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Dynamic of stochastic epidemic model based on the association between susceptible and recovered individuals

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Abstract: In this paper, we propose a new mathematical model based on the association between susceptible and recovered individual, where the association between susceptible and recovered individual is disturbed by white noise. This model is based on demographic changes and is used for long term behavior. We study the stability of equilibria of the deterministic model and prove the conditions for the extinction of diseases. Then, we investigate and obtain the critical condition of the stochastic epidemic model for the extinction and the permanence in mean of the disease with the white noise. To verify our results, we present some numerical simulations for real data related to disease.

Keywords: extinction; permanence in mean; stability; stochastic epidemic model

1. Introduction

Things are connected in the real world, and in the biological world as well. Epidemic diseases are commonly established through deterministic models in which populations that transmit disease are divided into three categories, such as susceptible, infected, and recovered individuals. In special situation, the connection between the three individuals will lead to their dynamic change.

The Kermack-McKendrick model [1] is a SIR model for the number of people infected with infectious diseases in closed populations. It assumes that the size of the population is fixed (such as not being born, dying from disease, or dying from natural causes), that the incubation period of infectious factors is immediate. It also assumes closed populations with no age, space or social structure. The Kermack-McKendrick model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = -\lambda S(t)I(t) \\ \frac{dI(t)}{dt} = (\lambda S(t)I(t) - \gamma I(t)) \\ \frac{dR(t)}{dt} = \gamma I(t) \end{cases} \quad (1.1)$$

where t is time, $S(t)$ is the number of susceptible people, $I(t)$ is the number of people infected, $R(t)$ is the number of people who have recovered and developed permanence immunity to the infection, λ is the infection rate, and γ is the recovery rate. In addition, we know that the constant $\mathcal{R}_0 = \frac{\lambda S_0}{\gamma}$, which is called the basic reproduction number, determines the most important quantities of epidemic behavior and potential in 1.1. In particular, \mathcal{R}_0 also determines whether an epidemic occurs or not.

A mathematical survey was carried out on the progress of the epidemic diseases in the same population. Overall, the threshold density of the population has been found, which depends on the epidemic's infectivity, recovery and mortality. If population density is below this threshold, there will

23 be no epidemic. In addition, if population density is slightly above the threshold, the impact of the
 24 epidemic will be to reduce density below the threshold [1]. Many articles apply this theory such as
 25 [3,4].

Some infections, such as those from the common cold and flu, do not confer any lasting immunity. The infection is not immunized after recovery from infection, and individuals become susceptible to infection[2]. Thus the model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = -\lambda S(t)I(t) + \gamma I(t) \\ \frac{dI(t)}{dt} = (\lambda S(t)I(t) - \gamma I(t)) \end{cases} \quad (1.2)$$

26 This model is called SIS model, also known as the contact process, is one of which has been
 27 conducted extensively from deterministic to stochastic point of view under a variety of assumptions
 28 [5–7]. Gray will discuss in this paper the effect of stochastic noise on the well-known SIS epidemic
 29 model [8]. The dynamics of a stochastic model with vaccination was discussed by [9–11,13]. Usually
 30 SDE can also be discussed with time delays influenced was discussed by [12–15].

31 In addition, there are infectious disease models such as SI(see,e.g.[16,17]), SIRS(see,e.g.[18][19]),
 32 SEIR(see,e.g.[20,21]), MSIR(see,e.g.[22]). A number of differential equation models were developed to
 33 describe AIDS(see,e.g.[23]) hepatitis B(see,e.g.[24–29]), and so on. Motivated by this fact, Tahir Khan,
 34 Amir Khan and Gul Zaman [29] considered the following hepatitis B epidemic model with the aid of
 35 diagram shown in Fig. 1:

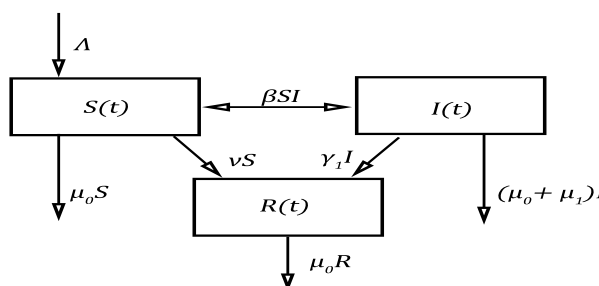


Figure 1. Transfer diagram of hepatitis B epidemic model

And deterministic epidemic model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta S(t)I(t) - (\mu_0 + \nu)S(t) \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu_0 + \mu_1 + \gamma_1)I(t) \\ \frac{dR(t)}{dt} = \gamma_1 I(t) + \nu S(t) - \mu_0 R(t) \end{cases} \quad (1.3)$$

36 where $S(t)$ denotes the number of members of a population who are susceptible to an infection at
 37 time t . $I(t)$ denotes the number of members who are infective at time t . $R(t)$ denotes the number of
 38 members who are recovered with an infection at time t as the result of vaccination. The parameters
 39 in the model are summarized in the following: β represents transmission rate between $S(t)$ and $I(t)$;
 40 Λ represents the per capita constant birth rate; μ_0 and μ_1 respectively represent the natural death
 41 rate and the disease induced death rate; ν represents the vaccination rate; γ_1 represents the constant
 42 recovery rate for the disease infected individual. All parameter values are assumed to be nonnegative,
 43 and $\mu_0, \Lambda > 0$.

44 It is now considered that the recovered individuals do not have permanent immunity. Because of
 45 environments, specific diseases, malnutritions, fatigue mental pressures and some drugs, the number

46 of recovered classes and susceptible classes can transfer to each other. This rate of transformation is
 47 assumed to be α , and the new model is shown below and the diagram shown in Fig.2,

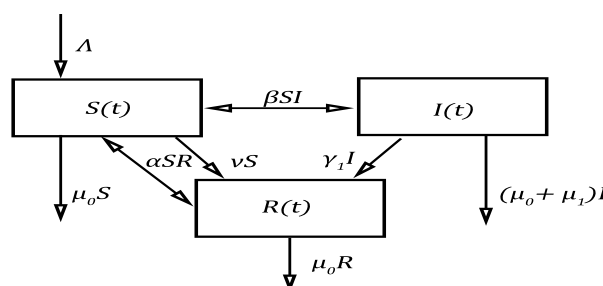


Figure 2. Transfer diagram of epidemic model based on the association between susceptible and recovered individuals

Based on the infectious disease model we introduced above, we first propose a deterministic epidemiological model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta S(t)I(t) + \alpha S(t)R(t) - (\mu_0 + \nu)S(t) \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu_0 + \mu_1 + \gamma_1)I(t) \\ \frac{dR(t)}{dt} = \gamma_1 I(t) + \nu S(t) - \mu_0 R(t) - \alpha S(t)R(t) \end{cases} \quad (1.4)$$

48 where α presents transmission rate between $S(t)$ and $R(t)$.

Taking the effect of randomly fluctuating environment into consideration in real world, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations transmission parameter α and β , i.e., $\beta \rightarrow \eta_1 \dot{B}(t)$ and $\alpha \rightarrow \eta_2 \dot{B}(t)$, where $B(t)$ is standard Brownian motion with the property $B(0) = 0$ and with the intensity of white noise $\eta_1^2 > 0$, $\eta_2^2 > 0$. Stochastic epidemic model as follows:

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \alpha S(t)R(t) - (\mu_0 + \nu)S(t)) dt - \eta_1 S(t)I(t) dB_1(t) \\ \quad - \eta_2 S(t)R(t) dB_2(t) \\ dI(t) = (\beta S(t)I(t) - (\mu_0 + \mu_1 + \gamma_1)I(t)) dt + \eta_1 S(t)I(t) dB_1(t) \\ dR(t) = (\gamma_1 I(t) + \nu S(t) - \mu_0 R(t) + \alpha S(t)R(t)) dt + \eta_2 S(t)R(t) dB_2(t) \end{cases} \quad (1.5)$$

49 It establishes a stochastic epidemic disease model based on the association between susceptible
 50 and recovered individuals with a varying population environment for a long term behavior. We
 51 discuss the disease extinction, the disease persistence in mean and obtain sufficient conditions for
 52 them. To verify our results, we present some numerical simulations for real data related to disease.

53 2. the Dynamics of deterministic system 1.4

54 From a mathematical point of view, throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be a complete
 55 probability space, $R_+^3 = \{x_i > 0, i = 1, 2, 3\}$. f is an integrable function on $[0, \infty)$, $\langle f(t) \rangle = \frac{1}{t} \int_0^t f(\theta) d\theta$.
 56 Then we have

Def 2.1. (i) The diseases $I(t)$ are said to be extinctive if $\lim_{t \rightarrow \infty} I(t) = 0$.

(ii) The diseases $I(t)$ are said to be permanent in mean if there exist two positive constants C such that

$$\inf \lim_{t \rightarrow \infty} \langle I(t) \rangle \geq C$$

Lemma 2.2. For any positive solution $(S(t), I(t), R(t))$ of system 1.1 or 1.2 with initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, we have

$$\max \left\{ \sup \lim_{t \rightarrow \infty} S(t), \sup \lim_{t \rightarrow \infty} I(t), \sup \lim_{t \rightarrow \infty} R(t) \right\} \leq \frac{\Lambda}{\mu_0}$$

58

Proof. From system 1.1 or the system 1.2, we have

$$\begin{aligned} \frac{d(S(t) + I(t) + R(t))}{dt} &= \Lambda - \mu_0(S(t) + I(t) + R(t)) - \mu_1 I(t) \\ &\leq \Lambda - \mu_0(S(t) + I(t) + R(t)) \end{aligned}$$

This implies that

$$\lim_{t \rightarrow \infty} (S(t) + I(t) + R(t)) \leq \frac{\Lambda}{\mu_0}$$

Then obviously we have

$$\sup \lim_{t \rightarrow \infty} S(t) \leq \frac{\Lambda}{\mu_0}, \sup \lim_{t \rightarrow \infty} I(t) \leq \frac{\Lambda}{\mu_0}, \sup \lim_{t \rightarrow \infty} R(t) \leq \frac{\Lambda}{\mu_0}$$

59 Since $S(t) > 0, I(t) > 0, R(t) > 0$. This completes the proof of lemma 2.2.

60

In system 1.1, let

$$\begin{aligned} f_1 &= \Lambda - \beta S(t)I(t) - \alpha S(t)R(t) - (\mu_0 + \nu)S(t) \\ f_2 &= \beta S(t)I(t) - (\mu_0 + \mu_1 + \gamma_1)I(t) \\ f_3 &= \gamma_1 I(t) + \nu S(t) - \mu_0 R(t) + \alpha S(t)R(t) \end{aligned}$$

61 Then setting $f_i = 0$ leads to the following equilibria:

$E_1 : (\tilde{S}, 0, \tilde{R})$ with

$$\tilde{S} = \frac{\Lambda}{\mu_0} - \tilde{R}, \tilde{R} = \frac{\left(\frac{\alpha\Lambda}{\mu_0} - (\nu + \mu_0)\right) + \sqrt{\left(\frac{\alpha\Lambda}{\mu_0} - (\nu + \mu_0)\right)^2 + \frac{4\alpha\nu\Lambda}{\mu_0}}}{2\alpha}$$

$E_2 : (S^*, I^*, R^*)$ with

$$\begin{aligned} S^* &= \frac{\mu_0 + \mu_1 + \mu_1}{\beta} \\ I^* &= \frac{\left(-\nu\beta\mu_0 - \mu_0(\beta\mu_0 - \alpha(\mu_0 + \mu_1 + \gamma_1)) - \alpha\beta\Lambda\right)(\mu_0 + \mu_1 + \gamma_1) + \Lambda\beta^2\mu_0}{\beta\left(\gamma_1\beta\mu_0 + (\mu_0 + \mu_1)(\beta\mu_0 - \alpha(\mu_0 + \mu_1 + \gamma_1))\right)} \\ R^* &= \frac{\Lambda\gamma_1\beta - (\mu_0 + \mu_1 + \gamma_1)(\mu_0\gamma_1 - \mu_0\nu - \mu_1\nu)}{\gamma_1\beta\mu_0 + (\mu_0 + \mu_1)(\beta\mu_0 - \alpha(\mu_0 + \mu_1 + \gamma_1))} \end{aligned}$$

From the expressions of S^*, I^* and R^* , we know of

$$\tilde{S} > 0, I^* > 0, R^* > 0$$

System 1.1 has positive equilibrium E^* , furthermore, let

$$\mathcal{R} = \frac{\beta\Lambda}{(\mu_0 + \mu_1 + \gamma_1)\mu_0}$$

62 **Theorem 2.3.** For system 1.1, the following conclusion are true:

63 (i) If $\mathcal{R} < 1$, then the disease I goes extinct.

64 (ii) E_2 is a unstable equilibrium.

Proof. The stability of the equilibrium point $(\tilde{S}, 0, \tilde{R})$ of system 1.1 is determined by the Jacobian

$$J_1 = \begin{pmatrix} -\alpha\tilde{R} - (\mu_0 + \nu) & 0 & \nu + \alpha\tilde{R} \\ -\beta\tilde{S} & \beta S - (\mu_0 + \mu_1 + \gamma_1) & \gamma_1 \\ -\alpha\tilde{S} & 0 & -\mu_0 + \alpha\tilde{S} \end{pmatrix}$$

Since $\mathcal{R} < 1$, one of three eigenvalues of matrix J_1 is given by

$$\lambda_1 = \beta\tilde{S} - (\mu_0 + \mu_1 + \gamma_1) < \frac{\beta\Lambda}{\mu_0} - (\mu_0 + \mu_1 + \gamma_1) < 0$$

The other two eigenvalues of matrix J_1 are root of the following equation

$$\lambda^2 + (-a_{11} - a_{33})\lambda + (-a_{11}a_{33} - a_{31}a_{13}) = 0$$

65 With the help of $\nu > \mu_0$, where

$$66 \quad a_{11} = -\alpha\tilde{R} - (\mu_0 + \nu) = -\frac{\alpha\Lambda}{2\mu_0} - \frac{1}{2}\sqrt{\left(\frac{\alpha\Lambda}{\mu_0} - (\nu + \mu_0)\right)^2 + \frac{4\alpha\nu\Lambda}{\mu_0}} < 0$$

$$a_{33} = -\mu_0 + \alpha\tilde{S}$$

$$67 \quad = \frac{-\mu_0 + \frac{\alpha\Lambda}{\mu_0} + \nu + \sqrt{\frac{\alpha^2\Lambda^2}{\mu_0^2} + (\nu + \mu_0)^2 + \frac{2(\nu - \mu_0)\alpha\Lambda}{\mu_0}}}{2}$$

$$> 0$$

$$68 \quad a_{13} = \nu + \alpha\tilde{R}, a_{31} = -\alpha\tilde{S}$$

69 Then $-a_{11} - a_{33} > 0$ and $-a_{11}a_{33} - a_{31}a_{13} > 0$. This implies $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus the
70 equilibrium E_1 is stable. This means the disease with relationship of between susceptible and recovered
71 individuals goes extinct.

Now let us prove instability of the equilibrium point E_2 . At E_2 the Jacobian takes the form of

$$J_2 = \begin{pmatrix} -\beta I^* - \alpha R^* - (\mu_0 + \nu) & \beta I^* & \nu + \alpha R^* \\ -(\mu_0 + \mu_1 + \gamma_1) & 0 & \gamma_1 \\ \frac{\alpha(\mu_0 + \mu_1 + \gamma_1)}{\beta} & 0 & -\mu_0 + \alpha S \end{pmatrix}$$

72 since $\lambda_1^* = \beta I^* > 0$, E_2 is a unstable equilibrium. The proof is completed.

73 3. Dynamics of stochastic system 1.5

74 In the following section, the extinction of the system 1.2 infectious disease under random
75 disturbance of white noise will be discussed. In order for SDE model 1.2 to have research value,
76 we need to at least prove that this SDE model does have a unique global solution. The existing general
77 existence-and-uniqueness theorem on SDEs is not applicable to this particular SDE in order to ensure
78 these properties. Therefore, new theories need to be established.

79 **Theorem 3.1.** For an initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. The solution $(S(t), I(t), R(t))$ of the proposed
80 stochastic epidemic model 1.2 is unique, for $t \geq 0$. Moreover, the solution remain in \mathbb{R}_+^3 with probability 1.

Proof. Our proof is motivated by the works of Mao et al. [30] and [31]. It is clear that the coefficients of the equation of the model are locally Lipschitz continuous for any given initial size of population $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. It follows that there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e]$, where τ_e is the explosion time (for detail see the reference [23]). To show that this solution is global, we prove that $\tau_e = \infty$ a.s. Let $k_0 \geq 0$ be sufficiently large, so that $S(t), I(t)$ and $R(t)$ all lie within the

interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$.

Define the stopping time

$$\begin{aligned} \tau_k = \{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \\ \leq \frac{1}{k} \text{ or } \max\{S(t), I(t), R(t)\}\} \end{aligned} \tag{3.1}$$

We set $\inf \emptyset = \infty$ as usual. According to the definition. τ_k increases as $k \rightarrow \infty$. set $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$. $\tau_\infty \leq \tau_e$ a.s.

If we can show that $\tau_\infty = \infty$ a.s. then $\tau_e = \infty$. and $(S(0), I(0), R(0)) \in \mathcal{R}_+^3$ a.s. $\forall t \geq 0$. We need to show that $\tau_e = \infty$ a.s.

If this statement is false. then there exist a pair of constants $T \geq 0$ and $\varepsilon \in (0, 1)$ s.t.

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{3.2}$$

81 Hence there is an integer $k_1 \geq k_0$. such that

$$82 P\{\tau_k \leq T\} > \varepsilon \forall k \geq k_1.$$

Let $N(t) = S(t) + I(t) + R(t)$ for $t \leq \tau_k$.

$$\begin{aligned} dN(t) &= d(S(t) + I(t) + R(t)) \\ &= (\Lambda - \alpha S(t)R(t) - \mu_0 S(t) - (\mu_0 + \mu_1)I(t) - \mu_0 R(t)) dt \\ &= (\Lambda - \alpha S(t)R(t) - \mu_0 N(t) - \mu_1 I(t)) dt \\ &\leq (\Lambda - \mu_0 N(t)) dt \end{aligned} \tag{3.3}$$

$$N(t) = \begin{cases} \frac{\Lambda}{\mu_0}, & \text{If } N(0) \leq \frac{\Lambda}{\mu_0} \\ N(0), & N(0) \geq \frac{\Lambda}{\mu_0} \end{cases} := M \tag{3.4}$$

Now, we define a \mathcal{C}^2 -function $V : \mathcal{R}_+^3 \rightarrow \mathcal{R}_+$. Such that

$$V(S, I, R) = S + I + R - 3 - (\ln S + \ln I + \ln R) \geq 0 \tag{3.5}$$

which can be seen from $y - 1 - \ln y \geq 0 \forall y > 0$.

Let $\forall k \geq k_0, \forall T > 0$. The application of *Itô formula*.

$$\begin{aligned} dV(S, I, R) &= \left(1 - \frac{1}{S}\right) dS + \frac{1}{2S^2} (dS)^2 + \left(1 - \frac{1}{I}\right) dI + \frac{1}{2I^2} (dI)^2 + \\ &\quad \left(1 - \frac{1}{R}\right) dR + \frac{1}{2R^2} (dR)^2 \\ &= LV(S, I, R) dt + \eta(1 - S) dB(t) \end{aligned} \tag{3.6}$$

$LV : \mathcal{R}_+^3 \rightarrow \mathcal{R}_+$ is defined by the following equation,

$$\begin{aligned} LV(S, I, R) &= \left(1 - \frac{1}{S}\right) (\Lambda - \beta S(t)I(t) - \alpha S(t)R(t) - (\mu_0 + \nu)S(t)) + \frac{1}{2}\eta_1^2 I^2 \\ &\quad + \frac{1}{2}\eta_2^2 R^2 + \left(1 - \frac{1}{I}\right) (\beta S(t)I(t) - (\mu_0 + \mu_1 + \gamma_1)I(t)) + \frac{1}{2}\eta_1^2 S^2 \\ &\quad + \left(1 - \frac{1}{R}\right) (\gamma_1 I(t) + \nu S(t) - \mu_0 R(t) + \alpha S(t)R(t)) + \frac{1}{2}\eta_2^2 S^2 \\ &\leq \Lambda + \beta I(t) + \alpha R(t) + (\mu_0 + \nu) + (\mu_0 + \mu_1 + \gamma_1) + \mu_0 \\ &\quad + \frac{1}{2}\eta_1^2 (S^2(t) + I^2(t)) + \frac{1}{2}\eta_2^2 (S^2(t) + R^2(t)) \\ &\leq \Lambda + 3\mu_0 + \nu + \mu_1 + \eta_1^2 M^2 + \eta_2^2 M^2 + (\beta + \alpha)M := K \end{aligned} \tag{3.7}$$

For the convenience of reader we cite the generalized *Itô formula* :If $V \in C^2(\mathcal{R}_+^3 \rightarrow \mathcal{R}_+)$, then for any stopping times $0 \leq \tau_1 \leq \tau_2 < \infty$

$$EV(x_1(\tau_2), x_2(\tau_2), x_3(\tau_2)) = EV(x_1(\tau_1), x_2(\tau_1), x_3(\tau_1)) + E \int_{\tau_1}^{\tau_2} LV(x_1(t), x_2(t), x_3(t)) dt$$

Consequently

$$\begin{aligned} & E[V(S(\tau_k \wedge T), I(\tau_k \wedge T), R(\tau_k \wedge T))] \\ & \leq E[V(S(0), I(0), R(0))] + E\left[\int_0^{\tau_k \wedge T} K dt\right] \\ & \leq V(S(0), I(0), R(0)) + KT \end{aligned} \quad (3.8)$$

Setting $\Omega_k = \tau_k \leq T$, for $k \geq k_1$. As a result, reads $P(\Omega_k) \geq \varepsilon$. Note that for every $\omega \in \Omega_k$, there exists at least one $S(\tau_k, \omega), I(\tau_k, \omega), R(\tau_k, \omega)$ that equal k or $\frac{1}{k}$, and hence $V(S(\tau_k), I(\tau_k), R(\tau_k))$ is not less than $k - 1 - \ln k$ or $\frac{1}{k} - 1 + \ln k$.

$$V(S(\tau_k), I(\tau_k), R(\tau_k)) \geq E(k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right) \quad (3.9)$$

It then follows from 3.2 and 3.8 that

$$\begin{aligned} E[V(S(0), I(0), R(0))] + kT & \geq E[1_{\Omega(\omega)} V(S(\tau_k), I(\tau_k), R(\tau_k))] \\ & \geq [(k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right)] \end{aligned} \quad (3.10)$$

$1_{\Omega(\omega)}$ is the indicator function of $\Omega, k \rightarrow \infty$.

$$\infty > V(S(0), I(0), R(0)) + MT = \infty$$

83 Which implies $\tau_\infty = \infty$ a.s.

Lemma 3.2. Let $(S(t), I(t), R(t))$ be a solution of system 1.2 with initial value $(S(0), I(0), R(0)) \in \mathcal{R}_+^3$. Then

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \eta_1 S(s) I(s) dB_1(s)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t \eta_1 S(s) R(s) dB_2(s)}{t} = 0$$

84

Proof. Let

$$X(t) = \int_0^t \eta_1 S(s) I(s) dB_1(s)$$

$$Y(t) = \int_0^t \eta_1 S(s) R(s) dB_2(s)$$

By the Burkholder-Davis-Gundy inequality in [30] and Lemma 2.3, we have

$$\begin{aligned} E\left[\sup_{0 \leq s \leq t} |X(s)|^\theta\right] & \leq C_\theta E\left[\int_0^t S^2(s) I^2(s) dB_1(s)\right]^{\frac{\theta}{2}} \\ & \leq C_\theta t^{\frac{\theta}{2}} E\left[\sup_{0 \leq s \leq t} S^\theta(s) I^\theta(s)\right] \\ & \leq M_\theta C_\theta t^{\frac{\theta}{2}} \end{aligned}$$

where $M_\theta = \frac{\Lambda^\theta}{\mu_0^\theta}$

Let ε be an arbitrary positive constant. Then

$$\begin{aligned} & P\{\omega : \sup_{k\delta \leq t \leq (k+1)\delta} |X(t)|^\theta > (k\delta)^{1+\varepsilon+\frac{\theta}{2}}\} \\ & \leq \frac{E(|X((k+1)\delta)|^\theta)}{(k\delta)^{1+\varepsilon+\frac{\theta}{2}}} \\ & \leq \frac{2^{\frac{\theta}{2}} M_\theta C_\theta}{(k\delta)^{1+\varepsilon}} \end{aligned}$$

By Doob's martingale inequality and the Borel-Cantelli lemma in [30], for almost all $\omega \in \Omega$. We get that

$$\sup_{1 < \delta \leq t \leq (k+1)\delta} |X(t)|^\theta \leq (k\delta)^{1+\varepsilon+\frac{\theta}{2}} \quad (3.11)$$

hold for all but finitely many k . Thus, there exists a positive $k_0(\omega)$. Hence, if $k \geq k_0(\omega)$ and $1 < \delta \leq t \leq (k+1)\delta$. for almost all $\omega \in \Omega$, then

$$\frac{\ln |X(t)|^\theta}{\ln t} \leq \frac{(1 + \varepsilon + \frac{\theta}{2}) \ln(k\delta)}{\ln(k\delta)} = 1 + \varepsilon + \frac{\theta}{2}$$

so, we have

$$\limsup_{t \rightarrow \infty} \frac{\ln |X(t)|}{\ln t} \leq \frac{1 + \varepsilon + \frac{\theta}{2}}{\theta}$$

Let $\varepsilon \rightarrow 0$, then we obtain that

$$\limsup_{t \rightarrow \infty} \frac{\ln |X(t)|}{\ln t} \leq \frac{1}{2} + \frac{1}{\theta}$$

Then, for arbitrary small positive constant $\varepsilon (\varepsilon \leq \frac{1}{2} - \frac{1}{\theta})$

There exist a constant $T(\omega)$ and a set Ω_ε , such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and for $t \geq T(\omega)$. $\omega \in \Omega_\varepsilon$.

$$\ln |X(t)| \leq (\frac{1}{2} + \frac{1}{\theta} + \varepsilon) \ln t$$

Therefore

$$\limsup_{t \rightarrow \infty} \frac{\ln |X(t)|}{t} \leq \limsup_{t \rightarrow \infty} \frac{t^{\frac{1}{2} + \frac{1}{\theta} + \varepsilon}}{t} = 0$$

Notice that

$$\liminf_{t \rightarrow \infty} \frac{\ln |X(t)|}{t} \geq 0$$

Then we have

$$\lim_{t \rightarrow \infty} \frac{\ln |X(t)|}{t} = 0$$

i.e.

$$\lim_{t \rightarrow \infty} \frac{\ln |X(t)|}{t} = \lim_{t \rightarrow \infty} \frac{\int_0^t \eta_1 S(s) I(s) dB_1(s)}{t} = 0$$

By the same argument, we can also obtain

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \eta_2 S(s) R(s) dB_2(s)}{t} = 0$$

85 **Lemma 3.3.** [9] Let $M = M_{t \geq 0}$ be a real valued continuous local martingale vanishing at $t = 0$, then

86 $\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty$ a.s. implies that $\lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0$ a.s. and also

87 $\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < 0$ a.s. implies that $\lim_{t \rightarrow \infty} \frac{M_t}{t} = 0$ a.s.

88 **Lemma 3.4.** [32] Let $f \in C[0, \infty) \times \Omega(0, \infty)$ and $F(t) \in C([0, \infty) \times \Omega, \mathbb{R})$, If there exist positive constants

89 λ_0, λ and T such that

90 $\ln f(t) \leq \lambda t - \lambda_0 \int_0^t f(s) ds + F(t)$ a.s. for all $t \geq T$ and

91 $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ a.s. then $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds \leq \frac{\lambda}{\lambda_0}$ a.s.

92 3.1. Extinct

Theorem 3.5. Let $(S(t), I(t), R(t))$ be the solution of 1.2. with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. if

$$\mathcal{R}_0 - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2(\mu_0 + \mu_1 + \gamma_1)} < 1$$

$$\frac{\eta_1^2 \Lambda}{\mu_0} < \beta$$

hold, then the disease goes to extinction almost surely, i.e.

$$\lim_{t \rightarrow \infty} I(t) \rightarrow 0$$

93 .

Proof. The integration of the proposed stochastic epidemic model leads to be following system of equations

$$\begin{aligned} \frac{S(t) - S(0)}{t} &= \Lambda - \beta \langle S(t)I(t) \rangle - \alpha \langle S(t)R(t) \rangle - (\mu_0 + \nu) \langle S(t) \rangle \\ &\quad - \frac{\eta_1}{t} \int_0^t S(s)I(s) dB_1(s) - \frac{\eta_2}{t} \int_0^t S(s)R(s) dB_2(s) \\ \frac{I(t) - I(0)}{t} &= \beta \langle S(t)I(t) \rangle - (\mu_0 + \mu_1 + \gamma_1) \langle I(t) \rangle + \frac{\eta_1}{t} \int_0^t S(s)I(s) dB_1(s) \\ \frac{R(t) - R(0)}{t} &= \gamma_1 \langle I(t) \rangle + \nu \langle S(t) \rangle - \mu_0 \langle R(t) \rangle + \alpha \langle S(t)R(t) \rangle \\ &\quad + \frac{\eta_2}{t} \int_0^t S(s)R(s) dB_2(s) \end{aligned} \quad (3.12)$$

Therefore

$$\begin{aligned} \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{R(t) - R(0)}{t} \\ = \Lambda - \mu_0 \langle S(t) \rangle - (\mu_0 + \mu_1) \langle I(t) \rangle - \mu_0 \langle R(t) \rangle \end{aligned}$$

so

$$\langle S(t) \rangle = \frac{1}{\mu_0} (\Lambda - (\mu_0 + \mu_1) \langle I(t) \rangle - \mu_0 \langle R(t) \rangle) + \Phi(t) \quad (3.13)$$

where $\Phi(t) = -\frac{1}{\mu_0} (\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{R(t)-R(0)}{t})$ Obviously $\Phi(t) \rightarrow 0$ a.s. $t \rightarrow \infty$. Applying to Itô formula to the second equation of 1.2, we arrive at

$$d \ln I(t) = \beta S(t) - (\mu_0 + \mu_1 + \gamma_1) - \frac{\eta_1^2 S^2(t)}{2} dt + \eta_1 S(t) dB_1(t) \quad (3.14)$$

The integration of 3.13 from 0 to t and division by t leads to the following equation

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &= \beta \langle S(t) \rangle - (\mu_0 + \mu_1 + \gamma_1) - \frac{\eta_1^2 \langle S^2(t) \rangle}{2} + \frac{\eta_1 \int_0^t S(s) dB_1(s)}{t} \\ &\leq \beta \langle S(t) \rangle - (\mu_0 + \mu_1 + \gamma_1) - \frac{\eta_1^2 \langle S(t) \rangle^2}{2} + \frac{\eta_1 \int_0^t S(s) dB_1(s)}{t} \end{aligned} \quad (3.15)$$

Substituting 3.12 in 3.13 and by the use of the local continuous martingale $M_1(t) = \eta_1 \int_0^t S(s) dB_1(s)$ with $M_1(0) = 0$.

We obtain

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &= \beta \left(\frac{\Lambda}{\mu_0} - \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} - \frac{\langle R(t) \rangle}{\mu_0} + \Phi(t) \right) - (\mu_0 + \mu_1 + \gamma_1) \\ &\quad - \frac{1}{2} \eta_1^2 \left(\frac{\Lambda}{\mu_0} - \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} - \langle R(t) \rangle + \Phi(t) \right)^2 + \frac{M_1(t)}{t} \\ &= \frac{\beta \Lambda}{\mu_0} - \frac{\beta(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} - \frac{\beta \langle R(t) \rangle}{\mu_0} - (\mu_0 + \mu_1 + \gamma_1) - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2} \\ &\quad - \frac{\eta_1^2 (\mu_0 + \mu_1)^2 \langle I(t) \rangle^2}{2\mu_0^2} - \frac{\eta_1^2 \langle R(t) \rangle^2}{2\mu_0^2} + \frac{\eta_1^2 \Lambda (\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0^2} \\ &\quad + \frac{\eta_1^2 \Lambda \langle R(t) \rangle}{\mu_0^2} - \frac{\eta_1 (\mu_0 + \mu_1) \langle I(t) R(t) \rangle}{\mu_0^2} + \frac{M_1(t)}{t} + \varphi(t) \\ &\leq \frac{\beta \Lambda}{\mu_0} - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2} - (\mu_0 + \mu_1 + \gamma_1) + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{\langle R(t) \rangle}{\mu_0} \\ &\quad + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} + \frac{M_1(t)}{t} + \varphi(t) \\ &\leq (\mu_0 + \mu_1 + \gamma_1) \left(\mathcal{R} - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2 (\mu_0 + \mu_1 + \gamma_1)} \right) + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{\langle R(t) \rangle}{\mu_0} \\ &\quad + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} + \frac{M_1(t)}{t} + \varphi(t) \end{aligned} \quad (3.16)$$

where

$$\varphi(t) = -\frac{\eta_1^2 \phi^2(t)}{2} + \beta \Phi(t) - \eta_1^2 \phi(t) \left(\frac{\Lambda}{\mu_0} - \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} - \frac{\langle R(t) \rangle}{\mu_0} \right)$$

Moreover, $\limsup_{t \rightarrow \infty} \frac{\langle M_1, M_1 \rangle_t}{t} \leq \frac{\eta_1^2 \Lambda^2}{\mu_0} < \infty$ a.s.

Now by Lemma 3.3 and using $\Phi(t) = 0$ a.s. $t \rightarrow \infty$. It may be verified that

$$\limsup_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0 \text{ and } \lim_{t \rightarrow \infty} \varphi(t) = 0 \text{ a.s.} \quad (3.17)$$

If the assumption is satisfied, then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &\leq (\mu_0 + \mu_1 + \gamma_1) \left(\mathcal{R} - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2 (\mu_0 + \mu_1 + \gamma_1)} \right) + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{\langle R(t) \rangle}{\mu_0} \\ &\quad + \left(\frac{\eta_1^2 \Lambda}{\mu_0} - \beta \right) \frac{(\mu_0 + \mu_1) \langle I(t) \rangle}{\mu_0} + \frac{I(0)}{t} \\ &< 0 \text{ a.s.} \end{aligned} \quad (3.18)$$

Now, 3.18 implies that

$$\lim_{t \rightarrow \infty} I(t) = 0 \text{ a.s.}$$

We observe from the proposed model that

$$\begin{aligned} dN(t) &= (\Lambda - \mu_0 N(t) - \mu_1 I(t)) dt \\ N(t) &= e^{-\mu_0 t} \left(N(0) + \int_0^t (\Lambda - \mu_1 I(s)) e^{-\mu_0 s} ds \right) \\ \lim_{t \rightarrow \infty} (S(t) + R(t)) &= \lim_{t \rightarrow \infty} \left(\frac{N(0) + \int_0^t (\Lambda - \mu_1 I(s)) e^{-\mu_0 s} ds}{e^{\mu_0 t}} - I(t) \right) = \frac{\Lambda}{\mu_0} \end{aligned}$$

Thus, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} (S(t) + R(t)) &= \frac{\Lambda}{\mu_0} \text{ a.s.} \\ \lim_{t \rightarrow \infty} S(t) &= \tilde{S} \quad \lim_{t \rightarrow \infty} R(t) = \tilde{R} \end{aligned}$$

3.2. Permanence in mean

theorem 3.6. Let $(S(t), I(t), R(t))$ be the solution of 1.2 with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. If

$$\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} > 1$$

hold, then the solution has the following property:

$$\liminf_{t \rightarrow \infty} \langle I(t) \rangle \geq \mathcal{C} \text{ a.s.}$$

then the diseases are said to be permanent in mean. a.s. where

$$\mathcal{C} = \frac{(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1) \left(\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} - 1 \right)}{\beta(\mu_0 + \mu_1 + \gamma_1)}$$

Proof. By the theorem 3.5 and the equality of 3.12 we arrived

$$\begin{aligned} & \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} \\ &= \Lambda - \alpha \langle S(t)R(t) \rangle - (\mu_0 + \nu)S(t) - (\mu_0 + \mu_1 + \gamma_1) \langle I(t) \rangle - \frac{M_2(t)}{t} \\ & \langle S(t) \rangle \\ &= \frac{1}{\mu_0 + \nu} \left(\Lambda - \alpha \langle S(t)R(t) \rangle - (\mu_0 + \nu)S(t) - (\mu_0 + \mu_1 + \gamma_1) \langle I(t) \rangle - \frac{M_2(t)}{t} \right) \end{aligned}$$

$$\begin{aligned}
 \frac{\ln I(t) - \ln I(0)}{t} &= \beta \langle S(t) \rangle - (\mu_0 + \mu_1 + \gamma_1) - \frac{1}{2} \eta_1^2 \langle S^2(t) \rangle + \frac{M_1(t)}{t} \\
 &\geq \frac{\beta}{\mu_0 + \nu} \left(\Lambda - \alpha \langle S(t)R(t) \rangle - (\mu_0 + \nu)S(t) - (\mu_0 + \mu_1 + \gamma_1) \langle I(t) \rangle \right. \\
 &\quad \left. - \frac{M_2(t)}{t} \right) - (\mu_0 + \mu_1 + \gamma_1) - \frac{\eta_1^2 \Lambda^2}{2(\mu_0 + \nu)^2} + \frac{M_1(t)}{t} \\
 &\geq \frac{\beta \Lambda}{\mu_0 + \nu} - \frac{\beta^3 \Lambda^2 \alpha}{(\mu_0 + \nu)^3} - \frac{\eta_1^2 \Lambda^2}{2(\mu_0 + \nu)^2} - \frac{\beta(\mu_0 + \mu_1 + \gamma_1)}{\mu_0 + \nu} \langle I(t) \rangle \\
 &\quad - (\mu_0 + \mu_1 + \gamma_1) - \frac{\beta M_2(t)}{(\mu_0 + \nu)t} + \frac{M_1(t)}{t} \\
 &\geq (\mu_0 + \mu_1 + \gamma_1) \left(\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} \right. \\
 &\quad \left. - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} - 1 \right) - \frac{\beta(\mu_0 + \mu_1 + \gamma_1)}{\mu_0 + \nu} \langle I(t) \rangle \\
 &\quad - (\mu_0 + \mu_1 + \gamma_1) - \frac{\beta M_2(t)}{(\mu_0 + \nu)t} + \frac{M_1(t)}{t}
 \end{aligned} \tag{3.19}$$

then

$$\begin{aligned}
 \langle I(t) \rangle &\geq \frac{(\mu_0 + \nu) \left(\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} - 1 \right)}{\beta} \\
 &\quad + \frac{\mu_0 + \nu}{\beta(\mu_0 + \mu_1 + \gamma_1)} \left(\frac{M_1(t)}{t} - \frac{\beta M_2(t)}{(\mu_0 + \nu)t} \right)
 \end{aligned} \tag{3.20}$$

Solving the 3.19 and 3.20 with the help of inequality 3.17 and taking the limit inferior of both side, we get

$$\begin{aligned}
 \liminf_{t \rightarrow \infty} \langle I(t) \rangle &\geq \frac{(\mu_0 + \nu) \left(\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} - 1 \right)}{\beta} \\
 &= \mathcal{C}
 \end{aligned} \tag{3.21}$$

95 4. Conclusion and simulations

96 In this section, we use the stochastic Euler method, we next present the computer simulations to
 97 support these results, illustrating extinction and persistence of the disease.

98 To verifies our analytical results, we take the paraments value as follows:

99

Paraments	Value
Λ	0.5
β	0.3
α	0.4
μ_0	0.1
μ_1	0.2
ν	0.4
γ_1	0.6
η_1	0.22
η_2	0.45

101 It is easy to ensure the conditions $\mathcal{R} - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2(\mu_0 + \mu_1 + \gamma_1)} = 0.9994 < 1$, $\frac{\eta_1^2 \Lambda}{\mu_0} = 0.242 < 0.3 = \beta$, see
 102 Figure:3 and Figure:4. This indicates the extinction of epidemic disease.

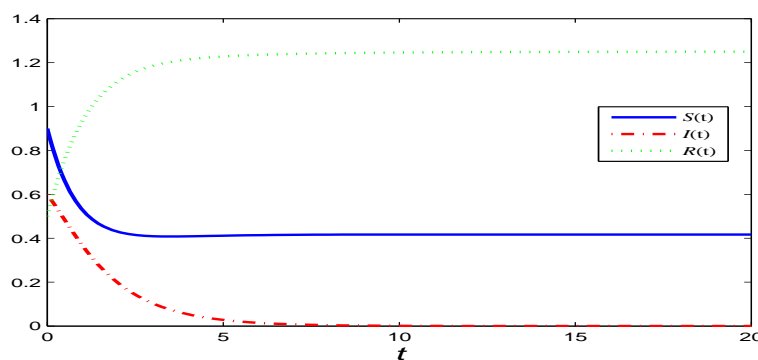


Figure 3. ODE Computer simulation $\Lambda = 0.5, \beta = 0.3, \alpha = 0.4, \mu_0 = 0.1, \mu_1 = 0.2, \nu = 0.4, \gamma_1 = 0.6, \eta_1 = 0.22, \eta_2 = 0.45, (S(0), I(0), R(0)) = (0.9, 0.6, 0.5)$.

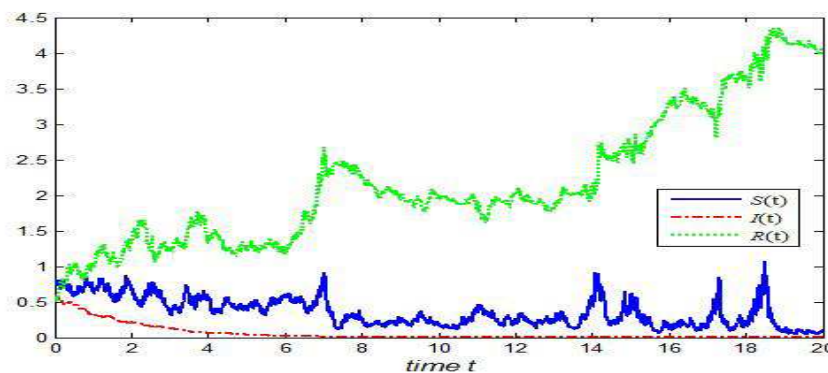


Figure 4. SDE Computer simulation $\Lambda = 0.5, \beta = 0.3, \alpha = 0.4, \mu_0 = 0.1, \mu_1 = 0.2, \nu = 0.4, \gamma_1 = 0.6, \eta_1 = 0.22, \eta_2 = 0.45, (S(0), I(0), R(0)) = (0.9, 0.6, 0.5), \mathcal{R}_0 - \frac{\eta_1^2 \Lambda^2}{2\mu_0^2(\mu_0 + \mu_1 + \gamma_1)} < 1, \frac{\eta_1^2 \Lambda}{\mu_0} < \beta$.

103
104

To verifies our analytical results, we take the paraments value as follows:

Paraments	Value
Λ	0.5
β	0.6
α	0.01
μ_0	0.1
μ_1	0.2
ν	0.15
γ_1	0.4
η_1	0.01
η_2	0.45

105

106
107

It is easy to ensure the conditions $\mathcal{R} - \frac{\beta \Lambda \nu}{\mu_0(\mu_0 + \nu)(\mu_0 + \mu_1 + \gamma_1)} - \frac{2\beta^3 \Lambda^2 \alpha + \eta_1^2 \Lambda^2 (\mu_0 + \nu)}{2(\mu_0 + \nu)^2 (\mu_0 + \mu_1 + \gamma_1)} = 1.7016 > 1$, see Figure: 5 and Figure:6. This indicates the permanence in mean of epidemic disease.

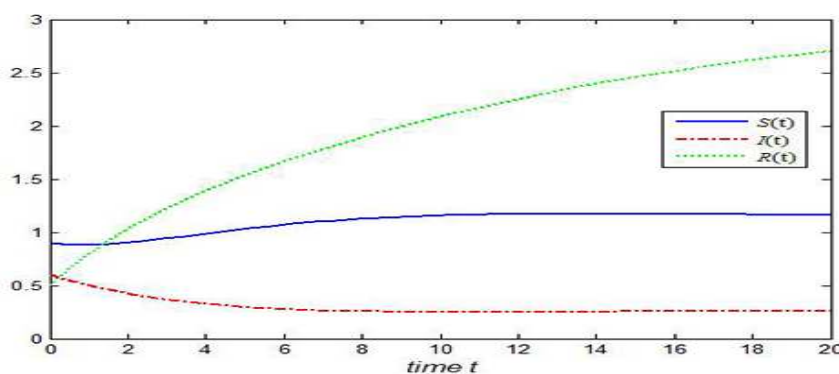


Figure 5. ODE Computer simulation $\Lambda = 0.5, \beta = 0.6, \alpha = 0.01, \mu_0 = 0.1, \mu_1 = 0.2, \nu = 0.15, \gamma_1 = 0.4, \eta_1 = 0.01, \eta_2 = 0.45, (S(0), I(0), R(0)) = (0.9, 0.6, 0.5)$.

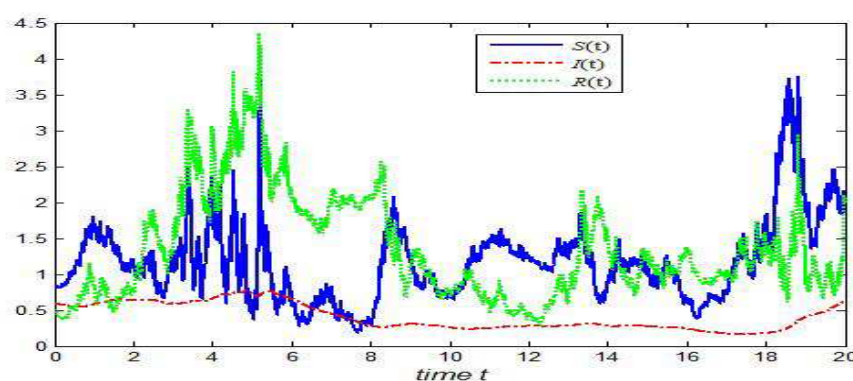


Figure 6. SDE Computer simulation $\Lambda = 0.5, \beta = 0.6, \alpha = 0.01, \mu_0 = 0.1, \mu_1 = 0.2, \nu = 0.15, \gamma_1 = 0.4, \eta_1 = 0.01, \eta_2 = 0.45, (S(0), I(0), R(0)) = (0.9, 0.6, 0.5), \mathcal{R} - \frac{\beta\Lambda\nu}{\mu_0(\mu_0+\nu)(\mu_0+\mu_1+\gamma_1)} - \frac{2\beta^3\Lambda^2\alpha+\eta_1^2\Lambda^2(\mu_0+\nu)}{2(\mu_0+\nu)^2(\mu_0+\mu_1+\gamma_1)} = 1.7016 > 1$.

108 Most real-world problems tend not to be deterministic, but to have stochastic effects. Based on the
 109 epidemic diseases of hepatitis B, this paper also considers the effects of environmental noise between
 110 susceptible individuals and recovered individuals. The dynamics of the model is given and sufficient
 111 conditions for its extinction and permanence in mean are discussed. Through numerical simulations,
 112 we clearly observed the behavior of infectious diseases. However, the critical condition of the model
 113 still needs to be proved and discussed.

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