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Article

Thermodynamic, non-extensive, or turbulent quasi equilibrium for space plasma environment

Peter H. Yoon [1,2,3] 0000-0001-8134-3790*

1 Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA; yoonp@umd.edu
2 School of Space Research, Kyung Hee University, Yongin, Gyeonggi 446-701, Korea
3 Korea Astronomy and Space Science Institute, Daejeon 34055, Korea
* Correspondence: yoonp@umd.edu

Abstract: The Boltzmann-Gibbs (BG) entropy has been used in a wide variety of problems for more than a century. It is well known that BG entropy is extensive, but for certain systems such as those dictated by long-range interactions, the entropy must be non-extensive. Tsallis entropy possesses non-extensive characteristics, which is parametrized by a variable q ($q = 1$ being the classic BG limit), but unless q is determined from microscopic dynamics, the model remains but a phenomenological tool. To this date very few examples have emerged in which q can be computed from first principles. This paper shows that the space plasma environment, which is governed by long-range collective electromagnetic interaction, represents a perfect example for which the q parameter can be computed from micro-physics. By taking the electron velocity distribution function measured in the heliospheric environment into account, and considering them to be in quasi equilibrium state with electrostatic turbulence known as the quasi-thermal noise, it is shown that the value corresponding to $q = 9/13 = 0.6923$ may be deduced. This prediction is verified against observation made by spacecraft, and it is shown to be in excellent agreement.

Keywords: non-extensive entropic principle; plasma turbulence; quasi equilibrium

1. Introduction

The celebrated Boltzmann-Gibbs (BG) definition for entropy [1–3]

$$S = k \log W \tag{1}$$

has been used in a wide variety of problems for more than a century. Here, W represents the combinatorial number of all possible micro states of a system, be it classical or quantum mechanical. The constant k is taken as the Boltzmann constant $k_B = 1.3806503 \times 10^{-23}\,\mathrm{m^2\,kg\,s^{-2}\,K^{-1}}$ for thermostatistics, and unity for information system, in which case, it is known as the Shannon entropy [4]. A more general form of Boltzmann-Gibbs-Shannon (BGS) entropy is expressed in terms of the probability, p_i, of the system being in a particular micro state, labeled i, namely,

$$S = -k \sum_{i=1}^{W} p_i \ln p_i . \tag{2}$$

For the particular case of equal probability, $p_i = 1/W$ for all i, we recover $S = k \log W$. This well-known expression for the entropy has been in use since the 1870s, not only in physics, but for a variety of problems in chemistry, mathematics, computational sciences, engineering, and elsewhere.

It should be noted that the BG entropy is not universally applicable to all situations. While the definition is eminently suitable for an ideal gas and systems dictated by short-range interactions, for

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systems interacting through long-range forces the applicability of BG entropy has been doubted by a number of individuals, including Boltzmann himself [5], Einstein [6], Fermi [7], and others. One of the characteristics of BG entropy is that it is additive, or extensive, that is, the entropy of the total system equals the sum of entropies of subsystems. If A and B represent two subsystems and $A+B$ the total system, then

$$S(A+B) = S(A) + S(B). \tag{3}$$

The non-extensive entropy violates this rule.

For short-range interactions, micro states are governed by neighboring particles. As such, the combinatorial number of possible micro states associated with the total system is simply the sum of combinatorial number of micro states associated with each sub system. This is because of particles interacting with short-ranged force are not aware of the presence of other particles belonging to other subsystems. The ionized gas, or plasma, is governed by long-ranged electromagnetic interaction among charged particles. As such, the combinatorial number of micro states may be more than that of the simple sum for sub systems, since charged particles in one system are affected by distant particles in other subsystem by virtue of the long-ranged nature of the interaction. For such a situation, the non-extensive entropic principle $S(A+B) \neq S(A) + S(B)$ is expected.

Over the past many years a number of attempts were made to generalize BG entropy to non-extensive situations. Among these is the 1988 paper by Tsallis [8], which has triggered a recent interest in the non-extensive thermostatics, although in a strict sense, Tsallis was not the first to suggest the particular functional form. As he acknowledges in his recent monograph [9], a number of previous attempts had already entertained similar functional form for the generalized entropy (see entry 107 in the Bibliography section of [9], p. 347, for full reference of early works.). Tsallis put forth a model entropy of the form

$$S_q = -\frac{k\left(1 - \sum_{i=1}^{W} p_i^q\right)}{1-q}. \tag{4}$$

It can be shown that

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)\,k^{-1} S_q(A)\,S_q(B) \neq S(A) + S(B), \tag{5}$$

where the parameter q is a measure of how far the system deviates from the BG statistics (for which $q = 1$). Note that a number of modifications and improvements of Tsallis' original formalism are available in the literature [10], but the original formalism by Tsallis sufficient for the present discussion.

An outstanding issue concerns the determination of q parameter from first principles. Chapter 7 of the monograph by Tsallis [9] lists applications of non-extensive statistical method to high-energy physics, turbulence, granular matter, geophysics and astrophysics, quantum chaos, etc., where the q parameter for each case is determined by empirical fitting method. For plasmas, on the other hand, the governing principle is simple enough, that is, laws of electromagnetics, that it is possible to compute the value of q parameter from first principles. The space environment is an almost perfect natural laboratory for plasma research, so we focus on examples from measurements made in space by artificial satellites in order investigate the basic property of space plasmas and possible connection to non-extensive statistical concepts.

In the 1960s *in situ* spacecraft measurements of charged particles became possible. It was realized then that the space plasma did not behave according to the laws of thermodynamic equilibrium. Instead of Maxwell-Boltzmann-Gauss distribution of particles, observed distributions typically featured suprathermal component, see, e.g., Refs. [11–13], for some early observations. In an attempt to fit the measured electron distributions, Vasyliunas [14] introduced a phenomenological model known as the kappa distribution,

$$f(v) \sim \left(1 + \frac{v^2}{\kappa v_T^2}\right)^{-(\kappa+1)}, \tag{6}$$

where $v_T = (2k_BT/m)^{1/2}$ is the Maxwellian thermal speed, that is, v_T would be thermal speed had $f(v)$ been given by the Maxwell-Boltzmann distribution, T and m being the temperature and mass of the charged particles, κ is a free fitting parameter, and $f(v)$ represents the velocity distribution function. When $\kappa \to \infty$ the model reduces to the Maxwellian-Boltzmann (or thermal) distribution,

$$f(v) \sim \exp\left(-\frac{v^2}{v_T^2}\right). \tag{7}$$

The kappa model enjoyed no first principle justification until it was realized that the most probable state defined in the context of the non-extensive entropic principle is related to the kappa distribution. For a continuous system, BG entropy can be written as

$$S_{BG} = -k_B \int d\mathbf{x} \int d\mathbf{v}\, f(\mathbf{v}) \ln f(\mathbf{v}), \tag{8}$$

where integration over space $\int d\mathbf{x}$ is understood as being normalized with respect to the total volume of the system, $\mathscr{V}^{-1} \int d\mathbf{x}$. Upon minimizing the Helmholtz free energy,

$$F = U - TS_{BG}, \tag{9}$$

where

$$U = \int d\mathbf{x} \int d\mathbf{v}\, \frac{mv^2}{2} f(\mathbf{v}) \tag{10}$$

is the total energy, we find that the Maxwell-Boltzmann distribution naturally emerges as the most probable state. In contrast, upon making use of the continuous version of the non-extensive (NE), or Tsallis entropy, to wit,

$$S_q = -\frac{k_B}{1-q} \int d\mathbf{x} \int d\mathbf{v} \left\{ f(\mathbf{v}) - [f(\mathbf{v})]^q \right\}, \tag{11}$$

and minimize the corresponding Helmholtz free energy, then the solution

$$f(\mathbf{v}) \sim \left[1 + \frac{(1-q)\,v^2}{v_T^2}\right]^{-1/(1-q)} \tag{12}$$

emerges as the most probable state. Upon defining

$$\kappa = \frac{1}{1-q}, \tag{13}$$

it is straightforward to see that this solution is Vasyliunas' kappa distribution. Note, however, that Vasyliunas' original model has the power index $\kappa + 1$ instead of κ. Nonetheless, the two models are practically equivalent. This has prompted an explosion of interest in the non-extensive statistical model in the framework of space plasmas [15–17].

Independent of these developments, Ref. [18] uncovered an interesting fact that the quasi steady state electrostatic turbulence generated by a weak electron beam propagating in the background plasma is characterized by a non-thermal tail population in the electron velocity distribution function, which superficially resembles the kappa distribution. Subsequently, Ref. [19] provided further proof that the kappa distribution is a unique solution that characterizes steady state electrostatic plasma turbulence. The finding that quasi steady-state plasma turbulence corresponds to the kappa distribution function strongly implies the profound inter-relationship with non-extensive statistical equilibrium. Quasi equilibrium state of plasma turbulence depict electrons interacting among themselves through long-range collective electrostatic fluctuations, which also describes the non-extensive charged-particle system interacting with long-range electrostatic force. In this regard, it is no surprise that the

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approach based upon plasma turbulence theory and that based on non-extensive statistical method are equivalent.

The solar wind electrons [20,21] can be modeled with multi-component velocity distribution functions: Maxwellian core; suprathermal halo; highly field-aligned strahl, which is typically associated with the high-speed wind; and highly energetic superhalo. As the superhalo electrons [22], which are observed in all solar wind conditions with nearly invariant velocity power law index, is at the high end of the velocity spectrum, comparing the asymptotic behavior of the kappa distribution with that of superhalo component can yield meaningful results, which we will do later in this paper. For the moment, we move on to the next section where we discuss the asymptotically steady state electrostatic plasma turbulence and the corresponding electron velocity distribution.

2. Steady State Plasma Turbulence and Electron Kappa Distribution

We begin the discussion with the kinetic equations of plasma turbulence, which are available in standard plasma physics literature [18,23–26]. For plasma in uniform medium free of external fields and subject to electrostatic interactions among charged particles including dynamic electrons and quasi stationary neutralizing background protons, the electron distribution function $f_e(\mathbf{v},t)$ obeys the kinetic equation given by

$$\begin{aligned}
\frac{\partial f_e}{\partial t} &= \frac{\partial}{\partial v_i}\left(A_i F_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v}), \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}.
\end{aligned} \tag{14}$$

In the above e and m_e stand for unit electric charge and electron mass, respectively, $\omega_{\mathbf{k}}^L = \omega_{pe}\,(1+\frac{3}{2}k^2\lambda_{De}^2)$ represents the dispersion relation satisfied by high frequency electrostatic wave in the plasma known as the Langmuir wave, $\lambda_{De} = T_e^{1/2}/(4\pi n e^2)^{1/2}$ being the Debye length, $\omega_{pe} = (4\pi n e^2/m_e)^{1/2}$ being the plasma oscillation frequency, T_e being the electron temperature, and $I_{\mathbf{k}}^{\sigma L}$ denotes the spectral electric field intensity associated with the Langmuir wave, $E_{\mathbf{k},\omega}^2 = \sum_{\sigma=\pm1} I_{\mathbf{k}}^{\sigma L}\delta(\omega-\sigma\omega_{\mathbf{k}}^L)$. The symbol $\sigma=\pm1$ denotes the sign of the wave phase and group velocities. The electron particle kinetic equation (14) is given in the form of Fokker-Planck equation where the velocity friction represented by coefficient $\mathbf{A}$ appears in balanced form against velocity diffusion with the associated diffusion coefficient D_{ij}. Since the velocity diffusion is dictated by the Langmuir wave spectral intensity $I_{\mathbf{k}}^{\sigma L}$, the particle equation must be considered in conjunction with the wave kinetic equation. When one considers the wave kinetic equation, one must take into account not only high-frequency Langmuir (L) wave but also low-frequency wave known as the ion-sound (S) wave, since L mode is nonlinear coupled to S mode via wave-wave resonant interaction. In short, the wave kinetic equations for L and S mode waves are to be solved as well. These are given by

$$\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} &= \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e+\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right) \\
&\quad +2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^{LS}\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\right. \\
&\quad \left.-\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right) \\
&\quad +\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,U_{\mathbf{k},\mathbf{k}'}^{LL}\left[\frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right.
\end{aligned} \tag{15}$$

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$$+\frac{ne^2}{\pi\omega_{pe}^2}\left(\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L} - \sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}}^{\sigma L}\right)(f_e + f_i)\Bigg],$$

$$\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} &= \frac{\pi\mu_{\mathbf{k}}\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})\left[\frac{ne^2}{\pi}(f_e + f_i)\right.\\
&\quad \left. +\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}}^{\sigma S}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(f_e + \frac{m_e}{m_i}f_i\right)\right]\\
&\quad + \sum_{\sigma',\sigma''=\pm 1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\, V_{\mathbf{k},\mathbf{k}'}^{SL}\left(\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\right.\\
&\quad \left. -\sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\, I_{\mathbf{k}}^{\sigma S} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma S}\right). \qquad (16)
\end{aligned}$$

These wave kinetic equations have been derived by various authors, and can be found in Refs. [18,23–26]. For L mode wave equation the first term on the right-hand side represents linear wave-particle resonant interaction between the electrons and Langmuir wave; the second and third lines collectively describe three wave or wave-wave nonlinear resonant processes among two Langmuir waves and an ion sound wave; the fourth and fifth lines together describe nonlinear wave-particle resonance among two Langmuir waves mediated by quasi stationary protons, whose velocity distribution is given by f_i. For the ion sound mode, the interpretations and designations of various terms are analogous to those of L mode wave, except that S mode wave kinetic equation does not have the term denoting the nonlinear wave-particle resonance. The ion sound or S mode enjoys the dispersion relation specified by

$$\omega_{\mathbf{k}}^S = \frac{kc_S\sqrt{1 + 3T_i/T_e}}{\sqrt{1 + k^2\lambda_{De}^2}}, \qquad (17)$$

where $c_S = (T_e/m_i)^{1/2}$ is the ion sound (or ion acoustic) speed and T_i stands for the ion (proton) temperature. Various objects which appear in the wave kinetic equations (15) and (16) are defined by

$$\begin{aligned}
\mu_{\mathbf{k}} &= k^3\lambda_{De}^3\sqrt{\frac{m_e}{m_i}}\sqrt{1 + \frac{3T_i}{T_e}},\\
V_{\mathbf{k},\mathbf{k}'}^{LS} &= \frac{\pi}{2}\frac{e^2}{T_e^2}\frac{\mu_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S),\\
V_{\mathbf{k},\mathbf{k}'}^{SL} &= \frac{\pi}{4}\frac{e^2}{T_e^2}\frac{\mu_{\mathbf{k}}[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2 k'^2\,|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L),\\
U_{\mathbf{k},\mathbf{k}'}^{LL} &= \frac{\pi}{\omega_{pe}^2}\frac{e^2}{m_e^2}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]. \qquad (18)
\end{aligned}$$

As is apparent from the definitions, nonlinear coupling coefficients, $V_{\mathbf{k},\mathbf{k}'}^{LS}$, $V_{\mathbf{k},\mathbf{k}'}^{SL}$, and $U_{\mathbf{k},\mathbf{k}'}^{LL}$, dictate the various wave-wave and nonlinear wave-particle resonant interactions, which are obvious from the delta function arguments.

Consider the particle kinetic equation for electrons (14). In what follows we assume that the wave dispersion relation depends only on the magnitude of $\mathbf{k}$, and that the forward- and backward wave intensities are identical and isotropic, $\omega_{\mathbf{k}}^L = \omega_k^L$ and $I_{\mathbf{k}}^{\sigma L} = I_L(k)$, which are valid assumptions, provided the electron distribution function is isotropic, $f_e(\mathbf{v}) = f_e(v)$. We assume the steady state, $\partial f_e/\partial t = 0$, by virtue of the velocity friction and diffusion terms balancing each other out,

$$0 = A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}. \qquad (19)$$

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Following the basic method pioneered in Ref. [27], the present author [19] demonstrated that the formal solution to the steady state particle kinetic equation is given by

$$f_e = C\,\exp\left(-\int dv\,\frac{m_e v}{4\pi^2}\,\frac{\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}}{\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\,I_L(k)}\right). \tag{20}$$

In the above the integral $\int_{\omega_{pe}/v}^{\infty} dk/k$ formally diverges for $k\to\infty$, but if we formally define

$$\mathscr{H}(v) = \int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}, \qquad \mathscr{H}(v)\,\mathscr{I}(v) = \int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\,I_L(k). \tag{21}$$

Then we may formally remove the divergence, so that we have

$$f_e = C\,\exp\left(-\int dv\,\frac{m_e v}{4\pi^2}\,\frac{1}{\mathscr{I}(v)}\right). \tag{22}$$

This solution show that a suitable model for $I_L(k)$, or for that matter, $\mathscr{I}(k)$, will lead to the suitable counterpart for f_e and vice versa. Apparently, there exists an infinite choice for coupled solutions $[f_e(v), I_L(k)]$, of which we are interested in a particular form of electron distribution velocity function that represents a kappa-like solution,

$$f_e(v) = \frac{C}{(1 + m_e v^2/2\kappa'\,\theta_e)^{\kappa+1}}. \tag{23}$$

The normalization constant C can be obtained by requiring the condition, $1 = 4\pi\int_0^\infty dv\,v^2 f_e$, and is thus given by

$$C = \frac{m_e^{3/2}}{(2\pi\,\theta_e)^{3/2}}\,\frac{\Gamma(\kappa+1)}{\kappa'^{3/2}\,\Gamma(\kappa-1/2)}. \tag{24}$$

Here $\Gamma(x)$ is the gamma function. The effective or kinetic temperature for this kappa-like model can be computed on the basis of definition, $T_e = \int d\mathbf{v}(m_e v^2/3) f_e$, and the result is

$$T_e = \theta_e\,\frac{\kappa'}{\kappa - 3/2}. \tag{25}$$

If we impose the model f_e given by (23), then it follows from (20) or (22) that the corresponding wave spectrum $I_L(k)$ can be deduced. First, it can be shown that the choice of

$$\mathscr{I}(v) = \frac{\theta_e}{4\pi^2}\,\frac{\kappa'}{\kappa+1}\left(1 + \frac{m_e v^2}{2\kappa'\theta_e}\right), \tag{26}$$

satisfies (22) with f_e given by (23). Then from (21), it follows that $I_L(k)$ is given by

$$\begin{aligned} I_L(k) &= \frac{\theta_e}{4\pi^2}\,\frac{\kappa'}{\kappa+1}\left(1 + \frac{m_e\omega_{pe}^2}{2\kappa' k^2\theta_e}\,[1 + 2\mathscr{H}(k)]\right), \\ \mathscr{H}(k) &= \int_k^\infty \frac{dk}{k}. \end{aligned} \tag{27}$$

We reiterate that the distribution (23) and the corresponding spectrum (27) are not unique, and that the indices κ and κ' are free parameters at this point. In order to prove the uniqueness as well as to determine the values for κ and κ', we next turn to the steady-state wave equations.

Consider the wave kinetic equations (15) and (16). We assume isotropic spectrum as in the above discussion on formal particle equation. We may ignore the S mode contribution as well as

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contributions from the three wave interaction process. Reference [19] discusses these assumptions and approximations in detail. The same reference also presents detailed modifications of nonlinear coupling coefficient when the underlying electron distribution is given by the kappa-like model (23). Consequently, by omitting the intermediate steps, we simply present the steady-state Langmuir wave equation without the wave-wave resonant interaction term,

$$\begin{aligned} 0 &= \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\omega_k-\mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e+\omega_k\,I_L(k)\,\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right) \\ &\quad -\left(\frac{\kappa-1/2}{\kappa'}\right)^2\frac{\omega_k}{4\pi nT_i}\sum_{+,-}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\omega_k\mp\omega_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] \\ &\quad \times\left(\frac{T_i}{4\pi^2}\left[\pm\omega_{k'}\,I_L(k)-\omega_k\,I_L(k')\right]+I_L(k')\,I_L(k)\,(\omega_k\mp\omega_{k'})\right)f_i, \end{aligned} \tag{28}$$

where we have taken advantage of the fact that the ion distribution is assumed to remain stationary and given by the Maxwellian form. Reference [19] shows that the first term on the right-hand side, which is dictated by the linear wave-particle resonance delta function condition, vanishes if f_e and $I_L(k)$ are chosen by (23) and (27), respectively. As a consequence, only the nonlinear wave-particle resonance term needs to be considered in the wave equation, which provides the necessary constraint, with which, we will be able to demonstrate that the kappa-like model (23) is indeed, a unique solution, and that κ and κ' can be determined. Thus, we consider the nonlinear term in (28), which upon setting equal to zero, reduces to

$$\begin{aligned} 0 &= \int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\omega_k-\omega_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] \\ &\quad \times\left(\frac{T_i}{4\pi^2}\left[\omega_{k'}\,I_L(k)-\omega_k\,I_L(k')\right]+I_L(k')\,I_L(k)\,(\omega_k-\omega_{k'})\right)f_i. \end{aligned} \tag{29}$$

As discussed in Ref. [19], the nonlinear resonance speed satisfying the condition $\omega_k-\omega_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}=0$ is given by $v_{\rm res}\sim 3\,(k-k')\,\theta_e/(2m_e\omega_{pe})\sim v_{Ti}\ll v_{Te}$. Consequently, $\mathbf{k}$ and $\mathbf{k}'$ must be sufficiently close to each other, or $|\mathbf{k}-\mathbf{k}'|\sim|\delta\mathbf{k}|\ll 1$. We thus employ the Taylor series expansion to obtain

$$\begin{aligned} 0 &= \int d(\delta\mathbf{k})\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\omega_k-\omega_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] \\ &\quad \times(\delta k)\left(\omega(k)\frac{dI_L(k)}{dk}+\frac{4\pi^2}{T_i}\frac{d\omega(k)}{dk}[I_L(k)]^2-\frac{d\omega(k)}{dk}I_L(k)\right)f_i. \end{aligned} \tag{30}$$

The necessary condition for equality leads to the spectrum,

$$I_L(k)=\frac{T_i}{4\pi^2}\left(1+\frac{2}{3}\frac{\kappa-3/2}{\kappa'\,k^2\theta_e/m_e\omega_{pe}^2}\right), \tag{31}$$

which is alternative to the earlier solution (27). Obviously the two expressions must be identical. If we identify

$$\kappa=\frac{9}{4}+\frac{3}{2}\mathscr{H}=2.25+1.5\mathscr{H}, \qquad \kappa'\theta_e=(\kappa+1)\,T_i, \tag{32}$$

then we may reconcile the two expressions, where we have treated $\mathscr{H}$ as constant. Such a reconciliation between (27) and (31) would not have been possible had we chosen f_e other than the kappa-like model (23). This amounts to the uniqueness proof for the kappa distribution as being associated with the steady state Langmuir turbulence.

To summarize the findings, the electron kappa distribution function represents a plasma state in quasi equilibrium with weak Langmuir turbulence, and the desired final form of f_e and $I_L(k)$ are given by

$$\begin{aligned} f_e(v) &= \frac{m_e^{3/2}}{(2\pi T_e)^{3/2}} \frac{\Gamma(\kappa+1)}{(\kappa-3/2)^{3/2}\Gamma(\kappa-1/2)} \left(1+\frac{1}{\kappa-3/2}\frac{m_e v^2}{2T_e}\right)^{-\kappa-1}, \\ I_L(k) &= \frac{T_e}{4\pi^2}\frac{\kappa-3/2}{\kappa+1}\left(1+\frac{1+2\mathscr{H}}{\kappa-3/2}\frac{2\pi n e^2}{k^2 T_e}\right), \\ \kappa &= \frac{9}{4}+\frac{3\mathscr{H}}{2} = 2.25+1.5\mathscr{H}, \qquad \frac{T_i}{T_e} = \frac{\kappa-3/2}{\kappa+1} = \frac{3+6\mathscr{H}}{13+6\mathscr{H}}. \end{aligned} \tag{33}$$

This solution is an indirect evidence that the turbulent equilibrium in plasmas may be equivalent to the non-extensive statistical state. As we have pointed out in the Introduction, the most probable state that maximizes the Tsallis non-extensive entropy is the kappa distribution function. The steady state of plasma turbulence is also characterized by the same kappa distribution, which thus indicates that the two approaches are describing the same statistical state.

The kappa electron velocity distribution function may also characterize the solar wind. For suprathermal velocity range, $v \gg v_{Te}$, the kappa electron distribution (33) behaves as an inverse power law distribution,

$$f_e \sim v^{-6.5}, \tag{34}$$

since $\kappa \approx 9/4 = 2.25$, assuming $\mathscr{H}$ can be ignored. If we recall that, while the solar wind electrons can be modeled by a combination of Maxwellian core, suprathermal halo, and superhalo, it is the comparison with superhalo, which is most useful, since these electrons are at the high end of the velocity spectrum [20,21]. Observation near Earth orbit shows that superhalo electrons behave as $f_e \sim v^{-5.0}$ to $v^{-8.7}$ with average behavior [22]

$$f_e^{\rm obs} \sim v^{-6.69}, \qquad v \gg v_{Te}. \tag{35}$$

This agrees quite well with (34). There exists further evidence to support our interpretation that the solar wind electrons are in turbulent equilibrium state with high-frequency Langmuir fluctuations, or equivalently, they can be characterized by the non-extensive statistical state. Reference [28] analyzed the solar wind halo electrons, and analyzed *Helios*, *Cluster*, and *Ulysses* spacecraft data. The authors show that the value of observed κ decreases from ~ 9 near 0.3 AU to ~ 4 near 1 AU, to ~ 2.25 near ~ 5 AU. This strongly implies that as the solar wind evolves radially and thus approaches the quasi equilibrium state, the distinction between the halo and superhalo electrons disappear, and the κ index approaches closer and closer to the theoretically predicted value.

3. The Question of True Thermodynamic Equilibrium for Space Plasma

We have thus far argued that the space plasma in the heliosphere may be in the state of quasi equilibrium in which the particles constantly exchange momentum and energy with the long-ranged collective fluctuations, thus maintaining the kappa distribution function. By inference with the non-extensive entropic principle, we have also made a conjecture that the turbulent quasi equilibrium for space plasma may be alternatively described within the framework of non-extensive statistical concept. The question that naturally arises is the problem of true thermodynamic equilibrium, and whether the space plasma can ever attain such a state. For collision-poor space plasmas the true thermodynamic equilibrium state may be reached, but from a theoretical point of view, turbulent quasi equilibrium state must eventually relax to the true thermodynamic equilibrium state through binary collisions. The road to true thermodynamic equilibrium can be discussed on the basis of kinetic plasma theory, but the analysis requires the knowledge of the time scales associated with the collisional relaxation, as opposed to the time scales that govern the formation of turbulent quasi equilibrium state.

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In general it is expected that the collisional relaxation time scale is much longer than that of turbulent equilibrium formation time scale, but the quantitative estimate is not so easy.

At the moment, we are not able to address the issue of time scales of collisional relaxation process. However, it is possible to discuss the theoretical framework that includes both turbulent quasi equilibrium state and collisionally relaxed thermodynamic state within a single framework of steady-state plasma equation. It is done by generalizing the particle kinetic equation (14) through addition of collisional operator. The basic theory may be developed on the basis of the electron kinetic equation that includes the influence of collective (Langmuir wave) fluctuations and binary collisions, which can be found in Ref. [29],

$$\begin{aligned}\frac{\partial f_e}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\left(A_v + A_v^c\right) f_e\right] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\left(D_{vv}+D_{vv}^c\right)\frac{\partial f_e}{\partial v}\right)\\ &+\frac{1}{v^2}\frac{\partial}{\partial \mu}\left(\left(1-\mu^2\right)\left(D_{\mu\mu}+D_{\mu\mu}^c\right)\frac{\partial f_e}{\partial \mu}\right),\end{aligned} \tag{36}$$

where the particle kinetic equation that generalizes (14) for the electrons is now expressed in spherical velocity coordinate. The velocity space friction and diffusion coefficients, $\mathbf{A}$ and D_{ij}, respectively, are the same those defined in (14), and the additional coefficients $\mathbf{A}^c$ and D_{ij}^c, respectively, pertain to collisional effects. Non-vanishing elements of these coefficients are given in spherical coordinate variables as follows:

$$\begin{aligned}A_v &= \frac{e^2\omega_{pe}^2}{m_e v^2}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k},\\ D_{vv} &= \frac{4\pi^2 e^2\omega_{pe}^2}{m_e^2 v^3}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\, I_L(k),\\ D_{\mu\mu} &= \frac{4\pi^2 e^2 (k^2v^2-\omega_{pe}^2)}{m_e^2 v}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\, I_L(k),\\ A_v^c &= \frac{4\pi n e^4 \ln\Lambda}{m_e^2}\frac{2}{v_{Te}^2}\left(G(x_e)+\frac{T_e}{T_i}G(x_i)\right),\\ D_{vv}^c &= \frac{4\pi n e^4\ln\Lambda}{m_e^2}\frac{G(x_e)+G(x_i)}{v},\\ D_{\mu\mu}^c &= \frac{4\pi n e^4\ln\Lambda}{m_e^2}\frac{1}{\sqrt{\pi}v}\left(\frac{e^{-x_e^2}}{x_e}+\frac{e^{-x_i^2}}{x_i}\right),\\ x_e &= \frac{v}{v_{Te}},\qquad x_i=\frac{v}{v_{Ti}},\qquad \Lambda = 4\pi n\lambda_{De}^3,\\ G(x) &= \frac{\mathrm{erf}(x)-(2/\sqrt{\pi})\,x\,e^{-x^2}}{2x^2}.\end{aligned} \tag{37}$$

In the collisional coefficients defined here, we took the approach of treating the collisional processes that involves electrons scattering off Maxwellian distribution of charged particles via Rosenbluth potential approximation [30].

If we assume that f_e is isotropic, then the steady-state solution is given by

$$\begin{aligned}f_e &= \text{const}\exp\left(-\int dv\,\frac{A_v+A_v^c}{D_{vv}+D_{vv}^c}\right)\\ &= C\exp\left(-\int dv\,\frac{v\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}+\frac{m_e v^3}{T_e}\ln\Lambda\left(G(x_e)+\frac{T_e}{T_i}G(x_i)\right)}{\displaystyle\frac{4\pi^2}{m_e}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\, I_L(k)+v^2\ln\Lambda\left[G(x_e)+G(x_i)\right]}\right).\end{aligned} \tag{38}$$

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This solution can be used to discuss either thermal equilibrium that is attained by collisional process, or turbulent quasi equilibrium state attained through collective fluctuations. If we ignore contribution from the collective fluctuations, that is, if we ignore the k integral terms in the numerator and denominator, then we have

$$\begin{aligned} f_e &= C\exp\left(-\frac{m_e}{T_e}\int dv v\,\frac{G(x_e)+\frac{T_e}{T_i}G(x_i)}{G(x_e)+G(x_i)}\right) \\ &= C\exp\left(-\frac{m_e v^2}{2T}\right), \end{aligned} \tag{39}$$

where in going from the first to second equality, we have assumed $T_e = T_i = T$. This is the thermal equilibrium distribution, as expected. On the other hand, if we ignore the collisional part dictated by $\ln\Lambda$, then we have

$$f_e = C\exp\left(-\frac{m_e}{4\pi^2}\int dv v\,\frac{\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}}{\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}I_L(k)}\right), \tag{40}$$

which is the same as (20). As we already saw, this formal solution leads to the kappa distribution, provided the fluctuation spectrum is specified by the mathematical form given in (33).

4. Discussion

To summarize the essential findings of the present paper, we have argued for an inter-relationship that may exist between the non-extensive statistical description of plasma, in which long-ranged electromagnetic force is involved, and the quasi steady state plasma turbulent state. Both descriptions share a common feature in that the equilibrium distribution function corresponds to the kappa distribution, or equivalently, the q distribution. In the non-extensive statistical approach, the q parameter is undetermined, but the plasma turbulence theory can be invoked in order to determine its value via the relationship $q = (\kappa - 1)/\kappa$. If we adopt $\kappa = 9/4 = 2.25$, then we find that $q = 5/9$. We have also verified the theoretical prediction of $\kappa = 9/4 = 2.25$ against spacecraft observations and found reasonable agreement. Finally, we have also briefly addressed the issue of including the effects of collisional relaxation in the general formalism.

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