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On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation

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Abstract. The general quintic functional equation and the general sextic functional equation are generalizations of many functional equations such as the additive function equation and the quadratic function equation. In this paper, we investigate Hyers-Ulam-Rassias stability of the general quintic functional equation and the general sextic functional equation.

AMS Subject Classification: 39B82; 39B52.

Key Words: stability of a functional equation; general quintic functional equation; a general quintic mapping; general sextic functional equation; a general sextic mapping.

1 Introduction

Let X be a real normed space and Y be a real Banach space. In 1940, Ulam [15] raised the question about the stability of group homomorphisms, and in the following year Hyers [5] solved this question about the additive functional equation, which gives a partial answer to Ulam's question. In 1978, Rassias [14] generalized Hyers' result (refer to [2, 3, 6, 8, 13] for a more generalized result). Since then, many mathematicians investigated the stability of different types of functional equations [4, 16]. Rassias [14] investigated the stability problem for approximately linear mappings controlled by the unbounded function $\theta(\|x\|^p + \|y\|^p)$ as follow:

Theorem 1.1 *Let $f : X \to Y$ be a mapping from a real normed vector space X into a Banach space Y satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in X\backslash\{0\}$, where θ and p are constants with $\theta > 0$ and $p < 1$. If $f(tx)$ is continuous in t for each fixed x, then there exists a unique linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta\,\|x\|^p}{|2 - 2^p|},$$

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for all $x \in X\backslash\{0\}$.

The functional equation is said to have Hyers-Ulam Rassis stablity when the stability can be proved under the control function $\theta(\|x\|^p + \|y\|^p)$.

A mapping $f : X \to Y$ is called a general quintic mapping if f satisfies the functional equation

$$\sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x + (i - 3y)) = 0 \tag{1.1}$$

which is called a general quintic functional equation. A mapping $f : X \to Y$ is called a general sextic mapping

$$\sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x + iy) = 0 \tag{1.2}$$

which is called a general sextic functional equation. For example, the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{i=0}^{5} a_i x^i$ and $g(x) = \sum_{i=0}^{6} a_i x^i$, $a_i \in \mathbb{R}$, satisfy the above functional equations. More detailed term for the concepts of "a general quintic mapping" and "a general sextic mapping" can be found in Baker's paper [1] by the terms "generalized polynomial mapping of degree at most 5" and "generalized polynomial mapping of degree at most 6", respectively. Kim etc. [7] has previously studied the stability of a general a general cubic functional equation and Lee [9, 10, 12] has studied the stability of a general quadratic functional equation, a general cubic functional equation, and a general quartic functional equation.

In section 2, we will investigate the Hyers-Ulam-Rassias stability of the general quintic functional equation. And in section 3, we will investigate the Hyers-Ulam-Rassias stability of the general sextic functional equation.

2 Stability of a general quintic functional equation

Throughout this section, for a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}, \\ Df(x, y) &:= \sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x + (i - 3y)), \\ \Gamma f(x) &:= Df_o(2x, 2x) + 6Df_o(3x, x) + 36Df_o(2x, x) + 70Df_o(x, x), \\ \Delta f(x) &:= Df_e(x, x) + 3Df_e(0, x) \end{aligned}$$

for all $x, y \in X$. By laborious computation we can get the equalities

$$\begin{aligned} \Gamma f(x) &= f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \\ \Delta f(x) &= f_e(4x) - 20f_e(2x) + 64f_e(x) \end{aligned} \tag{2.1}$$

for all $x \in X$.

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Lemma 2.1 *Let p be a fixed nonnegative real number such that $p \notin \{1,2,3,4,5\}$. For a given mapping $f : X \to Y$ with $f(0) = 0$, let $J_n f : X \to Y$ be the mappings defined by*

$$
J_n f(x) := \begin{cases}
-\frac{4^n}{3}\left(f_e\left(\frac{x}{2^n}\right) - 16 f_e\left(\frac{x}{2^{n+1}}\right)\right) + \frac{16^{n+1}}{12}\left(f_e\left(\frac{x}{2^n}\right) - 4 f_e\left(\frac{x}{2^{n+1}}\right)\right) \\
+\frac{2^n - 20\times 8^n + 64\times 32^n}{45} f_o\left(\frac{x}{2^n}\right) - \frac{40\times 2^n - 680\times 8^n + 640\times 32^n}{45} f_o\left(\frac{x}{2^{n+1}}\right) \\
+\frac{256\times 2^n - 1280\times 8^n + 1024\times 32^n}{45} f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 5 < p, \\
\\
-\frac{4^n}{3}\left(f_e\left(\frac{x}{2^n}\right) - 16 f_e\left(\frac{x}{2^{n+1}}\right)\right) + \frac{16^{n+1}}{12}\left(f_e\left(\frac{x}{2^n}\right) - 4 f_e\left(\frac{x}{2^{n+1}}\right)\right) \\
+\frac{2^n - 5\times 8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n - 170\times 8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n - 320\times 8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) \\
+\frac{4}{90\times 32^n}\left(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\right) & \text{if } 4 < p < 5, \\
\\
\frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} \\
+\frac{2^n - 5\times 8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n - 170\times 8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n - 320\times 8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) \\
+\frac{4}{90\times 32^n}\left(f_o(2^{n+1}x) - 10 f_o(2^n x) + 16 f_o(2^{n-1}x)\right) & \text{if } 3 < p < 4, \\
\\
\frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} \\
+\frac{4 f_o(2^{n+1}x)}{90\times 32^n} - \frac{40 f_o(2^n x)}{90\times 32^n} + \frac{64 f_o(2^{n-1}x)}{90\times 32^n} - \frac{5 f_o(2^{n+1}x)}{90\times 8^n} + \frac{170 f_o(2^n x)}{90\times 8^n} - \frac{320 f_o(2^{n-1}x)}{90\times 8^n} \\
+\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 2 < p < 3, \\
\\
\frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 4^n} - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} \\
+\frac{4 f_o(2^{n+1}x)}{90\times 32^n} - \frac{40 f_o(2^n x)}{90\times 32^n} + \frac{64 f_o(2^{n-1}x)}{90\times 32^n} - \frac{5 f_o(2^{n+1}x)}{90\times 8^n} + \frac{170 f_o(2^n x)}{90\times 8^n} - \frac{320 f_o(2^{n-1}x)}{90\times 8^n} \\
+\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times 2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times 2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 1 < p < 2, \\
\\
\frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 4^n} - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12\times 16^n} \\
+\frac{f_o(2^{n+2}x)}{720\times 32^n} - \frac{10 f_o(2^{n+1}x)}{720\times 32^n} + \frac{16 f_o(2^n x)}{720\times 32^n} - \frac{5 f_o(2^{n+2}x)}{720\times 8^n} + \frac{170 f_o(2^{n+1}x)}{720\times 8^n} - \frac{320 f_o(2^n x)}{720\times 8^n} \\
+\frac{f_o(2^{n+2}x) - 40 f_o(2^{n+1}x) + 256 f_o(2^n x)}{180\times 2^n} & \text{if } 0 \le p < 1
\end{cases}
$$

for all $x \in X$ and all nonnegative integers n. Then
$J_n f(x) - J_{n+1} f(x) =$

$$
\begin{cases}
\left(\frac{4^{2n+1}}{3} - \frac{4^n}{3}\right)\Delta f\left(\frac{x}{2^{n+2}}\right) + \left(\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45}\right)\Gamma f\left(\frac{x}{2^{n+3}}\right) & \text{if } 5 < p, \\
\left(\frac{4^{2n+1}}{3} - \frac{4^n}{3}\right)\Delta f\left(\frac{x}{2^{n+2}}\right) + \left(\frac{2^n}{90} - \frac{8^n}{18}\right)\Gamma f\left(\frac{x}{2^{n+2}}\right) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 4 < p < 5, \\
-\frac{4^n}{12}\Delta f\left(\frac{x}{2^{n+1}}\right) - \frac{\Delta f(2^n x)}{192\times 16^n} + \left(\frac{2^n}{90} - \frac{8^n}{18}\right)\Gamma f\left(\frac{x}{2^{n+2}}\right) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 3 < p < 4, \\
-\frac{4^n}{12}\Delta f\left(\frac{x}{2^{n+1}}\right) - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{2^n}{90}\Gamma f\left(\frac{x}{2^{n+2}}\right) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 2 < p < 3, \\
\frac{\Delta f(2^n x)}{48\times 4^n} - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{2^n}{90}\Gamma f\left(\frac{x}{2^{n+2}}\right) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \text{if } 1 < p < 2, \\
\frac{\Delta f(2^n x)}{48\times 4^n} - \frac{\Delta f(2^n x)}{192\times 16^n} + \frac{\Gamma f(2^n x)}{180\times 2^{n+1}} + \frac{\Gamma f(2^n x)}{144\times 8^{n+1}} - \frac{\Gamma f(2^n x)}{720\times 32^{n+1}} & \text{if } 0 \le p < 1
\end{cases} \quad (2.2)
$$

for all $x \in X$ and all nonnegative integers n.

Proof. For the case $2 < p < 3$, from the definition of $J_n f$ and the equalities (2.1), we

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obtain that

$$
\begin{aligned}
&J_nf(x) - J_{n+1}f(x)\\
&=\frac{4^n}{12}\left(16f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{x}{2^{n-1}}\right)\right) - \frac{4^{n+1}}{12}\left(16f_e\left(\frac{x}{2^{n+1}}\right) - f_e\left(\frac{x}{2^n}\right)\right)\\
&\quad - \frac{4f_e(2^nx) - f_e(2^{n+1}x)}{12\times 16^n} + \frac{4f_e(2^{n+1}x) - f_e(2^{n+2}x)}{12\times 16^{n+1}}\\
&\quad + \frac{4f_o(2^{n+1}x)}{90\times 32^n} - \frac{40f_o(2^nx)}{90\times 32^n} + \frac{64f_o(2^{n-1}x)}{90\times 32^n}\\
&\quad - \frac{4f_o(2^{n+2}x)}{90\times 32^{n+1}} + \frac{40f_o(2^{n+1}x)}{90\times 32^{n+1}} - \frac{64f_o(2^nx)}{90\times 32^{n+1}}\\
&\quad - \frac{5f_o(2^{n+1}x)}{90\times 8^n} + \frac{170f_o(2^nx)}{90\times 8^n} - \frac{320f_o(2^{n-1}x)}{90\times 8^n}\\
&\quad + \frac{5f_o(2^{n+2}x)}{90\times 8^{n+1}} - \frac{170f_o(2^{n+1}x)}{90\times 8^{n+1}} + \frac{320f_o(2^nx)}{90\times 8^{n+1}}\\
&\quad + \frac{2^n}{90}\left(f_o(\frac{x}{2^{n-1}}) - 40f_o(\frac{x}{2^n}) + 256f_o(\frac{x}{2^{n+1}})\right)\\
&\quad - \frac{2^{n+1}}{90}\left(f_o(\frac{x}{2^n}) - 40f_o(\frac{x}{2^{n+1}}) + 256f_o(\frac{x}{2^{n+2}})\right)\\
&=\frac{4^n}{12}\left(-f_e\left(\frac{x}{2^{n-1}}\right) + 20f_e\left(\frac{x}{2^n}\right) - 64f_e\left(\frac{x}{2^{n+1}}\right)\right)\\
&\quad - \frac{f_e(2^{n+2}x) - 20f_e(2^{n+1}x) + 64f_e(2^nx)}{12\times 16^{n+1}}\\
&\quad - \frac{4f_o(2^{n+2}x) - 168f_o(2^{n+1}x) + 1344f_o(2^nx) - 2048f_o(2^{n-1}x)}{90\times 32^{n+1}}\\
&\quad + \frac{5f_o(2^{n+2}x) - 210f_o(2^{n+1}x) + 1680f_o(2^nx) - 2560f_o(2^{n-1}x)}{90\times 8^{n+1}}\\
&\quad + \frac{2^n}{90}\left(f_o(\frac{x}{2^{n-1}}) - 42f_o(\frac{x}{2^n}) + 336f_o(\frac{x}{2^{n+1}}) - 512f_o(\frac{x}{2^{n+2}})\right)\\
&= -\frac{4^n}{12}\Delta f(\frac{x}{2^{n+1}}) - \frac{\Delta f(2^nx)}{192\times 16^n} + \frac{2^n}{90}\Gamma f(\frac{x}{2^{n+2}}) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}}
\end{aligned}
$$

for all $x \in X$ and all nonnegative integers n. Also we easily show that the equality (2.2) holds by the similar method for the other cases, either $0 \le p < 1$ or $1 < p < 2$ or $3 < p < 4$ or $4 < p < 5$ or $5 < p$. □

Lemma 2.2 *If $f : X \to Y$ is a mapping such that*

$$Df(x, y) = 0$$

for all $x, y \in X$ with $f(0) = 0$, then

$$J_nf(x) = f(x)$$

for all $x \in X$ and all positive integers n.

Proof. If $f : X \to Y$ is a mapping such that

$$Df(x,y) = 0$$

for all $x, y \in X$ with $f(0) = 0$, then it follows from the definitions of $\Delta f(x)$ and $\Gamma f(x)$ that $\Delta f(x) = 0$ and $\Gamma f(x) = 0$ for all $x \in X$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ and the equality (2.3), we conclude that

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n. □

From Lemma 2.2, we can prove the following stability theorem.

Theorem 2.3 *Let $p \neq 1, 2, 3, 4, 5$ be a fixed nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.3}$$

for all $x, y \in X$. Then there exists a general quintic mapping F such that $\|f(x) - f(0) - F(x)\| \le$

$$\begin{cases} \left(\frac{K}{45\times 2^{2p}(2^p-2)} + \frac{(128+44\times 2^p)K}{45(2^p-32)(2^p-8)2^{2p}} + \frac{5}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 5 < p, \\ \left(\frac{(2\times 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{5}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 4 < p < 5, \\ \left(\frac{(2\times 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{K}{90\times 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{K}{90\times 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{5}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 1 < p < 2, \\ \left(\frac{K}{180(2-2^p)} + \frac{(38-2^p)K}{180(8-2^p)(32-2^p)} + \frac{5}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 0 \le p < 1 \end{cases} \tag{2.4}$$

for all $x \in X$ and $F(0) = 0$, where $K = 182 + 38 \times 2^p + 6 \times 3^p$.

Proof. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x,y) = Df(x,y)$ and $\tilde{f}(0) = 0$. By (2.3) and the definitions of Γf and Δf, we have

$$\begin{aligned} \|\Gamma \tilde{f}(x)\| =& \big\|Df_o(2x,2x) + 6Df_o(3x,x) + 36Df_o(2x,x) + 70Df_o(x,x)\big\| \\ \le& \theta(182 + 38 \times 2^p + 6 \times 3^p)\|x\|^p, \\ \|\Delta \tilde{f}(x)\| =& \big\|Df_e(x,x) + 3Df_e(0,x)\big\| \le 5\theta\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (2.2) and (2.3) that

$\|J_n\tilde{f}(x)-J_{n+1}\tilde{f}(x)\|\leq$

$$\begin{cases} \left(\frac{2^nK}{45\times2^{(n+3)p}}-\frac{4\times8^nK}{9\times2^{(n+3)p}}+\frac{64\times32^nK}{45\times2^{(n+3)p}}+\frac{5(4^{2n+1}-4^n)}{3\times2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 5<p,\\ \left(\frac{(5\times8^n-2^n)K}{90\times2^{(n+2)p}}+\frac{2\times2^{(n-1)p}K)}{45\times32^{n+1}}+\frac{5(4^{2n+1}-4^n)}{3\times2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 4<p<5,\\ \left(\frac{(5\times8^n-2^n)K}{90\times2^{(n+2)p}}+\frac{2\times2^{(n-1)p}K)}{45\times32^{n+1}}+\frac{4^n}{12}\frac{5}{2^{(n+1)p}}+\frac{5\times2^{np}}{192\times16^n}\right)\theta\|x\|^p & \text{if } 3<p<4,\\ \left(\frac{2^nK}{90\times2^{(n+2)p}}+\frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K}{90\times32^{n+1}}+\frac{4^n}{12}\frac{5}{2^{(n+1)p}}+\frac{5\times2^{np}}{192\times16^n}\right)\theta\|x\|^p & \text{if } 2<p<3,\\ \left(\frac{2^nK}{90\times2^{(n+2)p}}+\frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K}{90\times32^{n+1}}+\frac{5(4^{n+1}-1)2^{np}}{192\times16^n}\right)\theta\|x\|^p & \text{if } 1<p<2,\\ \left(\frac{2^{np}K}{180\times2^{n+1}}+\frac{2^{np}K}{144\times8^{n+1}}-\frac{2^{np}K}{720\times32^{n+1}}+\frac{5(4^{n+1}-1)2^{np}}{192\times16^n}\right)\theta\|x\|^p & \text{if } 0\leq p<1 \end{cases}$$

for all $x\in X$. Together with the equality $J_n\tilde{f}(x)-J_{n+m}\tilde{f}(x)=\sum_{i=n}^{n+m-1}(J_i\tilde{f}(x)-J_{i+1}\tilde{f}(x))$ for all $x\in X$, we obtain that
$\|J_n\tilde{f}(x)-J_{n+m}\tilde{f}(x)\|\leq$

$$\begin{cases} \sum_{i=n}^{n+m-1}\left(\frac{(2^i-20\times8^i+64\times32^i)K}{45\times2^{(i+3)p}}+\frac{5(4^{2i+1}-4^i)}{3\times2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 5<p,\\ \sum_{i=n}^{n+m-1}\left(\frac{(5\times8^i-2^i)K}{90\times2^{(i+2)p}}+\frac{2\times2^{(i-1)p}K)}{45\times32^{i+1}}+\frac{5(4^{2i+1}-4^i)}{3\times2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 4<p<5,\\ \sum_{i=n}^{n+m-1}\left(\frac{(5\times8^i-2^i)K}{90\times2^{(i+2)p}}+\frac{2\times2^{(i-1)p}K)}{45\times32^{i+1}}+\frac{4^i}{12}\frac{5}{2^{(i+1)p}}+\frac{5\times2^{ip}}{192\times16^i}\right)\theta\|x\|^p & \text{if } 3<p<4,\\ \sum_{i=n}^{n+m-1}\left(\frac{32\times64^iK+(5\times4^{i+1}-4)\times2^{(2i+1)p}K}{90\times32^{i+1}\times2^{(i+2)p}}+\frac{4^i}{12}\frac{5}{2^{(i+1)p}}+\frac{5\times2^{ip}}{192\times16^i}\right)\theta\|x\|^p & \text{if } 2<p<3,\\ \sum_{i=n}^{n+m-1}\left(\frac{32\times64^iK+(5\times4^{i+1}-4)\times2^{(2i+1)p}K}{90\times32^{i+1}\times2^{(i+2)p}}+\frac{5(4^{i+1}-1)2^{ip}}{192\times16^i}\right)\theta\|x\|^p & \text{if } 1<p<2,\\ \sum_{i=n}^{n+m-1}\left(\frac{2^{ip}K}{180\times2^{i+1}}+\frac{2^{ip}K}{144\times8^{i+1}}-\frac{2^{ip}K}{720\times32^{i+1}}+\frac{5(4^{i+1}-1)2^{ip}}{192\times16^i}\right)\theta\|x\|^p & \text{if } 0\leq p<1 \end{cases} \tag{2.5}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. It follows from (2.5) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x\in X$. Hence we can define a mapping $F:X\to Y$ by

$$F(x):=\lim_{n\to\infty}J_n\tilde{f}(x)$$

for all $x\in X$. Note that $F(0)=0$ follows from $\tilde{f}(0)=0$. Moreover, letting $n=0$ and passing the limit $n\to\infty$ in (2.5) we get the inequality (2.4). For the case $2<p<3$, from the definition of F, we easily get

$$\begin{aligned}\|DF(x,y)\|=\lim_{n\to\infty}\Bigg\|&\frac{4^n}{12}\left(-Df_e\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+16Df_e\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right)\\ &+\frac{Df_e\left(2^{n+1}x,2^{n+1}y\right)-4Df_e\left(2^nx,2^ny\right)}{12\times16^n}\\ &+\frac{4Df_o(2^{n+1}x,2^{n+1}y)}{90\times32^n}-\frac{40Df_o(2^nx,2^ny)}{90\times32^n}+\frac{64Df_o(2^{n-1}x,2^{n-1}y)}{90\times32^n}\\ &-\frac{5[Df_o(2^{n+1}x,2^{n+1}y)-34Df_o(2^nx,2^ny)+64Df_o(2^{n-1}x,2^{n-1}y)]}{90\times8^n}\\ &+\frac{2^n}{90}\left[Df_o\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)-40Df_o\left(\frac{x}{2^n},\frac{y}{2^n}\right)+256Df_o\left(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}}\right)\right]\Bigg\|\end{aligned}$$

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$$
\begin{aligned}
&\leq \lim_{n\to\infty}\left(\frac{4^n(2^p+16)}{12\times 2^{np}}+\frac{2^{np}(2^p+4)}{12\times 16^n}+\frac{4(4^p+10\times 2^p+16)2^{(n-1)p}}{90\times 32^n}\right.\\
&\quad\left.+\frac{5(4^p+34\times 2^p+64)2^{(n-1)p}}{90\times 8^n}+\frac{2^n(4^p+40\times 2^p+256)}{90\times 2^{(n+1)p}}\right)\times\theta(\|x\|^p+\|y\|^p)\\
&=0
\end{aligned}
$$

for all $x, y \in X$. Also we easily show that $DF(x,y) = 0$ by the similar method for the other cases, either $0 \leq p < 1$ or $1 < p < 2$ or $3 < p < 4$ or $4 < p < 5$ or $5 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another general quintic mapping satisfying (2.4) and $F'(0) = 0$. By Lemma 2.2, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $2 < p < 3$, we have

$$
\begin{aligned}
\|J_n\tilde{f}(x)-F'(x)\| =&\|J_n\tilde{f}(x)-J_nF'(x)\|\\
\leq&\frac{4^{n+2}}{12}\left\|(\tilde{f}_e-F_e)\left(\frac{x}{2^n}\right)\right\|+\frac{4^n}{12}\left\|(\tilde{f}_e-F_e)\left(\frac{2x}{2^n}\right)\right\|+\frac{4\|(\tilde{f}_e-F_e)(2^nx)\|}{12\times 16^n}\\
&+\frac{\|(\tilde{f}_e-F_e)(2^{n+1}x)\|}{12\times 16^n}+\frac{4\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 32^n}+\frac{40\|(\tilde{f}_o-F_o)(2^nx)\|}{90\times 32^n}\\
&+\frac{64\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 32^n}+\frac{5\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 8^n}+\frac{170\|(\tilde{f}_o-F_o)(2^nx)\|}{90\times 8^n}\\
&+\frac{320\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 8^n}+\frac{2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{2x}{2^n}\right)\right\|\\
&+\frac{40\times 2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^n}\right)\right\|+\frac{256\times 2^n}{90}\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^{n+1}}\right)\right\|\\
\leq&\left(\frac{4^{n+2}+4^n2^p}{12\times 2^{np}}+\frac{4\times 2^{np}+2^{(n+1)p}}{12\times 16^n}+\frac{4(4^p+10\times 2^p+16)2^{(n-1)p}}{90\times 32^n}\right.\\
&\left.+\frac{5(4^p+34\times 2^p+64)2^{(n-1)p}}{90\times 8^n}+\frac{(4^p+40\times 2^p+256)2^n}{90\times 2^{(n+1)p}}\right)\\
&\times\left(\frac{K}{90\times 2^p(2^p-2)}+\frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p}+\frac{5}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p
\end{aligned}
$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we obtain the equality $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in X$, which means that $F(x) = F'(x)$ for all $x \in X$. Also we easily show that $F(x) = F'(x)$ by the similar method for the other cases, either $0 \leq p < 1$ or $1 < p < 2$ or $3 < p < 4$ or $4 < p < 5$ or $5 < p$. □

When n is a fixed number such that $n \in \{1, 2, 3, 4, 5\}$, if $f : \mathbb{R} \to \mathbb{R}$ is a solution of the functional equation $\sum_{i=0}^{n} {}_nC_i(-1)^i f(ix + y) - n!f(x) = 0$ for all $x, y \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is a solution of the functional equation $Df(x, y) = 0$ for all $x, y \in \mathbb{R}$.

So Example 1 in [11] shows that the assumption $p \neq 1, 2, 3, 4, 5$ cannot be omitted in Theorem 2.3.

Example 2.4 (Example 1 in [11]) *There is a mapping* $f : \mathbb{R} \to \mathbb{R}$

$$
\left|\sum_{i=0}^{n} {}_nC_i(-1)^i f(ix+y) - n!f(x)\right| \leq 4\times(n+1)!(n+1)^{2n}(|x|^n+|y|^n). \tag{2.6}
$$

for all $x, y \in \mathbb{R}$, *but there do not exist a mapping* $F : \mathbb{R} \to \mathbb{R}$ *and a constant* $d > 0$ *such that* $\sum_{i=0}^{n} {}_nC_i(-1)^i F(ix + y) - n!F(x) = 0$ *and* $|f(x) - F(x)| \le d|x|^n$ *for all* $x \in \mathbb{R}$.

3 Stability of a general sextic functional equation

Throughout this section, for a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned} Df(x,y) :=& \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy), \\ \Gamma(x) :=& Df_o(-6x, 2x) + 6Df_o(-x, x) + 42Df_o(-2x, x) + 112Df_o(-3x, x), \\ \Delta f(x) :=& Df_e(-6x, 2x) + 8Df_e(-x, x) + 56Df_e(-2x, x) + 112Df_e(-3x, x) \end{aligned}$$

for all $x, y \in X$. By laborious computation we can get the equalities

$$\Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \tag{3.1}$$
$$\Delta f(x) = f_e(8x) - 84f_e(4x) + 1344f_e(2x) - 4096f_e(x) \tag{3.2}$$

for all $x \in X$.

The proofs of the following two lemmas are very similar to the proofs of lemma 2.1 and lemma 2.2, so we omit them and just describe them.

Lemma 3.1 *Let* $p \neq 1, 2, 3, 4, 5, 6$ *be a fixed real number. For a given mapping* $f : X \to Y$

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with $f(0)=0$, *let* $J_nf : X \to Y$ *be the mappings defined by*

$$J_nf(x) := \begin{cases} -\frac{4^n-20\times16^n+64\times64^n}{45} f_e\left(\frac{x}{2^n}\right) - \frac{80\times4^n-1360\times16^n+1280\times64^n}{45} f_e\left(\frac{x}{2^{n+1}}\right) \\ +\frac{1024\times4^n-5120\times16^n+4096\times64^n}{45} f_e\left(\frac{x}{2^{n+2}}\right) \\ +\frac{2^n-20\times8^n+64\times32^n}{45} f_o\left(\frac{x}{2^n}\right) - \frac{40\times2^n-680\times8^n+640\times32^n}{45} f_o\left(\frac{x}{2^{n+1}}\right) \\ +\frac{256\times2^n-1280\times8^n+1024\times32^n}{45} f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 6<p, \\[2ex] \frac{4^n-5\times16^n}{180} f_e\left(\frac{x}{2^{n-1}}\right) - \frac{80\times4^n-340\times16^n}{180} f_e\left(\frac{x}{2^n}\right) + \frac{1024\times4^n-1280\times16^n}{180} f_e\left(\frac{x}{2^{n+1}}\right) \\ +\frac{4}{180\times64^n}\left(f_e(2^{n+1}x)-20f_e(2^nx)+64f_e(2^{n-1}x)\right) \\ +\frac{2^n-20\times8^n+64\times32^n}{45} f_o\left(\frac{x}{2^n}\right) - \frac{40\times2^n-680\times8^n+640\times32^n}{45} f_o\left(\frac{x}{2^{n+1}}\right) \\ +\frac{256\times2^n-1280\times8^n+1024\times32^n}{45} f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 5<p<6, \\[2ex] \frac{4^n-5\times16^n}{180} f_e\left(\frac{x}{2^{n-1}}\right) - \frac{80\times4^n-340\times16^n}{180} f_e\left(\frac{x}{2^n}\right) + \frac{1024\times4^n-1280\times16^n}{180} f_e\left(\frac{x}{2^{n+1}}\right) \\ +\frac{4}{180\times64^n}\left(f_e(2^{n+1}x)-20f_e(2^nx)+64f_e(2^{n-1}x)\right) \\ +\frac{2^n-5\times8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times2^n-170\times8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times2^n-320\times8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) \\ +\frac{4}{90\times32^n}\left(f_o(2^{n+1}x)-10f_o(2^nx)+16f_o(2^{n-1}x)\right) & \text{if } 4<p<5, \\[2ex] \frac{4f_e(2^{n+1}x)}{180\times64^n} - \frac{80f_e(2^nx)}{180\times64^n} + \frac{256f_e(2^{n-1}x)}{180\times64^n} - \frac{5f_e(2^{n+1}x)}{180\times16^n} + \frac{340f_e(2^nx)}{180\times16^n} - \frac{1280f_e(2^{n-1}x)}{180\times16^n} \\ +\frac{4^n}{180} f_e\left(\frac{x}{2^{n-1}}\right) - \frac{80\times4^n}{180} f_e\left(\frac{x}{2^n}\right) + \frac{1024\times4^n}{180} f_e\left(\frac{x}{2^{n+1}}\right) \\ +\frac{2^n-5\times8^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times2^n-170\times8^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times2^n-320\times8^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) \\ +\frac{4}{90\times32^n}\left(f_o(2^{n+1}x)-10f_o(2^nx)+16f_o(2^{n-1}x)\right) & \text{if } 3<p<4, \\[2ex] \frac{4f_e(2^{n+1}x)}{180\times64^n} - \frac{80f_e(2^nx)}{180\times64^n} + \frac{256f_e(2^{n-1}x)}{180\times64^n} - \frac{5f_e(2^{n+1}x)}{180\times16^n} + \frac{340f_e(2^nx)}{180\times16^n} - \frac{1280f_e(2^{n-1}x)}{180\times16^n} \\ +\frac{4^n}{180} f_e\left(\frac{x}{2^{n-1}}\right) - \frac{80\times4^n}{180} f_e\left(\frac{x}{2^n}\right) + \frac{1024\times4^n}{180} f_e\left(\frac{x}{2^{n+1}}\right) \\ +\frac{4f_o(2^{n+1}x)}{90\times32^n} - \frac{40f_o(2^nx)}{90\times32^n} + \frac{64f_o(2^{n-1}x)}{90\times32^n} - \frac{5f_o(2^{n+1}x)}{90\times8^n} + \frac{170f_o(2^nx)}{90\times8^n} - \frac{320f_o(2^{n-1}x)}{90\times8^n} \\ +\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 2<p<3, \\[2ex] \frac{f_e(2^{n+2}x)}{2880\times64^n} - \frac{20f_e(2^{n+1}x)}{2880\times64^n} + \frac{64f_e(2^nx)}{2880\times64^n} - \frac{5f_e(2^{n+2}x)}{2880\times16^n} + \frac{340f_e(2^{n+1}x)}{2880\times16^n} - \frac{1280f_e(2^nx)}{2880\times16^n} \\ +\frac{4f_e(2^{n+2}x)-320f_e(2^{n+1}x)+4096f_e(2^nx)}{2880\times4^n} \\ +\frac{4f_o(2^{n+1}x)}{90\times32^n} - \frac{40f_o(2^nx)}{90\times32^n} + \frac{64f_o(2^{n-1}x)}{90\times32^n} - \frac{5f_o(2^{n+1}x)}{90\times8^n} + \frac{170f_o(2^nx)}{90\times8^n} - \frac{320f_o(2^{n-1}x)}{90\times8^n} \\ +\frac{2^n}{90} f_o\left(\frac{x}{2^{n-1}}\right) - \frac{40\times2^n}{90} f_o\left(\frac{x}{2^n}\right) + \frac{256\times2^n}{90} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 1<p<2, \\[2ex] \frac{f_e(2^{n+2}x)}{2880\times64^n} - \frac{20f_e(2^{n+1}x)}{2880\times64^n} + \frac{64f_e(2^nx)}{2880\times64^n} - \frac{5f_e(2^{n+2}x)}{2880\times16^n} + \frac{340f_e(2^{n+1}x)}{2880\times16^n} - \frac{1280f_e(2^nx)}{2880\times16^n} \\ +\frac{4f_e(2^{n+2}x)-320f_e(2^{n+1}x)+4096f_e(2^nx)}{2880\times4^n} \\ +\frac{f_o(2^{n+2}x)}{720\times32^n} - \frac{10f_o(2^{n+1}x)}{720\times32^n} + \frac{16f_o(2^nx)}{720\times32^n} - \frac{5f_o(2^{n+2}x)}{720\times8^n} + \frac{170f_o(2^{n+1}x)}{720\times8^n} - \frac{320f_o(2^nx)}{720\times8^n} \\ +\frac{f_o(2^{n+2}x)-40f_o(2^{n+1}x)+256f_o(2^nx)}{180\times2^n} & \text{if } p<1 \end{cases}$$

for all $x \in X$ *and all nonnegative integers* n. *Then*

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$J_n f(x) - J_{n+1} f(x) =$

$$\begin{cases} \left(\frac{4^n}{45} - \frac{4\times 16^n}{9} + \frac{64\times 64^n}{45}\right)\Delta f\left(\frac{x}{2^{n+3}}\right) + \left(\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45}\right)\Gamma f\left(\frac{x}{2^{n+3}}\right) & \\ \hfill \text{if } 6 < p, \\ \left(\frac{4^n}{180} - \frac{16^n}{36}\right)\Delta f\left(\frac{x}{2^{n+2}}\right) - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + \left(\frac{2^n}{45} - \frac{4\times 8^n}{9} + \frac{64\times 32^n}{45}\right)\Gamma f\left(\frac{x}{2^{n+3}}\right) & \\ \hfill \text{if } 5 < p < 6, \\ \left(\frac{4^n}{180} - \frac{16^n}{36}\right)\Delta f\left(\frac{x}{2^{n+2}}\right) - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + \left(\frac{2^n}{90} - \frac{8^n}{18}\right)\Gamma f\left(\frac{x}{2^{n+2}}\right) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \\ \hfill \text{if } 4 < p < 5, \\ \frac{4^n}{180}\Delta f\left(\frac{x}{2^{n+2}}\right) + \frac{\Delta f(2^{n-1}x)}{36\times 16^{n+1}} - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + \left(\frac{2^n}{90} - \frac{8^n}{18}\right)\Gamma f\left(\frac{x}{2^{n+2}}\right) - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \\ \hfill \text{if } 3 < p < 4, \\ \frac{4^n}{180}\Delta f\left(\frac{x}{2^{n+2}}\right) + \frac{\Delta f(2^{n-1}x)}{36\times 16^{n+1}} - \frac{\Delta f(2^{n-1}x)}{45\times 64^{n+1}} + \frac{2^n}{90}\Gamma f\left(\frac{x}{2^{n+2}}\right) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \\ \hfill \text{if } 2 < p < 3, \\ -\frac{\Delta f(2^n x)}{720\times 4^{n+1}} + \frac{\Delta f(2^n x)}{576\times 16^{n+1}} - \frac{\Delta f(2^n x)}{2880\times 64^{n+1}} + \frac{2^n}{90}\Gamma f\left(\frac{x}{2^{n+2}}\right) + \frac{\Gamma f(2^{n-1}x)}{18\times 8^{n+1}} - \frac{2\Gamma f(2^{n-1}x)}{45\times 32^{n+1}} & \\ \hfill \text{if } 1 < p < 2, \\ -\frac{\Delta f(2^n x)}{720\times 4^{n+1}} + \frac{\Delta f(2^n x)}{576\times 16^{n+1}} - \frac{\Delta f(2^n x)}{2880\times 64^{n+1}} - \frac{\Gamma f(2^n x)}{180\times 2^{n+1}} + \frac{\Gamma f(2^n x)}{144\times 8^{n+1}} - \frac{\Gamma f(2^n x)}{720\times 32^{n+1}} & \\ \hfill \text{if } p < 1 \end{cases} \tag{3.3}$$

for all $x \in X$ and all nonnegative integers n.

Lemma 3.2 *If $f : X \to Y$ is a mapping such that $Df(x, y) = 0$ for all $x, y \in X$, then $J_n f(x) = f(x)$ for all $x \in X$ and all positive integers n.*

Lemma 3.3 *If $f : X \to Y$ is a mapping such that $f(0) = 0$ and $Df(x, y) = 0$ for all $x, y \in X \backslash \{0\}$, then $Df(x, y) = 0$ for all $x, y \in X$.*

Proof. Since $Df(x, 0) = 0$ and $Df(0, y) = Df(7y, -y) = 0$ for all $x, y \in X\backslash\{0\}$, the equality $Df(x, y) = 0$ holds for all $x, y \in X$. □

From Lemma 3.2 and Lemma 3.3, we can prove the following Hyers-Ulam-Rassias stability of the sextic functional equation.

Theorem 3.4 *Let $p \neq 1, 2, 3, 4, 5, 6$ be a fixed real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.4}$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general sextic mapping F such that

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$\|f(x) - f(0) - F(x)\| \leq$

$$\begin{cases} \left(\frac{K}{(2^p-2)} + \frac{(128+44\times2^p)K}{(2^p-32)(2^p-8)} + \frac{K'}{(2^p-4)} + \frac{(256+44\times2^p)K'}{(2^p-64)(2^p-16)}\right)\frac{\theta\|x\|^p}{45\times2^{2p}} & \textit{if } 6 < p, \\ \left(\frac{K}{2^p(2^p-2)} + \frac{(128+44\times2^p)K}{(2^p-32)(2^p-8)2^p} + \frac{(2^p-1)K'}{(2^p-4)(2^p-16)} + \frac{K'}{(64-2^p)}\right)\frac{\theta\|x\|^p}{45\times2^p} & \textit{if } 5 < p < 6, \\ \left(\frac{(2\times2^p-1)K}{(2^p-8)(2^p-2)} + \frac{2K}{(32-2^p)} + \frac{(2^p-1)K'}{(2^p-4)(2^p-16)} + \frac{K'}{(64-2^p)}\right)\frac{\theta\|x\|^p}{45\times2^p} & \textit{if } 4 < p < 5, \\ \left(\frac{(2\times2^p-1)K}{(2^p-8)(2^p-2)} + \frac{2K}{(32-2^p)} + \frac{K'}{4(2^p-4)} + \frac{(256-2^p)K'}{4(64-2^p)(16-2^p)}\right)\frac{\theta\|x\|^p}{45\times2^p} & \textit{if } 3 < p < 4, \\ \left(\frac{K}{(2^p-2)} + \frac{(128-2^p)K}{(32-2^p)(8-2^p)} + \frac{K'}{2(2^p-4)} + \frac{(256-2^p)K'}{2(64-2^p)(16-2^p)}\right)\frac{\theta\|x\|^p}{90\times2^p} & \textit{if } 2 < p < 3, \\ \left(\frac{K}{2^p(2^p-2)} + \frac{(128-2^p)K}{(32-2^p)(8-2^p)2^p} + \frac{(44+2^p)K'}{32(16-2^p)(4-2^p)} + \frac{K'}{32(64-2^p)}\right)\frac{\theta\|x\|^p}{90} & \textit{if } 1 < p < 2, \\ \left(\frac{(22+2^p)K}{(8-2^p)(2-2^p)} + \frac{K}{(32-2^p)} + \frac{(44+2^p)K'}{4(16-2^p)(4-2^p)} + \frac{K'}{4(64-2^p)}\right)\frac{\theta\|x\|^p}{720} & \textit{if } p < 1 \end{cases} \tag{3.5}$$

for all $x \in X\backslash\{0\}$ *and* $F(0) = 0$, *where* $K := 166 + 43 \times 2^p + 112 \times 3^p + 6^p$ *and* $K' := 184 + 57 \times 2^p + 112 \times 3^p + 6^p$.

Proof. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then $D\tilde{f}(x, y) = Df(x, y)$ and $\tilde{f}(0) = 0$. By (3.4) and the definitions of $\Gamma\tilde{f}$ and $\Delta\tilde{f}$, we have

$$\begin{aligned} \|\Gamma\tilde{f}(x)\| =& \big\|Df_o(-6x, 2x) + 6Df_o(-x, x) + 42Df_o(-2x, x) + 112Df_o(-3x, x)\big\| \\ \leq& \theta(166 + 43 \times 2^p + 112 \times 3^p + 6^p)\|x\|^p, \\ \|\Delta\tilde{f}(x)\| =& \big\|Df_e(-6x, 2x) + 8Df_e(-x, x) + 56Df_e(-2x, x) + 112Df_e(-3x, x)\big\| \\ \leq& \theta(184 + 57 \times 2^p + 112 \times 3^p + 6^p)\|x\|^p \end{aligned}$$

for all $x \in X\backslash\{0\}$. It follows from (3.3) and (3.4) that
$\|J_n\tilde{f}(x) - J_{n+1}\tilde{f}(x)\| \leq$

$$\begin{cases} \left(\frac{2^nK}{45\times2^{(n+3)p}} - \frac{4\times8^nK}{9\times2^{(n+3)p}} + \frac{64\times32^nK}{45\times2^{(n+3)p}} + \frac{4^nK'}{45\times2^{(n+3)p}} - \frac{4\times16^nK'}{9\times2^{(n+3)p}} + \frac{64\times64^nK'}{45\times2^{(n+3)p}}\right)\theta\|x\|^p & \text{if } 6 < p, \\ \left(\frac{2^nK}{45\times2^{(n+3)p}} - \frac{4\times8^nK}{9\times2^{(n+3)p}} + \frac{64\times32^nK}{45\times2^{(n+3)p}} + \frac{(5\times16^n-4^n)K'}{180\times2^{(n+2)p}} + \frac{2\times2^{(n-1)p}K')}{90\times64^{n+1}}\right)\theta\|x\|^p & \text{if } 5 < p < 6, \\ \left(\frac{(5\times8^n-2^n)K}{90\times2^{(n+2)p}} + \frac{2\times2^{(n-1)p}K)}{45\times32^{n+1}} + \frac{(5\times16^n-4^n)K'}{180\times2^{(n+2)p}} + \frac{2\times2^{(n-1)p}K')}{90\times64^{n+1}}\right)\theta\|x\|^p & \text{if } 4 < p < 5, \\ \left(\frac{(5\times8^n-2^n)K}{90\times2^{(n+2)p}} + \frac{2\times2^{(n-1)p}K)}{45\times32^{n+1}} + \frac{4^nK'}{180\times2^{(n+2)p}} + \frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K'}{180\times64^{n+1}}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{2^nK}{90\times2^{(n+2)p}} + \frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K}{90\times32^{n+1}} + \frac{4^nK'}{180\times2^{(n+2)p}} + \frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K'}{180\times64^{n+1}}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{2^nK}{90\times2^{(n+2)p}} + \frac{(5\times4^{n+1}-4)\times2^{(n-1)p}K}{90\times32^{n+1}} + \frac{2^{np}K'}{720\times4^{n+1}} - \frac{2^{np}K'}{576\times16^{n+1}} + \frac{2^{np}K'}{2880\times64^{n+1}}\right)\theta\|x\|^p & \text{if } 1 < p < 2, \\ \left(\frac{2^{np}K}{180\times2^{n+1}} - \frac{2^{np}K}{144\times8^{n+1}} + \frac{2^{np}K}{720\times32^{n+1}} + \frac{2^{np}K'}{720\times4^{n+1}} - \frac{2^{np}K'}{576\times16^{n+1}} + \frac{2^{np}K'}{2880\times64^{n+1}}\right)\theta\|x\|^p & \text{if } p < 1 \end{cases}$$

for all $x \in X\backslash\{0\}$. Together with the equality $J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x) = \sum_{i=n}^{n+m-1}(J_i\tilde{f}(x) - J_{i+1}\tilde{f}(x))$ for all $x \in X$, we obtain that

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$$\|J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x)\| \leq$$

$$\begin{cases} \sum_{i=n}^{n+m-1} \left(\frac{2^iK-20\times 8^iK+64\times 32^iK+4^iK'-20\times 16^iK'+64\times 64^iK'}{45\times 2^{(i+3)p}}\right)\theta\|x\|^p & \text{if } 6<p, \\ \sum_{i=n}^{n+m-1} \left(\frac{2^iK-20\times 8^iK+64\times 32^iK}{45\times 2^{(i+3)p}} + \frac{(5\times 16^i-4^i)K'}{180\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K')}{90\times 64^{i+1}}\right)\theta\|x\|^p & \text{if } 5<p<6, \\ \sum_{i=n}^{n+m-1} \left(\frac{(5\times 8^i-2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{(5\times 16^i-4^i)K'}{180\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K')}{90\times 64^{i+1}}\right)\theta\|x\|^p & \text{if } 4<p<5, \\ \sum_{i=n}^{n+m-1} \left(\frac{(5\times 8^i-2^i)K}{90\times 2^{(i+2)p}} + \frac{2\times 2^{(i-1)p}K)}{45\times 32^{i+1}} + \frac{4^iK'}{180\times 2^{(i+2)p}} + \frac{(5\times 4^{i+1}-4)\times 2^{(i-1)p}K'}{180\times 64^{i+1}}\right)\theta\|x\|^p & \text{if } 3<p<4, \\ \sum_{i=n}^{n+m-1} \left(\frac{32\times 64^iK+(5\times 4^{i+1}-4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{4^{4i+3}+(5\times 4^{i+1}-4)\times 2^{(2i+1)p}K'}{180\times 64^{i+1}2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 2<p<3, \\ \sum_{i=n}^{n+m-1} \left(\frac{32\times 64^iK+(5\times 4^{i+1}-4)\times 2^{(2i+1)p}K}{90\times 32^{i+1}\times 2^{(i+2)p}} + \frac{(4^{i+2}-5)2^{ip}K'}{2880\times 16^{i+1}} + \frac{2^{ip}K'}{2880\times 64^{i+1}}\right)\theta\|x\|^p & \text{if } 1<p<2, \\ \sum_{i=n}^{n+m-1} \left(\frac{2^{ip}K}{180\times 2^{i+1}} - \frac{2^{ip}K}{144\times 8^{i+1}} + \frac{2^{ip}K}{720\times 32^{i+1}} + \frac{(4^{i+2}-5)2^{ip}K'}{2880\times 16^{i+1}} + \frac{2^{ip}K'}{2880\times 64^{i+1}}\right)\theta\|x\|^p & \text{if } p<1 \end{cases} \tag{3.6}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (3.6) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $\tilde{f}(0) = 0$, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n\tilde{f}(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (3.6) we get the inequality (3.5). For the case $2 < p < 3$, from the definition of F, we easily get

$$\begin{aligned} \|DF(x,y)\| = \lim_{n\to\infty} \Bigg\| & \frac{Df_e(2^{n+1}x, 2^{n+1}y)}{45\times 64^n} - \frac{20Df_e(2^nx, 2^ny)}{45\times 64^n} + \frac{64Df_o(2^{n-1}x, 2^{n-1}y)}{45\times 64^n} \\ & - \frac{Df_e(2^{n+1}x, 2^{n+1}y) - 68Df_e(2^nx, 2^ny) + 256Df_e(2^{n-1}x, 2^{n-1}y)}{36\times 16^n} \\ & + \frac{4^n}{180}\left[Df_e\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) - 80Df_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 1024Df_e\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right] \\ & + \frac{4Df_o(2^{n+1}x, 2^{n+1}y)}{90\times 32^n} - \frac{40Df_o(2^nx, 2^ny)}{90\times 32^n} + \frac{64Df_o(2^{n-1}x, 2^{n-1}y)}{90\times 32^n} \\ & - \frac{5[Df_o(2^{n+1}x, 2^{n+1}y) - 34Df_o(2^nx, 2^ny) + 64Df_o(2^{n-1}x, 2^{n-1}y)]}{90\times 8^n} \\ & + \frac{2^n}{90}\left[Df_o\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) - 40Df_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 256Df_o\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right]\Bigg\| \\ \leq \lim_{n\to\infty} \Bigg(& \frac{(4^p + 20\times 2^p + 64)2^{(n-1)p}}{45\times 64^n} + \frac{(4^p + 68\times 2^p + 256)2^{(n-1)p}}{36\times 16^n} \\ & + \frac{4^n(4^p + 80\times 2^p + 1024)}{180\times 2^{(n+1)p}} + \frac{4(4^p + 10\times 2^p + 16)2^{(n-1)p}}{90\times 32^n} \\ & + \frac{(4^p + 34\times 2^p + 64)2^{(n-1)p}}{18\times 8^n} + \frac{2^n(4^p + 40\times 2^p + 256)}{90\times 2^{(n+1)p}}\Bigg) \times \theta(\|x\|^p + \|y\|^p) \\ = & 0 \end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Since $DF(x,y) = 0$ for all $x, y \in X\backslash\{0\}$, $F : X \to Y$ satisfies the equality $DF(x,y) = 0$ for all $x, y \in X$ by Lemma 3.3. Also we easily show that

$DF(x,y)=0$ by the similar method for the other cases, either $p<1$ or $1<p<2$ or $3<p<4$ or $4<p<5$ or $5<p<6$ or $6<p$. To prove the uniqueness of F, let $F':X\to Y$ be another sextic mapping satisfying (3.5) and $F'(0)=0$. By Lemma 3.2, the equality $F'(x)=J_nF'(x)$ holds for all $n\in\mathbb{N}$. For the case $2<p<3$, we have

$$\begin{aligned}
&\|J_n\tilde{f}(x)-F'(x)\|\\
&\quad=\|J_n\tilde{f}(x)-J_nF'(x)\|\\
&\quad\le\frac{\|(\tilde{f}_e-F_e)(2^{n+1}x)\|}{45\times 64^n}+\frac{20\|(\tilde{f}_e-F_e)(2^{n}x)\|}{45\times 64^n}+\frac{64\|(\tilde{f}_e-F_e)(2^{n-1}x)\|}{45\times 64^n}\\
&\qquad+\frac{\|(\tilde{f}_e-F_e)(2^{n+1}x)\|}{36\times 16^n}+\frac{68\|(\tilde{f}_e-F_e)(2^{n}x)\|}{36\times 16^n}+\frac{256\|(\tilde{f}_e-F_e)(2^{n-1}x)\|}{36\times 16^n}\\
&\qquad+\frac{4^n}{180}\left[\left\|(\tilde{f}_e-F_e)\left(\frac{2x}{2^n}\right)\right\|+80\left\|(\tilde{f}_e-F_e)\left(\frac{x}{2^n}\right)\right\|+1024\left\|(\tilde{f}_e-F_e)\left(\frac{x}{2^{n+1}}\right)\right\|\right]\\
&\qquad+\frac{4\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 32^n}+\frac{40\|(\tilde{f}_o-F_o)(2^{n}x)\|}{90\times 32^n}+\frac{64\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 32^n}\\
&\qquad+\frac{5\|(\tilde{f}_o-F_o)(2^{n+1}x)\|}{90\times 8^n}+\frac{170\|(\tilde{f}_o-F_o)(2^{n}x)\|}{90\times 8^n}+\frac{320\|(\tilde{f}_o-F_o)(2^{n-1}x)\|}{90\times 8^n}\\
&\qquad+\frac{2^n}{90}\left[\left\|(\tilde{f}_o-F_o)\left(\frac{2x}{2^n}\right)\right\|+40\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^n}\right)\right\|+256\left\|(\tilde{f}_o-F_o)\left(\frac{x}{2^{n+1}}\right)\right\|\right]\\
&\quad\le\left(\frac{(4^p+20\times 2^p+64)2^{(n-1)p}}{45\times 64^n}+\frac{(4^p+68\times 2^p+256)2^{(n-1)p}}{36\times 16^n}\right.\\
&\qquad+\frac{(4^p+80\times 2^p+1024)2^n}{180\times 2^{(n+1)p}}+\frac{2(4^p+10\times 2^p+16)2^{(n-1)p}}{45\times 32^n}\\
&\qquad\left.+\frac{(4^p+34\times 2^p+64)2^{(n-1)p}}{18\times 8^n}+\frac{(4^p+40\times 2^p+256)2^n}{90\times 2^{(n+1)p}}\right)\\
&\qquad\times\left(\frac{K}{2^p-2}+\frac{(128-2^p)K}{(32-2^p)(8-2^p)}+\frac{K'}{2(2^p-4)}+\frac{(256-2^p)K'}{2(64-2^p)(16-2^p)}\right)\frac{\theta\|x\|^p}{90\times 2^p}
\end{aligned}$$

for all $x\in X\backslash\{0\}$ and all positive integer n. Taking the limit in the above inequality as $n\to\infty$, we obtain the equality $F'(x)=\lim_{n\to\infty}J_n\tilde{f}(x)$ for all $x\in X\backslash\{0\}$, which means that $F(x)=F'(x)$ for all $x\in X\backslash\{0\}$. Also we easily show that $F(x)=F'(x)$ by the similar method for the other cases, either $p<1$ or $1<p<2$ or $3<p<4$ or $4<p<5$ or $5<p<6$ or $6<p$. □

From Theorem 3.4, we also prove the hyperstability of the sextic functional equation when $p<0$.

Theorem 3.5 *Let $p<0$ be a real number. If a mapping $f:X\to Y$ satisfies the inequality (3.4) for all $x,y\in X\backslash\{0\}$, then $f:X\to Y$ is a sextic mapping itself.*

Proof. According to Theorem 3.4, there is a unique sextic mapping F of the functional equation $DF(x,y)=0$ such that

$$\|\tilde{f}(x)-F(x)\|\le\left(\frac{(22+2^p)K}{(8-2^p)(2-2^p)}+\frac{K}{(32-2^p)}+\frac{(44+2^p)K'}{4(16-2^p)(4-2^p)}+\frac{K'}{4(64-2^p)}\right)\frac{\theta\|x\|^p}{720}$$

for all $x \in X\backslash\{0\}$ and $F(0)=0$. From the equality

$$\begin{aligned} D\tilde{f}(nx,-(n-1)x) =& D\tilde{f}(nx,-(n-1)x) - DF(nx,-(n-1)x) \\ =& \sum_{i=0}^{7} {}_7C_i(-1)^{7-i}(\tilde{f}-F)(nx-i(n-1)x) \end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}$, we have the inequality

$$\begin{aligned} \|{}_7C_1(\tilde{f}-F)(x)\| =& \Bigg\| Df(nx,-(n-1)x) + (\tilde{f}-F)(nx) \\ & - \sum_{i=2}^{7} {}_7C_i(-1)^{7-i}(\tilde{f}-F)(nx-i(n-1)x) \Bigg\| \\ \leq& \theta\|x\|^p \Bigg[n^p + (n-1)^p + \Big(n^p + \sum_{i=2}^{7} {}_7C_i(i(n-1)-n)^p \Big) \\ & \times \Bigg(\frac{(22+2^p)K}{(8-2^p)(2-2^p)} + \frac{K}{(32-2^p)} + \frac{(44+2^p)K'}{4(16-2^p)(4-2^p)} + \frac{K'}{4(64-2^p)} \Bigg) \Bigg] \end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}\backslash\{1,2\}$. Since $\tilde{f}(0)=F(0)$ and n^p, $\sum_{i=2}^{7} {}_7C_i(i(n-1)-n)^p$, and $(n-1)^p$ tend to 0 as $n \to \infty$, we get $\tilde{f}(x)=F(x)$ for all $x \in X$. Therefore $Df(x,y)=D\tilde{f}(x,y)=DF(x,y)=0$ for all $x,y \in X$. □

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