \pi Q^2 \alpha \omega_e^2}{128 \omega_\infty^2 \eta^4 b^2} + \frac{\pi Q^2 \alpha}{64 b^2 \eta^4} + 1/6 \frac{\alpha \omega_e^2}{\omega_\infty^2 \eta^3 b} + 1/6 \frac{\alpha}{b \eta^3} \end{aligned} \quad (64)$$

The proposed deflection angle shows that the photon rays are moving in a medium of homogeneous plasma.

VI. CONCLUSION

In this paper, we obtain the deflection angle of photon to the spherically symmetric and asymptotically flat space-time of hairy BH with Einstein-Maxwell-dilaton system in weak field limit. To this end, we set the photon rays on the equatorial plane in the black hole spacetime. For this purpose, we have used the GBT and obtain the deflection angle of photon by integrating a domain outside the impact parameter. Moreover, we also found the deflection angle of photon by asymptotically flat hairy BH in plasma medium. We examined that the proposed deflection angle shows that gravitational lensing can be affected from the hair of the black hole and it is a global effect as well as a valuable tool to study the nature of singularities of black holes.

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