Article

A Soft Embedding Lemma for Soft Topological Spaces

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Abstract: In 1999, Molodtsov initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties in many fields of applied sciences. In 2011, Shabir and Naz introduced and studied the notion of soft topological spaces, also defining and investigating many new soft properties as generalization of the classical ones. In this paper, we introduce the notions of soft separation between soft points and soft closed sets in order to obtain a generalization of the well-known Embedding Lemma to the class of soft topological spaces.

Keywords: soft set; soft sets theory; soft topology; embedding lemma; soft mapping; soft topological product; soft slab; soft continuous mapping; soft diagonal mapping

1. Introduction

Almost every branch of sciences and many practical problems in engineering, economics, computer science, physics, meteorology, statistics, medicine, sociology, etc. have its own uncertainties and ambiguities because they depend on the influence of many parameters and, due to the inadequacy of the existing theories of parameterization in dealing with uncertainties, it is not always easy to model such a kind of problems by using classical mathematical methods. In 1999, Molodtsov [1] initiated the novel concept of Soft Sets Theory as a new mathematical tool and a completely different approach for dealing with uncertainties while modelling problems in a large class of applied sciences.

In the past few years, the fundamentals of soft set theory have been studied by many researchers. Starting from 2002, Maji, Biswas and Roy [2,3] studied the theory of soft sets initiated by Molodtsov, defining notion as the equality of two soft sets, the subset and super set of a soft set, the complement of a soft set, the union and the intersection of soft sets, the null soft set and absolute soft set, and they gave many examples. In 2005, Pei and Miao [4] and Chen et al. [5] improved the work of Maji. Further contributions to the Soft Sets Theory were given by Yang [6], Ali et al. [7], Fu [8], Qin and Hong [9], Sezgin and Atagün [10], Neog and Sut [11], Ahmad and Kharal [12], Babitha and Sunil [13], Ibrahim and Yosuf [14], Singh and Onyeozili [15], Feng and Li [16], Onyeozili and Gwary [17], Çağman [18].

In 2011, Shabir and Naz [19] introduced the concept of soft topological spaces, also defining and investigating the notions of soft closed sets, soft closure, soft neighborhood, soft subspace and some separation axioms. Some other properties related to soft topology were studied by Çağman, Karataş and Enginoglu in [20]. In the same year Hussain and Ahmad [21] investigated the properties of soft closed sets, soft neighbourhoods, soft interior, soft exterior and soft boundary, while Kharal and Ahmad [22] defined the notion of a mapping on soft classes and studied several properties of images and inverse images. The notion of soft interior, soft neighborhood and soft continuity were also object of study by Zorlutuna, Akdag, Min and Atmaca in [23]. Some other relations between these notions was proved by Ahmad and Hussain in [24]. The neighbourhood properties of a soft topological space were investigated in 2013 by Nazmul and Samanta [25]. The class of soft Hausdorff spaces was extensively studied by Varol and Aygün in [26]. In 2012, Aygünoğlu and Aygün [27] defined and studied the notions of soft continuity and soft product topology. Some years later, Zorlutuna and Çaku [28] gave some new characterizations of soft continuity, soft openness and soft closedness of soft mappings, also generalizing the Pasting Lemma to the soft topological spaces. Soft first countable and soft second countable spaces were instead defined and studied by Rong in [29]. Furthermore, the notion of soft continuity between soft topological spaces was independently introduced and investigated by Hazra, Majumdar and Samanta in [30]. Soft connectedness was also studied in 2015 by Al-Khafaj [31] and Hussain [32]. In the same year, Das and Samanta [33,34] introduced and extensively studied the soft metric spaces. In 2015, Hussain and Ahmad [35] redefined and explored several properties of soft T_i (with $i = 0, 1, 2, 3, 4$) separation axioms and discuss some soft invariance properties namely soft topological property and soft hereditary property. In [36], Xie introduced the concept of soft points and proved that soft sets can be translated into soft points so that they may conveniently dealt as same as ordinary sets. In 2016, Tantawy, El-Sheikh and Hamde [37] continued the study of soft T_i-spaces (for $i = 0, 1, 2, 3, 4, 5$) also discussing the hereditary and topological properties for such spaces. In 2017, Fu, Fu and You [38] investigated some basic properties concerning the soft topological product space. Further contributions to the theory of soft sets and that of soft topology were added in 2011, by Min [39], in 2012, by Janaki [40], and by Varol, Shostak and Aygün [41], in 2013 and 2014, by Peyghan, Samadi and Tayebi [42], by Wardowski [43], by Nazmul and Samanta [44], by Peyghan [45], and by Georgiou, Megaritis and Petropoulos [46,47], in 2015 by Uluçay, Şahin, Olgun and Kiliçman [48], and by Shi and Pang [49], in 2016 by Wadkar, Bhardwaj, Mishra and Singh [50], by Matejdes [51], and by Fu and Fu [38], in 2017 by Bdaiwi [52], and, more recently, by Bayramov and Aras [53], and by Nordo [54,55].

In the present paper we will present the notions of family of soft mappings soft separating soft points and soft points from soft closed sets in order to give a generalization of the well-known Embedding Lemma for soft topological spaces.

2. Preliminaries

In this section we present some basic definitions and results on soft sets and suitably exemplify them. Terms and undefined concepts are used as in [56].

Definition 1. [1] *Let $\mathbb{U}$ be an initial universe set and $\mathbb{E}$ be a nonempty set of parameters (or abstract attributes) under consideration with respect to $\mathbb{U}$ and $A \subseteq \mathbb{E}$, we say that a pair (F, A) is a* ***soft set*** *over $\mathbb{U}$ if F is a set-valued mapping $F : A \to \mathbb{P}(\mathbb{U})$ which maps every parameter $e \in A$ to a subset $F(e)$ of $\mathbb{U}$.*

In other words, a soft set is not a real (crisp) set but a parameterized family $\{F(e)\}_{e \in A}$ of subsets of the universe $\mathbb{U}$. For every parameter $e \in A$, $F(e)$ may be considered as the set of *e-approximate elements* of the soft set (F, A).

Remark 1. *In 2010, Ma, Yang and Hu [57] proved that every soft set (F, A) is equivalent to the soft set $(F, \mathbb{E})$ related to the whole set of parameters $\mathbb{E}$, simply considering empty every approximations of parameters which are missing in A, that is extending in a trivial way its set-valued mapping, i.e. setting $F(e) = \emptyset$, for every $e \in \mathbb{E} \setminus A$.*
For such a reason, in this paper we can consider all the soft sets over the same parameter set $\mathbb{E}$ as in [58] and we will redefine all the basic operations and relations between soft sets originally introduced in [1–3] as in [25], that is by considering the same parameter set.

Definition 2. [23] *The set of all the soft sets over a universe $\mathbb{U}$ with respect to a set of parameters $\mathbb{E}$ will be denoted by $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$.*

Definition 3. [25] *Let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$ and a common set of parameters $\mathbb{E}$, we say that $(F, \mathbb{E})$ is a* ***soft subset*** *of $(G, \mathbb{E})$ and we write $(F, \mathbb{E}) \tilde{\subseteq} (G, \mathbb{E})$ if $F(e) \subseteq G(e)$ for every $e \in \mathbb{E}$.*

Definition 4. [25] *Let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$, we say that $(F, \mathbb{E})$ and $(G, \mathbb{E})$ are* ***soft equal*** *and we write $(F, \mathbb{E}) \tilde{=} (G, \mathbb{E})$ if $(F, \mathbb{E}) \tilde{\subseteq} (G, \mathbb{E})$ and $(G, \mathbb{E}) \tilde{\subseteq} (F, \mathbb{E})$.*

Remark 2. *If $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ are two soft sets over $\mathbb{U}$, it is a trivial matter to note that $(F,\mathbb{E})\tilde{=}(G,\mathbb{E})$ if and only if it results $F(e) = G(e)$ for every $e \in \mathbb{E}$.*

Definition 5. [25] *A soft set $(F,\mathbb{E})$ over a universe $\mathbb{U}$ is said to be the **null soft set** and it is denoted by $(\tilde{\emptyset},\mathbb{E})$ if $F(e) = \emptyset$ for every $e \in \mathbb{E}$.*

Definition 6. [25] *A soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ over a universe $\mathbb{U}$ is said to be the **absolute soft set** and it is denoted by $(\tilde{\mathbb{U}},\mathbb{E})$ if $F(e) = \mathbb{U}$ for every $e \in \mathbb{E}$.*

Definition 7. *Let $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$ and V be a nonempty subset of U, the **constant soft set** of V, denoted by $(\tilde{V},\mathbb{E})$) (or, sometimes, by $\tilde{V}$), is the soft set $(\underline{V},\mathbb{E})$, where $\underline{V} : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the constant set-valued mapping defined by $\underline{V}(e) = V$, for every $e \in \mathbb{E}$.*

Definition 8. [25] *Let $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$, the **soft complement** (or more exactly the soft relative complement) of $(F,\mathbb{E})$, denoted by $(F,\mathbb{E})^{\complement}$, is the soft set $\left(F^{\complement},\mathbb{E}\right)$ where $F^{\complement} : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $F^{\complement}(e) = F(e)^{\complement} = \mathbb{U} \setminus F(e)$, for every $e \in \mathbb{E}$.*

Definition 9. [25] *Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$, the **soft difference** of $(F,\mathbb{E})$ and $(G,\mathbb{E})$, denoted by $(F,\mathbb{E})\tilde{\setminus}(G,\mathbb{E})$, is the soft set $(F \setminus G,\mathbb{E})$ where $F \setminus G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \setminus G)(e) = F(e) \setminus G(e)$, for every $e \in \mathbb{E}$.*

Clearly, for every soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, it results $(F,\mathbb{E})^{\complement} \tilde{=} (\tilde{\mathbb{U}},\mathbb{E})\tilde{\setminus}(F,\mathbb{E})$.

Definition 10. [25] *Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, the **soft union** of $(F,\mathbb{E})$ and $(G,\mathbb{E})$, denoted by $(F,\mathbb{E})\tilde{\cup}(G,\mathbb{E})$, is the soft set $(F \cup G,\mathbb{E})$ where $F \cup G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \cup G)(e) = F(e) \cup G(e)$, for every $e \in \mathbb{E}$.*

Definition 11. [25] *Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, the **soft intersection** of $(F,\mathbb{E})$ and $(G,\mathbb{E})$, denoted by $(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E})$, is the soft set $(F \cap G,\mathbb{E})$ where $F \cap G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \cap G)(e) = F(e) \cap G(e)$, for every $e \in \mathbb{E}$.*

Proposition 1. [18] *For every soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, the following hold:*

(1) $(F,\mathbb{E})\,\tilde{\cup}\,(F,\mathbb{E}) \tilde{=} (F,\mathbb{E})$.
(2) $(F,\mathbb{E})\,\tilde{\cup}\,(\tilde{\emptyset},\mathbb{E}) \tilde{=} (F,\mathbb{E})$.
(3) $(F,\mathbb{E})\,\tilde{\cup}\,(\tilde{\mathbb{U}},\mathbb{E}) \tilde{=} (\tilde{\mathbb{U}},\mathbb{E})$.
(4) $(F,\mathbb{E})\,\tilde{\cap}\,(F,\mathbb{E}) \tilde{=} (F,\mathbb{E})$.
(5) $(F,\mathbb{E})\,\tilde{\cap}\,(\tilde{\emptyset},\mathbb{E}) \tilde{=} (\tilde{\emptyset},\mathbb{E})$.
(6) $(F,\mathbb{E})\,\tilde{\cap}\,(\tilde{\mathbb{U}},\mathbb{E}) \tilde{=} (F,\mathbb{E})$.

Definition 12. [31] *Two soft sets $(F,\mathbb{E})$ and $(G,\mathbb{E})$ over a common universe $\mathbb{U}$ are said to be **soft disjoint** if their soft intersection is the soft null set, i.e. if $(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E}) \tilde{=} (\tilde{\emptyset},\mathbb{E})$. If two soft sets are not soft disjoint, we also say that they **soft meet** each other. In particular, if $(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E}) \tilde{\neq} (\tilde{\emptyset},\mathbb{E})$ we say that $(F,\mathbb{E})$ **soft meets** $(G,\mathbb{E})$.*

Proposition 2. [19] *Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, we have that $(F,\mathbb{E})\tilde{\setminus}(G,\mathbb{E})\tilde{=}(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E})^{\complement}$.*

The notions of soft union and intersection admit some obvious generalizations to a family with any number of soft sets.

Definition 13. [25] *Let $\{(F_i, \mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, the (generalized)* ***soft union*** *of $\{(F_i, \mathbb{E})\}_{i\in I}$, denoted by $\widetilde{\bigcup}_{i\in I}(F_i, \mathbb{E})$, is defined by $(\bigcup_{i\in I} F_i, \mathbb{E})$ where $\bigcup_{i\in I} F_i : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $\left(\bigcup_{i\in I} F_i\right)(e) = \bigcup_{i\in I} F_i(e)$, for every $e \in \mathbb{E}$.*

Definition 14. [25] *Let $\{(F_i, \mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, the (generalized)* ***soft intersection*** *of $\{(F_i, \mathbb{E})\}_{i\in I}$, denoted by $\widetilde{\bigcap}_{i\in I}(F_i, \mathbb{E})$, is defined by $(\bigcap_{i\in I} F_i, \mathbb{E})$ where $\bigcap_{i\in I} F_i : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $\left(\bigcap_{i\in I} F_i\right)(e) = \bigcap_{i\in I} F_i(e)$, for every $e \in \mathbb{E}$.*

Proposition 3. *Let $\{(F_i, \mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, it results:*

(1) $\left(\widetilde{\bigcup}_{i\in I}(F_i, \mathbb{E})\right)^{\complement} \tilde{=} \widetilde{\bigcap}_{i\in I}(F_i, \mathbb{E})^{\complement}$.

(2) $\left(\widetilde{\bigcap}_{i\in I}(F_i, \mathbb{E})\right)^{\complement} \tilde{=} \widetilde{\bigcup}_{i\in I}(F_i, \mathbb{E})^{\complement}$.

Definition 15. [36] *A soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ over a universe $\mathbb{U}$ is said to be a* ***soft point*** *over U if it has only one non-empty approximation which is a singleton, i.e. if there exists some parameter $\alpha \in \mathbb{E}$ and an element $p \in \mathbb{U}$ such that $F(\alpha) = \{p\}$ and $F(e) = \emptyset$ for every $e \in \mathbb{E} \setminus \{\alpha\}$. Such a soft point is usually denoted by $(p_\alpha, \mathbb{E})$. The singleton $\{p\}$ is called the support set of the soft point and α is called the expressive parameter of $(p_\alpha, \mathbb{E})$.*

Remark 3. *In other words, a soft point $(p_\alpha, \mathbb{E})$ is a soft set corresponding to the set-valued mapping $p_\alpha : \mathbb{E} \to (U)$ that, for any $e \in \mathbb{E}$, is defined by*

$$p_\alpha(e) = \begin{cases} \{p\} & \text{if } e = \alpha \\ \emptyset & \text{if } e \in \mathbb{E} \setminus \{\alpha\} \end{cases} .$$

Definition 16. [36] *The set of all the soft points over a universe $\mathbb{U}$ with respect to a set of parameters $\mathbb{E}$ will be denoted by $\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$.*

Since any soft point is a particular soft set, it is evident that $\mathcal{SP}(\mathbb{U})_{\mathbb{E}} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$.

Definition 17. [36] *Let $(p_\alpha, \mathbb{E}) \in \mathcal{SP}(\mathbb{U})_{\mathbb{E}}$ and $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft point and a soft set over a common universe $\mathbb{U}$, respectively. We say that* ***the soft point*** *$(p_\alpha, \mathbb{E})$* ***soft belongs to the soft set*** *$(F, \mathbb{E})$ and we write $(p_\alpha, \mathbb{E})\tilde{\in}(F, \mathbb{E})$, if the soft point is a soft subset of the soft set, i.e. if $(p_\alpha, \mathbb{E})\tilde{\subseteq}(F, \mathbb{E})$ and hence if $p \in F(\alpha)$. We also say that* ***the soft point*** *$(p_\alpha, \mathbb{E})$* ***does not belongs to the soft set*** *$(F, \mathbb{E})$ and we write $(p_\alpha, \mathbb{E})\tilde{\notin}(F, \mathbb{E})$, if the soft point is not a soft subset of the soft set, i.e. if $(p_\alpha, \mathbb{E})\tilde{\nsubseteq}(F, \mathbb{E})$ and hence if $p \notin F(\alpha)$.*

Definition 18. [33] *Let $(p_\alpha, \mathbb{E}), (q_\beta, \mathbb{E}) \in \mathcal{SP}(\mathbb{U})_{\mathbb{E}}$ be two soft points over a common universe $\mathbb{U}$, we say that $(p_\alpha, \mathbb{E})$ and $(q_\beta, \mathbb{E})$ are* ***soft equal****, and we write $(p_\alpha, \mathbb{E})\tilde{=}(q_\beta, \mathbb{E})$, if they are equals as soft sets and hence if $p = q$ and $\alpha = \beta$.*

Definition 19. [33] *We say that two soft points $(p_\alpha, \mathbb{E})$ and $(q_\beta, \mathbb{E})$ are* ***soft distincts****, and we write $(p_\alpha, \mathbb{E})\tilde{\neq}(q_\beta, \mathbb{E})$, if and only if $p \neq q$ or $\alpha \neq \beta$.*

The notion of soft point allows us to express the soft inclusion in a more familiar way.

Proposition 4. *Let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$ respect to a parameter set $\mathbb{E}$, then $(F, \mathbb{E})\tilde{\subseteq}(G, \mathbb{E})$ if and only if for every soft point $(p_\alpha, \mathbb{E})\tilde{\in}(F, \mathbb{E})$ it follows that $(p_\alpha, \mathbb{E})\tilde{\in}(G, \mathbb{E})$.*

Proof. Suppose that $(F, \mathbb{E})\tilde{\subseteq}(G, \mathbb{E})$. Then, for every $(p_\alpha, \mathbb{E})\tilde{\in}(F, \mathbb{E})$, by Definition 17, we have that $p \in F(\alpha)$. Since, by Definition 3, we have in particular that $F(\alpha) \subseteq G(\alpha)$, it follows that $p \in G(\alpha)$, which, by Definition 17, is equivalent to say that $(p_\alpha, \mathbb{E})\tilde{\in}(G, \mathbb{E})$.

Conversely, suppose that for every soft point $(p_\alpha, \mathbb{E}) \tilde{\in} (F, \mathbb{E})$ it follows $(p_\alpha, \mathbb{E}) \tilde{\in} (G, \mathbb{E})$. Then, for every $e \in \mathbb{E}$ and any $p \in F(e)$, by Definition 17, we have that the soft point $(p_e, \mathbb{E}) \tilde{\in} (F, \mathbb{E})$. So, by our hypotesis, it follows that $(p_e, \mathbb{E}) \tilde{\in} (G, \mathbb{E})$ which is equivalent to $p \in G(e)$. This proves that $F(e) \subseteq G(e)$ for every $e \in \mathbb{E}$ and so that $(F, \mathbb{E}) \tilde{\subseteq} (G, \mathbb{E})$. □

Definition 20. [21] *Let $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$ and V be a nonempty subset of $\mathbb{U}$, the **sub soft set** of $(F, \mathbb{E})$ over V, is the soft set $\left({}^{V}F, \mathbb{E}\right)$, where ${}^{V}F : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by ${}^{V}F(e) = F(e) \cap V$, for every $e \in \mathbb{E}$.*

Remark 4. *Using Definitions 7 and 11, it is a trivial matter to verify that a sub soft set of $(F, \mathbb{E})$ over V can also be expressed as $\left({}^{V}F, \mathbb{E}\right) \tilde{=} (F, \mathbb{E}) \tilde{\cap} (\tilde{V}, \mathbb{E})$.*
Furthermore, it is evident that the sub soft set $\left({}^{V}F, \mathbb{E}\right)$ above defined belongs to the set of all the soft sets over V with respect to the set of parameters $\mathbb{E}$, which is contained in the set of all the soft sets over the universe $\mathbb{U}$ with respect to $\mathbb{E}$, that is $\left({}^{V}F, \mathbb{E}\right) \in \mathcal{SS}(V)_{\mathbb{E}} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$.

Definition 21. [13,59] *Let $\{(F_i, \mathbb{E}_i)\}_{i \in I}$ be a family of soft sets over a universe set $\mathbb{U}_i$ with respect to a set of parameters $\mathbb{E}_i$ (with $i \in I$), respectively. Then the **soft product** (or, more precisely, the **soft cartesian product**) of $\{(F_i, \mathbb{E}_i)\}_{i \in I}$, denoted by $\widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i)$, is the soft set $(\prod_{i \in I} F_i, \prod_{i \in I} \mathbb{E}_i)$ over the (usual) cartesian product $\prod_{i \in I} \mathbb{U}_i$ and with respect to the set of parameters $\prod_{i \in I} \mathbb{E}_i$, where $\prod_{i \in I} F_i : \prod_{i \in I} \mathbb{E}_i \to \mathbb{P}\left(\prod_{i \in I} \mathbb{U}_i\right)$ is the set-valued mapping defined by $\prod_{i \in I} F_i\left(\langle e_i \rangle_{i \in I}\right) = \prod_{i \in I} F_i(e_i)$, for every $\langle e_i \rangle_{i \in I} \in \prod_{i \in I} \mathbb{E}_i$.*

Proposition 5. [60] *Let $\widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i)$ be the soft product of a family $\{(F_i, \mathbb{E}_i)\}_{i \in I}$ of soft sets over a universe set $\mathbb{U}_i$ with respect to a set of parameters $\mathbb{E}_i$ (with $i \in I$), and let $(p_\alpha, \prod_{i \in I} \mathbb{E}_i) \in \mathcal{SP}(\prod_{i \in I} \mathbb{U}_i)_{\prod_{i \in I} \mathbb{E}_i}$ be a soft point of the product $\prod_{i \in I} \mathbb{U}_i$, where $p = \langle p_i \rangle_{i \in I} \in \prod_{i \in I} \mathbb{U}_i$ and $\alpha = \langle \alpha_i \rangle_{i \in I} \in \prod_{i \in I} \mathbb{E}_i$, then $(p_\alpha, \prod_{i \in I} \mathbb{E}_i) \tilde{\in} \widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i)$ if and only if $((p_i)_{\alpha_i}, \mathbb{E}_i) \tilde{\in} (F_i, \mathbb{E}_i)$ for every $i \in I$.*

Proof. In fact, by using Definitions 21 and 17, $(p_\alpha, \prod_{i \in I} \mathbb{E}_i) \tilde{\in} \widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i)$ means that $p \in \left(\prod_{i \in I} F_i\right)(\alpha)$ that is $\langle p_i \rangle_{i \in I} \in \left(\prod_{i \in I} F_i\right)\left(\langle \alpha_i \rangle_{i \in I}\right)$ which corresponds to say tat $p_i \in F_i(\alpha_i)$ for every $i \in I$ which, by Definition 17, is equivalent to $((p_i)_{\alpha_i}, \mathbb{E}_i) \tilde{\in} (F_i, \mathbb{E}_i)$ for every $i \in I$. □

Corollary 1. [60] *The soft product of a family $\{(F_i, \mathbb{E}_i)\}_{i \in I}$ of soft sets over a universe set $\mathbb{U}_i$ with respect to a set of parameters $\mathbb{E}_i$ (with $i \in I$) is null if and only if at least one of its soft sets is null, that is $\widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i) \tilde{=} (\tilde{\emptyset}, \prod_{i \in I} \mathbb{E}_i)$ iff there exists some $j \in I$ such that $(F_j, \mathbb{E}_j) \tilde{=} (\tilde{\emptyset}, \mathbb{E})$.*

Proposition 6. [60] *Let $\{(F_i, \mathbb{E}_i)\}_{i \in I}$ and $\{(G_i, \mathbb{E}_i)\}_{i \in I}$ be two families of soft sets over a universe set $\mathbb{U}_i$ with respect to a set of parameters $\mathbb{E}_i$ (with $i \in I$), such that $(F_i, \mathbb{E}_i) \tilde{\subseteq} (G_i, \mathbb{E}_i)$ for every $i \in I$, then their respective soft products are such that $\widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i) \tilde{\subseteq} \widetilde{\prod}_{i \in I}(G_i, \mathbb{E}_i)$.*

Proposition 7. [59] *Let $\{(F_i, \mathbb{E}_i)\}_{i \in I}$ and $\{(G_i, \mathbb{E}_i)\}_{i \in I}$ be two families of soft sets over a universe set $\mathbb{U}_i$ with respect to a set of parameters $\mathbb{E}_i$ (with $i \in I$), then it results:*

$$\widetilde{\prod}_{i \in I}\left((F_i, \mathbb{E}_i) \tilde{\cap} (G_i, \mathbb{E}_i)\right) \tilde{=} \widetilde{\prod}_{i \in I}(F_i, \mathbb{E}_i) \tilde{\cap} \widetilde{\prod}_{i \in I}(G_i, \mathbb{E}_i).$$

According to Remark 1 the following notions by Kharal and Ahmad have been simplified and slightly modified for soft sets defined on a common parameter set.

Definition 22. [22] *Let $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ and $\mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be two sets of soft open sets over the universe sets $\mathbb{U}$ and $\mathbb{U}'$ with respect to the sets of parameters $\mathbb{E}$ and $\mathbb{E}'$, respectively. and consider a mapping $\varphi : \mathbb{U} \to \mathbb{U}'$ between the two universe sets and a mapping $\psi : \mathbb{E} \to \mathbb{E}'$ between the two set of parameters. The mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ which maps every soft set $(F, \mathbb{E})$ of $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ to a soft set $(\varphi_\psi(F), \mathbb{E}')$ of $\mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$, denoted by $\varphi_\psi(F, \mathbb{E})$, where $\varphi_\psi(F) : \mathbb{E}' \to \mathbb{P}(\mathbb{U}')$ is the set-valued mapping defined by $\varphi_\psi(F)(e') =$*

$\bigcup\{\varphi(F(e)) : e \in \psi^{-1}(\{e'\})\}$ *for every* $e' \in \mathbb{E}'$, *is called a* ***soft mapping*** *from* $\mathbb{U}$ *to* $\mathbb{U}'$ *induced by the mappings* φ *and* ψ, *while the soft set* $\varphi_\psi(F,\mathbb{E}) \widetilde{=} (\varphi_\psi(F),\mathbb{E}')$ *is said to be the* ***soft image*** *of the soft set* $(F,\mathbb{E})$ *under the soft mapping* $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$.
The soft mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ *is said* ***injective*** *(respectively* ***surjective***, ***bijective****) if the mappings* $\varphi : \mathbb{U} \to \mathbb{U}'$ *and* $\psi : \mathbb{E} \to \mathbb{E}'$ *are both injective (resp. surjective, bijective).*

Remark 5. *In other words a soft mapping* $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ *matches every set-valued mapping* $F : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ *to a set-valued mapping* $\varphi_\psi(F) : \mathbb{E}' \to \mathbb{P}(\mathbb{U}')$ *which, for every* $e' \in \mathbb{E}'$, *is defined by*

$$\varphi_\psi(F)(e') = \begin{cases} \bigcup_{e\in\psi^{-1}(\{e'\})} \varphi(F(e)) & \text{if } \psi^{-1}(\{e'\}) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases} .$$

In particular, if the soft mapping φ_ψ *is bijective, the set-valued mapping* $\varphi_\psi(F) : \mathbb{E}' \to \mathbb{P}(\mathbb{U}')$ *is defined simply by* $\varphi_\psi(F)(e') = \varphi\left(F\left(\psi^{-1}(e')\right)\right)$, *for every* $e' \in \mathbb{E}'$.
Let us also note that in some paper (see, for example, [26]) the soft mapping φ_ψ *is denoted with* $(\varphi,\psi) : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$.

It is worth noting that soft mappings between soft sets behaves similarly to usual (crisp) mappings in the sense that they maps soft points to soft points, as proved in the following property.

Proposition 8. *Let* $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ *be a soft mapping induced by the mappings* $\varphi : \mathbb{U} \to \mathbb{U}'$ *and* $\psi : \mathbb{E} \to \mathbb{E}'$ *between the two sets* $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}, \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ *of soft sets. and consider a soft point* $(p_\alpha,\mathbb{E})$ *of* $\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$. *Then the soft image* $\varphi_\psi(p_\alpha,\mathbb{E})$ *of the soft point* $(p_\alpha,\mathbb{E})$ *under the soft mapping* φ_ψ *is the soft point* $\left(\varphi(p)_{\psi(\alpha)},\mathbb{E}'\right)$, *i.e.* $\varphi_\psi(p_\alpha,\mathbb{E}) \widetilde{=} \left(\varphi(p)_{\psi(\alpha)},\mathbb{E}'\right)$.

Proof. Let $(p_\alpha,\mathbb{E})$ be a soft point of $\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$, by Definition 22, its soft image $\varphi_\psi(p_\alpha,\mathbb{E})$ under the soft mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ is the soft set $(\varphi_\psi(p_\alpha),\mathbb{E}')$ corresponding to the set-valued mapping $\varphi_\psi(p_\alpha) : \mathbb{E}' \to \mathbb{P}(\mathbb{U}')$ which, for every $e' \in \mathbb{E}'$, is defined by $\varphi_\psi(p_\alpha)(e') = \bigcup\{\varphi(p_\alpha(e)) : e \in \psi^{-1}(\{e'\})\}$. Now, if $e' = \psi(\alpha)$ we have that:

$$\begin{aligned}
\varphi_\psi(p_\alpha)(\psi(\alpha)) &= \bigcup\left\{\varphi(p_\alpha(e)) : e \in \psi^{-1}(\{\psi(\alpha)\})\right\} \\
&= \bigcup\{\varphi(p_\alpha(e)) : e \in \mathbb{E},\, \psi(e) = \psi(\alpha)\} \\
&= \bigcup\{\varphi(p_\alpha(e)) : e = \alpha\} \cup \bigcup\{\varphi(p_\alpha(e)) : e \in \mathbb{E}\setminus\{\alpha\}, \psi(e) = \psi(\alpha)\} \\
&= \{\varphi(p_\alpha(\alpha))\} \cup \bigcup\{\varphi(p_\alpha(e)) = \emptyset : e \in \mathbb{E}\setminus\{\alpha\},\, \psi(e) = e'\} \\
&= \{\varphi(p)\} \cup \emptyset \\
&= \{\varphi(p)\}
\end{aligned}$$

while, for every $e' \in \mathbb{E}' \setminus \{\psi(\alpha)\}$, we have that $\psi(\alpha) \neq e'$ and so it follows that:

$$\begin{aligned}
\varphi_\psi(p_\alpha)(e') &= \bigcup\left\{\varphi(p_\alpha(e)) : e \in \psi^{-1}(\{e'\})\right\} \\
&= \bigcup\{\varphi(p_\alpha(e)) : e \in \mathbb{E},\, \psi(e) = e'\} \\
&= \bigcup\{\varphi(p_\alpha(e)) : e \in \mathbb{E}\setminus\{\alpha\},\, \psi(e) = e'\} \\
&= \bigcup\{\varphi(p_\alpha(e)) = \emptyset : e \in \mathbb{E}\setminus\{\alpha\},\, \psi(e) = e'\} \\
&= \emptyset\,.
\end{aligned}$$

This proves that the set-valued mapping $\varphi_\psi(p_\alpha) : \mathbb{E}' \to \mathbb{P}(\mathbb{U}')$ sends the parameter $\psi(\alpha)$ to the singleton $\{\varphi(p)\}$ and maps every other parameters of $\mathbb{E}' \setminus \{\psi(e)\}$ to the empty set, and so, by

Definition 15, this means that the soft image $\varphi_\psi(p_\alpha, \mathbb{E})$ of the soft point $(p_\alpha, \mathbb{E}) \in \mathcal{SP}(\mathbb{U})_\mathbb{E}$ under the soft mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ is the soft point in $\mathcal{SP}(\mathbb{U}')_{\mathbb{E}'}$ having $\{\varphi(p)\}$ as support set and $\psi(\alpha)$ as expressive parameter, that is $\left(\varphi(p)_{\psi(\alpha)}, \mathbb{E}'\right)$. □

Corollary 2. *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$ between the two sets $\mathcal{SS}(\mathbb{U})_\mathbb{E}, \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ of soft sets, then φ_ψ is injective if and only if its soft images of every distinct pair of soft points are distinct too, i.e. if for every $(p_\alpha, \mathbb{E}), (q_\beta, \mathbb{E}) \in \mathcal{SP}(\mathbb{U})_\mathbb{E}$ such that $(p_\alpha, \mathbb{E}) \tilde{\neq} (q_\beta, \mathbb{E})$ it follows that $\varphi_\psi(p_\alpha, \mathbb{E}) \tilde{\neq} \varphi_\psi(q_\beta, \mathbb{E})$.*

Proof. It easily derives from Definitions 19 and 22, and Proposition 8. □

Definition 23. [22] *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$ between the two sets $\mathcal{SS}(\mathbb{U})_\mathbb{E}, \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ of soft sets and consider a soft set $(G, \mathbb{E}')$ of $\mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$. The **soft inverse image** of $(G, \mathbb{E}')$ under the soft mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$, denoted by $\varphi_\psi^{-1}(G, \mathbb{E}')$ is the soft set $(\varphi_\psi^{-1}(G), \mathbb{E}')$ of $\mathcal{SS}(\mathbb{U})_\mathbb{E}$ where $\varphi_\psi^{-1}(G) : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $\varphi_\psi^{-1}(G)(e) = \varphi^{-1}(G(\psi(e)))$ for every $e \in \mathbb{E}$.*

Proposition 9. [22,23,27] *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$ and let $(F, \mathbb{E}), (F_i, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_\mathbb{E}$ and $(G, \mathbb{E}'), (G_i, \mathbb{E}') \in \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be soft sets over $\mathbb{U}$ and $\mathbb{U}'$, respectively, then the following hold:*

(1) $\varphi_\psi(\tilde{\emptyset}, \mathbb{E}) \cong (\tilde{\emptyset}, \mathbb{E}')$.
(2) $\varphi_\psi^{-1}(\tilde{\emptyset}, \mathbb{E}') \cong (\tilde{\emptyset}, \mathbb{E})$.
(3) $\varphi_\psi^{-1}(\tilde{\mathbb{U}}', \mathbb{E}') \cong (\tilde{\mathbb{U}}, \mathbb{E})$.
(4) $(F, \mathbb{E}) \tilde{\subseteq} \varphi_\psi^{-1}(\varphi_\psi(F, \mathbb{E}))$ *and the soft equality holds when φ_ψ is injective.*
(5) $\varphi_\psi\left(\varphi_\psi^{-1}(G, \mathbb{E}')\right) \tilde{\subseteq} (G, \mathbb{E}')$ *and the soft equality holds when φ_ψ is surjective.*
(6) $\varphi_\psi^{-1}\left((G, \mathbb{E}')^{\complement}\right) \cong \left(\varphi_\psi^{-1}(G, \mathbb{E}')\right)^{\complement}$.
(7) *if* $(F_1, \mathbb{E}) \tilde{\subseteq} (F_2, \mathbb{E})$. *then* $\varphi_\psi(F_1, \mathbb{E}) \tilde{\subseteq} \varphi_\psi(F_2, \mathbb{E})$.
(8) *if* $(G_1, \mathbb{E}') \tilde{\subseteq} (G_2, \mathbb{E}')$. *then* $\varphi_\psi^{-1}(G_1, \mathbb{E}') \tilde{\subseteq} \varphi_\psi^{-1}(G_2, \mathbb{E}')$.
(9) $\varphi_\psi\left(\tilde{\bigcup}_{i \in I}(F_i, \mathbb{E})\right) \cong \tilde{\bigcup}_{i \in I} \varphi_\psi(F_i, \mathbb{E})$.
(10) $\tilde{\bigcap}_{i \in I} \varphi_\psi(F_i, \mathbb{E}) \tilde{\subseteq} \varphi_\psi\left(\tilde{\bigcap}_{i \in I}(F_i, \mathbb{E})\right)$.
(11) $\varphi_\psi^{-1}\left(\tilde{\bigcup}_{i \in I}(G_i, \mathbb{E}')\right) \cong \tilde{\bigcup}_{i \in I} \varphi_\psi^{-1}(G_i, \mathbb{E}')$.
(12) $\varphi_\psi^{-1}\left(\tilde{\bigcap}_{i \in I}(G_i, \mathbb{E}')\right) \cong \tilde{\bigcap}_{i \in I} \varphi_\psi^{-1}(G_i, \mathbb{E}')$.

Proposition 10. [22] *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$ and let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_\mathbb{E}$ and $(F', \mathbb{E}'), (G', \mathbb{E}') \in \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be soft sets over $\mathbb{U}$ and $\mathbb{U}'$, respectively, then the following hold:*

(1) $(F, \mathbb{E}) \tilde{\subsetneq} (G, \mathbb{E})$ *implies* $\varphi_\psi(F, \mathbb{E}) \tilde{\subsetneq} \varphi_\psi(G, \mathbb{E})$.
(2) $(F', \mathbb{E}) \tilde{\subsetneq} (G', \mathbb{E}')$ *implies* $\varphi_\psi^{-1}(F', \mathbb{E}') \tilde{\subsetneq} \varphi_\psi^{-1}(G', \mathbb{E}')$.

Corollary 3. *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_\mathbb{E} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$. If $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_\mathbb{E}$ and $(F', \mathbb{E}') \in \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ are soft sets over $\mathbb{U}$ and $\mathbb{U}'$, respectively and $(p_\alpha, \mathbb{E}) \in \mathcal{SP}(\mathbb{U})_\mathbb{E}$ and $(q_\beta, \mathbb{E}') \in \mathcal{SP}(\mathbb{U}')_{\mathbb{E}'}$ are soft points over $\mathbb{U}$ and $\mathbb{U}'$, respectively, then the following hold:*

(1) $(p_\alpha, \mathbb{E}) \tilde{\in} (F, \mathbb{E})$ *implies* $\varphi_\psi(p_\alpha, \mathbb{E}) \tilde{\in} \varphi_\psi(F, \mathbb{E})$.
(2) $(q_\beta, \mathbb{E}') \tilde{\in} (F', \mathbb{E}')$ *implies* $\varphi_\psi^{-1}(q_\beta, \mathbb{E}') \tilde{\subseteq} \varphi_\psi^{-1}(F', \mathbb{E}')$.

Definition 24. *Let $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ be a bijective soft mapping induced by the mappings $\varphi : \mathbb{U} \to \mathbb{U}'$ and $\psi : \mathbb{E} \to \mathbb{E}'$. The* ***soft inverse mapping*** *of φ_ψ, denoted by φ_ψ^{-1}, is the soft mapping $\varphi_\psi^{-1} = (\varphi^{-1})_{\psi^{-1}} : \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'} \to \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ induced by the inverse mappings $\varphi^{-1} : \mathbb{U}' \to \mathbb{U}$ and $\psi^{-1} : \mathbb{E}' \to \mathbb{E}$ of the mappings φ and ψ, respectively.*

Remark 6. *Evidently, the soft inverse mapping $\varphi_\psi^{-1} : \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'} \to \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ of a bijective soft mapping $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ is also bijective and its soft image of a soft set in $\mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ coincides with the soft inverse image of the corresponding soft set under the soft mapping φ_ψ.*

Definition 25. [27] *Let $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}, \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$ and $\mathcal{SS}(\mathbb{U}'')_{\mathbb{E}''}$ be three sets of soft open sets over the universe sets $\mathbb{U}, \mathbb{U}', \mathbb{U}''$ with respect to the sets of parameters $\mathbb{E}, \mathbb{E}', \mathbb{E}''$, respectively, and $\varphi_\psi : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}'}$, $\gamma_\delta : \mathcal{SS}(\mathbb{U})_{\mathbb{E}'} \to \mathcal{SS}(\mathbb{U}')_{\mathbb{E}''}$ be two soft mappings between such sets, then the* ***soft composition*** *of the soft mappings φ_ψ and γ_δ, denoted by $\gamma_\delta \tilde{\circ} \varphi_\psi$ is the soft mapping $(\gamma \circ \varphi)_{\delta \circ \psi} : \mathcal{SS}(\mathbb{U})_{\mathbb{E}} \to \mathcal{SS}(\mathbb{U}'')_{\mathbb{E}''}$ induced by the compositions $\gamma \circ \varphi : \mathbb{U} \to \mathbb{U}''$ of the mappings φ and γ between the universe sets and $\delta \circ \psi : \mathbb{E} \to \mathbb{E}''$ of the mappings ψ and δ between the parameter sets.*

The notion of soft topological spaces as topological spaces defined over a initial universe with a fixed set of parameters was introduced in 2011 by Shabir and Naz [19].

Definition 26. [19] *Let X be an initial universe set, $\mathbb{E}$ be a nonempty set of parameters with respect to X and $\mathcal{T} \subseteq \mathcal{SS}(X)_{\mathbb{E}}$ be a family of soft sets over X, we say that $\mathcal{T}$ is a* ***soft topology*** *on X with respect to $\mathbb{E}$ if the following four conditions are satisfied:*

(i) *the null soft set belongs to $\mathcal{T}$, i.e. $(\tilde{\emptyset}, \mathbb{E}) \in \mathcal{T}$.*
(ii) *the absolute soft set belongs to $\mathcal{T}$, i.e. $(\tilde{X}, \mathbb{E}) \in \mathcal{T}$.*
(iii) *the soft intersection of any two soft sets of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e. for every $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{T}$ then $(F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E}) \in \mathcal{T}$.*
(iv) *the soft union of any subfamily of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$, i.e. for every $\{(F_i, \mathbb{E})\}_{i \in I} \subseteq \mathcal{T}$ then $\tilde{\bigcup}_{i \in I} (F_i, \mathbb{E}) \in \mathcal{T}$.*

The triplet $(X, \mathcal{T}, \mathbb{E})$ is called a ***soft topological space*** *(or soft space, for short) over X with respect to $\mathbb{E}$. In some case, when it is necessary to better specify the universal set and the set of parameters, the topology will be denoted by $\mathcal{T}(X, \mathbb{E})$.*

Definition 27. [19] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X with respect to $\mathbb{E}$, then the members of $\mathcal{T}$ are said to be* ***soft open set*** *in X.*

Definition 28. [30] *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two soft topologies over a common universe set X with respect to a set of paramters $\mathbb{E}$. We say that $\mathcal{T}_2$ is* ***finer*** *(or stronger) than $\mathcal{T}_1$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ where $\subseteq$ is the usual set-theoretic relation of inclusion between crisp sets. In the same situation, we also say that $\mathcal{T}_1$ is* ***coarser*** *(or weaker) than $\mathcal{T}_2$.*

Definition 29. [19] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X and $(F, \mathbb{E})$ be a soft set over X. We say that $(F, \mathbb{E})$ is* ***soft closed set*** *in X if its complement $(F, \mathbb{E})^{\complement}$ is a soft open set, i.e. if $(F, \mathbb{E})^{\complement} \in \mathcal{T}$.*

Notation 1. *The family of all soft closed sets of a soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X with respect to $\mathbb{E}$ will be denoted by σ, or more precisely with $\sigma(X, \mathbb{E})$ when it is necessary to specify the universal set X and the set of parameters $\mathbb{E}$.*

Proposition 11. [19] *Let σ be the family of soft closed sets of a soft topological space $(X, \mathcal{T}, \mathbb{E})$, the following hold:*

(1) *the null soft set is a soft closed set, i.e. $(\tilde{\emptyset}, \mathbb{E}) \in \sigma$.*
(2) *the absolute soft set is a soft closed set, i.e. $(\tilde{X}, \mathbb{E}) \in \sigma$.*

(3) *the soft union of any two soft closed sets is still a soft closed set, i.e. for every* $(C,\mathbb{E}),(D,\mathbb{E}) \in \sigma$ *then* $(C,\mathbb{E})\tilde{\cup}(D,\mathbb{E}) \in \sigma$.

(4) *the soft intersection of any subfamily of soft closed sets is still a soft closed set, i.e. for every* $\{(C_i,\mathbb{E})\}_{i\in I} \subseteq \sigma$ *then* $\widetilde{\bigcap}_{i\in I}(C_i,\mathbb{E}) \in \sigma$.

Definition 30. [27] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space over* X *and* $\mathcal{B} \subseteq \mathcal{T}$ *be a non-empty subset of soft open sets. We say that* $\mathcal{B}$ *is a* ***soft open base*** *for* $(X,\mathcal{T},\mathbb{E})$ *if every soft open set of* $\mathcal{T}$ *can be expressed as soft union of a subfamily of* $\mathcal{B}$*, i.e. if for every* $(F,\mathbb{E}) \in \mathcal{T}$ *there exists some* $\mathcal{A} \subset \mathcal{B}$ *such that* $(F,\mathbb{E}) = \widetilde{\bigcup}\{(A,\mathbb{E}) : (A,\mathbb{E}) \in \mathcal{A}\}$.

Proposition 12. [25] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space over* X *and* $\mathcal{B} \subseteq \mathcal{T}$ *be a family of soft open sets of* X*. Then* $\mathcal{B}$ *is a soft open base for* $(X,\mathcal{T},\mathbb{E})$ *if and only if for every soft open set* $(F,\mathbb{E}) \in \mathcal{T}$ *and any soft point* $(x_\alpha,\mathbb{E})\tilde{\in}(F,\mathbb{E})$ *there exists some soft open set* $(B,\mathbb{E}) \in \mathcal{B}$ *such that* $(x_\alpha,\mathbb{E})\tilde{\in}(B,\mathbb{E})\tilde{\subseteq}(F,\mathbb{E})$.

Definition 31. [23] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space,* $(N,\mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ *be a soft set and* $(x_\alpha,\mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ *be a soft point over a common universe* X*. We say that* $(N,\mathbb{E})$ *is a* ***soft neighbourhood*** *of the soft point* $(x_\alpha,\mathbb{E})$ *if there is some soft open set soft containing the soft point and soft contained in the soft set, that is if there exists some soft open set* $(A,\mathbb{E}) \in \mathcal{T}$ *such that* $(x_\alpha,\mathbb{E})\tilde{\in}(A,\mathbb{E})\tilde{\subseteq}(N,\mathbb{E})$.

Notation 2. *The family of all soft neighbourhoods (sometimes also called soft neighbourhoods system) of a soft point* $(x_\alpha,\mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ *in a soft topological space* $(X,\mathcal{T},\mathbb{E})$ *will be denoted by* $\mathcal{N}_{(x_\alpha,\mathbb{E})}$ *(or more precisely with* $\mathcal{N}^{\mathcal{T}}_{(x_\alpha,\mathbb{E})}$ *if it is necessary to specify the topology).*

Definition 32. [19] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space over* X *and* $(F,\mathbb{E})$ *be a soft set over* X*. Then the* ***soft closure*** *of the soft set* $(F,\mathbb{E})$ *with respect to the soft space* $(X,\mathcal{T},\mathbb{E})$*, denoted by* s-cl$_X(F,\mathbb{E})$*, is the soft intersection of all soft closed set over* X *soft containing* $(F,\mathbb{E})$*, that is*

$$\text{s-cl}_X(F,\mathbb{E}) \hat{=} \widetilde{\bigcap}\left\{(C,\mathbb{E}) \in \sigma(X,\mathbb{E}) : (F,\mathbb{E})\tilde{\subseteq}(C,\mathbb{E})\right\}.$$

Proposition 13. [19] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space over* X*, and* $(F,\mathbb{E})$ *be a soft set over* X*. Then the following hold:*

(1) s-cl$_X(\tilde{\emptyset},\mathbb{E}) \hat{=} (\tilde{\emptyset},\mathbb{E})$.
(2) s-cl$_X(\tilde{X},\mathbb{E}) \hat{=} (\tilde{X},\mathbb{E})$.
(3) $(F,\mathbb{E}) \tilde{\subseteq}$ s-cl$_X(F,\mathbb{E})$.
(4) $(F,\mathbb{E})$ *is a soft closed set over* X *if and only if* s-cl$_X(F,\mathbb{E}) \hat{=} (F,\mathbb{E})$.
(5) s-cl$_X$(s-cl$_X(F,\mathbb{E})) \hat{=}$ s-cl$_X(F,\mathbb{E})$.

Proposition 14. [19] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space and* $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ *be two soft sets over a common universe* X*. Then the following hold:*

(1) $(F,\mathbb{E}) \tilde{\subseteq} (G,\mathbb{E})$ *implies* s-cl$_X(F,\mathbb{E}) \tilde{\subseteq}$ s-cl$_X(G,\mathbb{E})$.
(2) s-cl$_X((F,\mathbb{E}) \tilde{\cup} (G,\mathbb{E})) \hat{=}$ s-cl$_X(F,\mathbb{E}) \tilde{\cup}$ s-cl$_X(G,\mathbb{E})$.
(3) s-cl$_X((F,\mathbb{E}) \tilde{\cap} (G,\mathbb{E})) \tilde{\subseteq}$ s-cl$_X(F,\mathbb{E}) \tilde{\cap}$ s-cl$_X(G,\mathbb{E})$.

Definition 33. [36] *Let* $(X,\mathcal{T},\mathbb{E})$ *be a soft topological space,* $(F,\mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ *and* $(x_\alpha,\mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ *be a soft set and a soft point over the common universe* X *with respect to the sets of parameters* $\mathbb{E}$*, respectively. We say that* $(x_\alpha,\mathbb{E})$ *is a* ***soft adherent point*** *(sometimes also called* ***soft closure point****) of* $(F,\mathbb{E})$ *if it soft meets every soft neighbourhood of the soft point, that is if for every* $(N,\mathbb{E}) \in \mathcal{N}_{(x_\alpha,\mathbb{E})}$, $(F,\mathbb{E}) \tilde{\cap} (N,\mathbb{E}) \tilde{\neq} (\tilde{\emptyset},\mathbb{E})$.

As in the classical topological space, it is possible to prove that the soft closure coincides with the set of all its soft adherent points.

Proposition 15. [36] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space, $(F, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ and $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ be a soft set and a soft point over the common universe X with respect to the sets of parameters $\mathbb{E}$, respectively. Then $(x_\alpha, \mathbb{E}) \tilde{\in}$ s-cl$_X(F, \mathbb{E})$ if and only if $(x_\alpha, \mathbb{E})$ is a soft adherent point of $(F, \mathbb{E})$.*

Having in mind the Definition 20 we can recall the following proposition.

Proposition 16. [21] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and Y be a nonempty subset of X, then the family $\mathcal{T}_Y$ of all sub soft sets of $\mathcal{T}$ over Y, i.e.*

$$\mathcal{T}_Y = \left\{ \left({}^Y F, \mathbb{E} \right) : (F, \mathbb{E}) \in \mathcal{T} \right\}$$

is a soft topology on Y.

Definition 34. [21] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and let Y be a nonempty subset of X, the soft topology $\mathcal{T}_Y = \left\{ \left({}^Y F, \mathbb{E} \right) : (F, \mathbb{E}) \in \mathcal{T} \right\}$ is said to be the **soft relative topology** of $\mathcal{T}$ on Y and $(Y, \mathcal{T}_Y, \mathbb{E})$ is called the **soft topological subspace** of $(X, \mathcal{T}, \mathbb{E})$ on Y.*

Proposition 17. *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and $(Y, \mathcal{T}_Y, \mathbb{E})$ be its soft topological subspace over the subset $Y \subseteq X$, then a soft set $(D, \mathbb{E}) \in \mathcal{SS}(Y)_{\mathbb{E}}$ is a soft closed set respect to the soft subspace $(Y, \mathcal{T}_Y, \mathbb{E})$ if and only if it is a sub soft set of some soft closed set of the soft space $(X, \mathcal{T}, \mathbb{E})$, i.e.*

$$(D, \mathbb{E}) \in \sigma(Y, \mathbb{E}) \quad \Longleftrightarrow \quad \exists (C, \mathbb{E}) \in \sigma(X, \mathbb{E}) : (D, \mathbb{E}) \tilde{=} \left({}^Y C, \mathbb{E} \right).$$

Proof. It easily follows from Definitions 29 and 34, Remark 4, and Proposition 2. □

Proposition 18. [61] *Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, $(Y, \mathcal{T}_Y, \mathbb{E})$ be its soft topological subspace on the subset $Y \subseteq X$, and $(G, \mathbb{E}) \in \mathcal{SS}(Y)_{\mathbb{E}}$ be a soft set over Y respect to the set of parameter $\mathbb{E}$. Then the soft closure of $(G, \mathbb{E})$ respect to the soft subspace $(Y, \mathcal{T}_Y, \mathbb{E})$ coincides with the soft intersection of its soft closure respect to the soft space $(X, \mathcal{T}, \mathbb{E})$ and of the absolute soft set $(\tilde{Y}, \mathbb{E})$ of the subspace, that is*

$$\text{s-cl}_Y(G, \mathbb{E}) \tilde{=} \text{s-cl}_X(G, \mathbb{E}) \tilde{\cap} (\tilde{Y}, \mathbb{E}).$$

Definition 35. [23] *Let $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ be a soft mapping between two soft topological spaces $(X, \mathcal{T}, \mathbb{E})$ and $(X', \mathcal{T}', \mathbb{E}')$ induced by the mappings $\varphi : X \to X'$ and $\psi : \mathbb{E} \to \mathbb{E}'$ and $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ be a soft point over X. We say that the soft mapping φ_ψ is **soft continuous at the soft point** $(x_\alpha, \mathbb{E})$ if for each soft neighbourhood $(G, \mathbb{E}')$ of $\varphi_\psi(x_\alpha, \mathbb{E})$ in $(X', \mathcal{T}', \mathbb{E}')$ there exists some soft neighbourhood $(F, \mathbb{E})$ of $(x_\alpha, \mathbb{E})$ in $(X, \mathcal{T}, \mathbb{E})$ such that $\varphi_\psi(F, \mathbb{E}) \tilde{\subseteq} (G, \mathbb{E}')$.*
*If φ_ψ is soft continuous at every soft point $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$, then $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ is called **soft continuous** on X.*

Proposition 19. [23] *Let $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ be a soft mapping between two soft topological spaces $(X, \mathcal{T}, \mathbb{E})$ and $(X', \mathcal{T}', \mathbb{E}')$ induced by the mappings $\varphi : X \to X'$ and $\psi : \mathbb{E} \to \mathbb{E}'$. Then the soft mapping φ_ψ is soft continuous if and only if every soft inverse image of a soft open set in X' is a soft open set in X, that is, if for each $(G, \mathbb{E}') \in \mathcal{T}'$ we have that $\varphi_\psi^{-1}(G, \mathbb{E}') \in \mathcal{T}$.*

Proposition 20. [23] *Let $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ be a soft mapping between two soft topological spaces $(X, \mathcal{T}, \mathbb{E})$ and $(X', \mathcal{T}', \mathbb{E}')$ induced by the mappings $\varphi : X \to X'$ and $\psi : \mathbb{E} \to \mathbb{E}'$. Then the soft mapping φ_ψ is soft continuous if and only if every soft inverse image of a soft closed set in X' is a soft closed set in X, that is, if for each $(C, \mathbb{E}') \in \sigma(X', \mathbb{E}')$ we have that $\varphi_\psi^{-1}(C, \mathbb{E}') \in \sigma(X, \mathbb{E})$.*

Definition 36. [23] *Let $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ be a soft mapping between two soft topological spaces $(X, \mathcal{T}, \mathbb{E})$ and $(X', \mathcal{T}', \mathbb{E}')$ induced by the mappings $\varphi : X \to X'$ and $\psi : \mathbb{E} \to \mathbb{E}'$, and let Y be a nonempty* *subset of X, the* ***restriction*** *of the soft mapping* φ_ψ *to* Y*, denoted by* $\varphi_{\psi_{|Y}}$*, is the soft mapping* $(\varphi_{|Y})_\psi : \mathcal{SS}(Y)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *induced by the restriction* $\varphi_{|Y} : Y \to X'$ *of the mapping* φ *between the universe sets and by the same mapping* $\psi : \mathbb{E} \to \mathbb{E}'$ *between the parameter sets.*

Proposition 21. [23] *If* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *is a soft continuous mapping between two soft topological spaces* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$*, then its restriction* $\varphi_{\psi_{|Y}} : \mathcal{SS}(Y)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *to a nonempty subset* Y *of* X *is soft continuous too.*

Proposition 22. *If* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *is a soft continuous mapping between two soft topological spaces* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$*, then its corestriction* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \varphi_\psi(\mathcal{SS}(X)_{\mathbb{E}})$ *is soft continuous too.*

Proof. It easily follows from Definitions 22 and 23, and Proposition 19. □

Definition 37. [27] *Let* $(X, \mathcal{T}, \mathbb{E})$ *be a soft topological space over* X *and* $\mathcal{S} \subseteq \mathcal{T}$ *be a non-empty subset of soft open sets. We say that* $\mathcal{S}$ *is a* ***soft open subbase*** *for* $(X, \mathcal{T}, \mathbb{E})$ *if the family of all finite soft intersections of members of* $\mathcal{S}$ *forms a soft open base for* $(X, \mathcal{T}, \mathbb{E})$*.*

Proposition 23. [27] *Let* $\mathcal{S} \subseteq \mathcal{SS}(X)_{\mathbb{E}}$ *be a family of soft sets over* X*, containing both the null soft set* $(\tilde{\emptyset}, \mathbb{E})$ *and the absolute soft set* $(\tilde{X}, \mathbb{E})$*. Then the family* $\mathcal{T}(\mathcal{S})$ *of all soft union of finite soft intersections of soft sets in* $\mathcal{S}$ *is a soft topology having* $\mathcal{S}$ *as soft open subbase.*

Definition 38. [27] *Let* $\mathcal{S} \subseteq \mathcal{SS}(X)_{\mathbb{E}}$ *be a a family of soft sets over* X *respect to a set of parameters* $\mathbb{E}$ *and such that* $(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}) \in \mathcal{S}$*, then the soft topology* $\mathcal{T}(\mathcal{S})$ *of the above Proposition 23 is called the* ***soft topology generated*** *by the soft open subbase* $\mathcal{S}$ *over* X *and* $(X, \mathcal{T}(\mathcal{S}), \mathbb{E})$ *is said to be the* ***soft topological space generated by*** $\mathcal{S}$ *over* X*.*

Definition 39. [27] *Let* $\mathcal{SS}(X)_{\mathbb{E}}$ *be the set of all the soft sets over a universe set* X *with respect to a set of parameter* $\mathbb{E}$ *and consider a family of soft topological spaces* $\{(Y_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i \in I}$ *and a corresponding family* $\{(\varphi_\psi)_i\}_{i \in I}$ *of soft mappings* $(\varphi_\psi)_i = (\varphi_i)_{\psi_i} : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(Y_i)_{\mathbb{E}_i}$ *induced by the mappings* $\varphi_i : X \to Y_i$ *and* $\psi_i : \mathbb{E} \to \mathbb{E}_i$ *(with* $i \in I$*). Then the soft topology* $\mathcal{T}(\mathcal{S})$ *generated by the soft open subbase* $\mathcal{S} = \left\{(\varphi_\psi)_i^{-1}(G, \mathbb{E}_i) : (G, \mathbb{E}_i) \in \mathcal{T}_i,\ i \in I\right\}$ *of all soft inverse images of soft open sets of* $\mathcal{T}_i$ *under the soft mappings* $(\varphi_\psi)_i$ *is called the* ***initial soft topology*** *induced on* X *by the family of soft mappings* $\{(\varphi_\psi)_i\}_{i \in I}$ *and it is denoted by* $\mathcal{T}_{ini}(X, \mathbb{E}, Y_i, \mathbb{E}_i, (\varphi_\psi)_i; i \in I)$*.*

Proposition 24. [27] *The initial soft topology* $\mathcal{T}_{ini}(X, \mathbb{E}, Y_i, \mathbb{E}_i, (\varphi_\psi)_i; i \in I)$ *induced on* X *by the family of soft mappings* $\{(\varphi_\psi)_i\}_{i \in I}$ *is the coarsest soft topology on* $\mathcal{SS}(X)_{\mathbb{E}}$ *for which all the soft mappings* $(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(Y_i)_{\mathbb{E}_i}$ *(with* $i \in I$*) are soft continuous.*

Definition 40. [27] *Let* $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i \in I}$ *be a family of soft topological spaces over the universe sets* X_i *with respect to the sets of parameters* $\mathbb{E}_i$*, respectively. For every* $i \in I$*, the soft mapping* $(\pi_i)_{\rho_i} : \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *induced by the canonical projections* $\pi_i : \prod_{i \in I} X_i \to X_i$ *and* $\rho_i : \prod_{i \in I} \mathbb{E}_i \to \mathbb{E}_i$ *is said the* ***i-th soft projection mapping*** *and, by setting* $(\pi_\rho)_i = (\pi_i)_{\rho_i}$*, it will be denoted by* $(\pi_\rho)_i : \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$*.*

Definition 41. [27] *Let* $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i \in I}$ *be a family of soft topological spaces and let* $\{(\pi_\rho)_i\}_{i \in I}$ *be the corresponding family of soft projection mappings* $(\pi_\rho)_i : \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *(with* $i \in I$*). Then, the initial soft topology* $\mathcal{T}_{ini}(\prod_{i \in I} X_i, \mathbb{E}, X_i, \mathbb{E}_i, (\pi_\rho)_i; i \in I)$ *induced on* $\prod_{i \in I} X_i$ *by the family of soft projection mappings* $\{(\pi_\rho)_i\}_{i \in I}$ *is called the* ***soft product topology*** *of the soft topologies* $\mathcal{T}_i$ *(with* $i \in I$*) and denoted by* $\mathcal{T}(\prod_{i \in I} X_i)$*.*

The triplet $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ *will be said the* ***soft topological product space*** *of the soft topological spaces* $(X_i, \mathcal{T}_i, \mathbb{E}_i)$.

The following statement easily derives from Definition 41 and Proposition 24.

Corollary 4. *The soft product topology* $\mathcal{T}(\prod_{i\in I} X_i)$ *is the coarsest soft topology over* $\mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ *for which all the soft projection mappings* $(\pi_\rho)_i : \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *(with* $i \in I$*) are soft continuous.*

Definition 42. [60] *Let* $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ *be the soft topological product space of the soft topological spaces* $(X_i, \mathcal{T}_i, \mathbb{E}_i)$ *(with* $i \in I$*) and let* $(\pi_\rho)_i : \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *be the i-th the soft projection mapping. The inverse soft image of a soft open set* $(F_i, \mathbb{E}_i) \in \mathcal{T}_i$ *under the soft projection mapping* $(\pi_\rho)_i$*, that is* $(\pi_\rho)_i^{-1}(F_i, \mathbb{E}_i)$ *is called a* ***soft slab*** *and it is denoted by* $\langle (F_i, \mathbb{E}_i) \rangle$.

Definitions 37, 39, 41 and 42 give immediately the following property.

Proposition 25. [60] *The family* $\mathcal{S} = \{\langle (F_i, \mathbb{E}_i) \rangle : (F_i, \mathbb{E}_i) \in \mathcal{T}_i, \ i \in I\}$ *of all soft slabs of soft open sets of* $\mathcal{T}_i$ *is a soft open subbase of the soft topological product space* $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$.

Proposition 26. [60] *Let* $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ *be the soft topological product space of the soft topological spaces* $(X_i, \mathcal{T}_i, \mathbb{E}_i)$*, with* $i \in I$ *and let* $(F_j, \mathbb{E}_j) \in \mathcal{T}_j$ *be a soft open set of* X_j*, then its soft slab* $\langle (F_j, \mathbb{E}_j) \rangle$ *coincides with a soft cartesian product in which only the j-th component is the soft set* $(F_j, \mathbb{E}_j)$ *and the other ones are the absolute soft sets* $(\tilde{X}_i, \mathbb{E}_i)$*, that is*

$$\langle (F_j, \mathbb{E}_j) \rangle \doteq \widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \quad \text{where} \quad (A_i, \mathbb{E}_i) = \begin{cases} (F_j, \mathbb{E}_j) & \text{if } i = j \\ (\tilde{X}_i, \mathbb{E}_i) & \text{otherwise} \end{cases}.$$

Proof. By Definitions 42 and 23, we have that

$$\langle (F_j, \mathbb{E}_j) \rangle \doteq (\pi_\rho)_j^{-1}(F_j, \mathbb{E}_j) \doteq \left((\pi_\rho)_j^{-1}(F_j), \prod_{i\in I} \mathbb{E}_i \right)$$

where $(\pi_\rho)_j^{-1}(F_j) : \prod_{i\in I} \mathbb{E}_i \to \mathbb{P}(\prod_{i\in I} X_i)$ is the set-valued mapping defined by $\left((\pi_\rho)_j^{-1}(F_j)\right)(e) = \pi_j^{-1}\left(F_j\left(\rho_j(e)\right)\right)$ for every $e = \langle e_i \rangle_{i\in I} \in \prod_{i\in I} \mathbb{E}_i$.

On the other hand, by Definition 21, it results

$$\widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \doteq \left(\prod_{i\in I} A_i, \prod_{i\in I} \mathbb{E}_i \right)$$

where $\prod_{i\in I} A_i : \prod_{i\in I} \mathbb{E}_i \to \mathbb{P}(\prod_{i\in I} X_i)$ is the set-valued mapping defined by $(\prod_{i\in I} A_i)(e) = \prod_{i\in I} A_i(e_i)$ for every $e = \langle e_i \rangle_{i\in I} \in \prod_{i\in I} \mathbb{E}_i$ and since $A_j(e_j) = F_j(e_j)$ and $A_i(e_i) = X_i$ for every $i \in I \setminus \{j\}$, it follows that $(\prod_{i\in I} A_i)(e) = \langle F_j(e_j) \rangle$, where the last set is the classical slab of the set $F_j(e_j)$ in the usual cartesian product $\prod_{i\in I} X_i$. Thus, we also have that $(\prod_{i\in I} A_i)(e) = \pi_j^{-1}(F_j(e_j)) = \pi_j^{-1}(F_j(\rho_j(e))) = \left((\pi_\rho)_j^{-1}(F_j)\right)(e)$ for every $e \in \mathbb{E}$, and so, by Remark 2, the soft equality holds. □

Definition 43. [60] *The soft intersection of a finite family of slab* $\langle (F_{i_1}, \mathbb{E}_{i_1}) \rangle$ *of soft open sets* $(F_{i_k}, \mathbb{E}_{i_k}) \in \mathcal{T}_{i_k}$ *(with* $k = 1, \ldots n$*), that is* $\widetilde{\bigcap}_{k=1}^{n} \langle (F_{i_k}, \mathbb{E}_{i_k}) \rangle$ *is said to be a* ***soft*** n***-slab*** *and it is denoted by* $\langle (F_{i_1}, \mathbb{E}_{i_1}), \ldots (F_{i_n}, \mathbb{E}_{i_n}) \rangle$.

Definitions 30, 37, 39, 41 and 43 allow us to obtain the following property.

Proposition 27. [60] *The family*

$$\mathcal{B} = \{\langle (F_{i_1}, \mathbb{E}_{i_1}), \dots (F_{i_n}, \mathbb{E}_{i_n}) \rangle : (F_{i_k}, \mathbb{E}_{i_k}) \in \mathcal{T}_{i_k}, \ i_k \in I, n \in \mathbb{N}^* \}$$

of all soft n-slabs of soft open sets of $\mathcal{T}_i$ is a soft open base of the soft topological product space $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$.

Proposition 28. [60] *Let $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ be the soft topological product space of the soft topological spaces $(X_i, \mathcal{T}_i, \mathbb{E}_i)$, with $i \in I$ and let $(F_{i_k}, \mathbb{E}_{i_k}) \in \mathcal{T}_{i_k}$ be a finite family of soft open sets of X_{i_k}, with $k = 1, \dots n$, respectively, then the soft n-slab $\langle (F_{i_1}, \mathbb{E}_{i_1}), \dots (F_{i_n}, \mathbb{E}_{i_n}) \rangle$ coincides with a soft cartesian product in which only the i_k-th components (with $k = 1, \dots n$) are the soft sets $(F_{i_k}, \mathbb{E}_{i_k})$ and the other ones are the absolute soft sets $(\tilde{X}_i, \mathbb{E}_i)$, that is*

$$\langle (F_{i_1}, \mathbb{E}_{i_1}), \dots (F_{i_n}, \mathbb{E}_{i_n}) \rangle \cong \widetilde{\prod}_{i\in I} (A_i, \mathbb{E}_i)$$

where

$$(A_i, \mathbb{E}_i) = \begin{cases} (F_{i_k}, \mathbb{E}_{i_k}) & \text{if } i = i_k \text{ for some } k = 1, \dots n \\ (\tilde{X}_i, \mathbb{E}_i) & \text{otherwise} \end{cases} .$$

Proof. Similarly to the proof of Proposition 26, by applying Definitions 43, 11, 42 and 23, we have that

$$\begin{aligned} \langle (F_{i_1}, \mathbb{E}_{i_1}), \dots (F_{i_n}, \mathbb{E}_{i_n}) \rangle &\cong \widetilde{\bigcap}_{k=1}^n \langle (F_{i_k}, \mathbb{E}_{i_k}) \rangle \\ &\cong \widetilde{\bigcap}_{k=1}^n (\pi_\rho)_{i_k}^{-1} (F_{i_k}, \mathbb{E}_{i_k}) \\ &\cong \left(\bigcap_{k=1}^n (\pi_\rho)_{i_k}^{-1} (F_{i_k}), \prod_{i\in I} \mathbb{E}_i \right) \end{aligned}$$

where $\bigcap_{k=1}^n (\pi_\rho)_{i_k}^{-1} (F_{i_k}) : \prod_{i\in I} \mathbb{E}_i \to \mathbb{P}(\prod_{i\in I} X_i)$ is the set-valued mapping defined by $\left(\bigcap_{k=1}^n (\pi_\rho)_{i_k}^{-1} (F_{i_k}) \right)(e) = \bigcap_{k=1}^n \pi_{i_k}^{-1} \left(F_{i_k} \left(\rho_{i_k}(e) \right) \right)$ for every $e = \langle e_i \rangle_{i\in I} \in \prod_{i\in I} \mathbb{E}_i$.

On the other hand, by Definition 21, it results

$$\widetilde{\prod}_{i\in I} (A_i, \mathbb{E}_i) \cong \left(\prod_{i\in I} A_i, \prod_{i\in I} \mathbb{E}_i \right)$$

where $\prod_{i\in I} A_i : \prod_{i\in I} \mathbb{E}_i \to \mathbb{P}(\prod_{i\in I} X_i)$ is the set-valued mapping defined by $(\prod_{i\in I} A_i)(e) = \prod_{i\in I} A_i(e_i)$ for every $e = \langle e_i \rangle_{i\in I} \in \prod_{i\in I} \mathbb{E}_i$ and since $A_{i_k}(e_{i_k}) = F_{i_k}(e_{i_k})$ for every $k = 1, \dots n$ and $A_i(e_i) = X_i$ for every $i \in I \setminus \{i_1, \dots i_n\}$, it follows that $(\prod_{i\in I} A_i)(e) = \langle F_{i_1}(e_{i_1}), \dots F_{i_n}(e_{i_n}) \rangle$, where the last set is the classical n-slab of the sets $F_{i_k}(e_{i_k})$ (for $k = 1, \dots n$) in the usual cartesian product $\prod_{i\in I} X_i$. Thus, we also have that $(\prod_{i\in I} A_i)(e) = \bigcap_{k=1}^n \langle F_{i_k}(e_{i_k}) \rangle = \bigcap_{k=1}^n \pi_{i_k}^{-1} (F_{i_k}(e_{i_k})) = \bigcap_{k=1}^n \pi_{i_k}^{-1} (F_{i_k}(\rho_{i_k}(e))) = \left(\bigcap_{k=1}^n (\pi_\rho)_{i_k}^{-1} (F_{i_k}) \right)(e)$ for every $e \in \mathbb{E}$, and so, by Remark 2, the proposition is proved. □

Proposition 29. [27] *Let $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i\in I}$ be a family of soft topological spaces, $(X, \mathcal{T}(X), \mathbb{E})$ be the soft topological product of such soft spaces induced on the product $X = \prod_{i\in I} X_i$ of universe sets with respect to the product $\mathbb{E} = \prod_{i\in I} \mathbb{E}_i$ of the sets of parameters, $(Y, \mathcal{T}', \mathbb{E}')$ be a soft topological space and $\varphi_\psi : \mathcal{SS}(Y)_{\mathbb{E}'} \to \mathcal{SS}(X)_{\mathbb{E}}$ be a soft mapping induced by the mappings $\varphi : Y \to X$ and $\psi : \mathbb{E}' \to \mathbb{E}$. Then the soft mappings φ_ψ is soft continuous if and only if, for every $i \in I$, the soft compositions $(\pi_\rho)_i \,\tilde{\circ}\, \varphi_\psi$ with the soft projection mappings $(\pi_\rho)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ are soft continuous mappings.*

Let us note that the soft cartesian product $\widetilde{\prod}_{i\in I} (F_i, \mathbb{E}_i)$ of a family $\{(F_i, \mathbb{E}_i)\}_{i\in I}$ of soft sets over a set X_i with respect to a set of parameters $\mathbb{E}_i$, respectively, as introduced in Definition 21, is a soft set of the soft topological product space $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ i.e. that $\widetilde{\prod}_{i\in I} (F_i, \mathbb{E}_i) \in \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I} \mathbb{E}_i}$ and the following statement holds.

Proposition 30. [60] *Let $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ be the soft topological product space of a family $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i\in I}$ of soft topological spaces and let $\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)$ be the soft product in $\mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ of a family $\{(F_i, \mathbb{E}_i)\}_{i\in I}$ of soft sets of $\mathcal{SS}(X_i)_{\mathbb{E}_i}$, for every $i \in I$. Then the soft closure of $\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)$ in the soft topological product $(\prod_{i\in I} X_i, \mathcal{T}(\prod_{i\in I} X_i), \prod_{i\in I} \mathbb{E}_i)$ coincides with the soft product of the corresponding soft closures of the soft sets $(F_i, \mathbb{E}_i)$ in the corresponding soft topological spaces $(X_i, \mathcal{T}_i, \mathbb{E}_i)$, that is:*

$$\text{s-cl}_{\prod_{i\in I} X_i}\Big(\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)\Big) \cong \widetilde{\prod}_{i\in I} \text{s-cl}_{X_i}(F_i, \mathbb{E}_i).$$

Proof. Let $(x_\alpha, \prod_{i\in I} \mathbb{E}_i)$ be a soft point of $\mathcal{SP}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$, with $x = \langle x_i\rangle_{i\in I}$ and $\alpha = \langle \alpha_i\rangle_{i\in I}$, such that $(x_\alpha, \prod_{i\in I} \mathbb{E}_i) \tilde{\in} \text{s-cl}_{\prod_{i\in I} X_i}\Big(\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)\Big)$. For any $j \in I$, let us consider a soft open set $(N_j, \mathbb{E}_j) \in \mathcal{T}_j$ such that $\Big((x_j)_{\alpha_j}, \mathbb{E}_j\Big) \tilde{\in} (N_j, \mathbb{E}_j)$. By Proposition 25, the soft slab $\langle (N_j, \mathbb{E}_j)\rangle$ is a soft open set of the soft open subbase of the soft topological product space $\prod_{i\in I} X_i$. By Proposition 26, we know that

$$\langle (N_j, \mathbb{E}_j)\rangle \cong \widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \quad \text{where} \quad (A_i, \mathbb{E}_i) = \begin{cases} (N_j, \mathbb{E}_j) & \text{if } i = j \\ (\tilde{X}_j, \mathbb{E}_j) & \text{otherwise} \end{cases}$$

and so that $(x_\alpha, \prod_{i\in I} \mathbb{E}_i) \tilde{\in} \langle (N_i, \mathbb{E}_i)\rangle$. Thus, by our hypothesis, it follows that

$$\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i) \tilde{\cap} \widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \not\cong \left(\tilde{\emptyset}, \prod_{i\in I} \mathbb{E}_i\right)$$

which, by Proposition 7, is equivalent to

$$\widetilde{\prod}_{i\in I} \left((F_i, \mathbb{E}_i) \tilde{\cap} (A_i, \mathbb{E}_i)\right) \not\cong \left(\tilde{\emptyset}, \prod_{i\in I} \mathbb{E}_i\right)$$

and hence, by Corollary 1, it follows in particular that

$$(F_j, \mathbb{E}_j) \tilde{\cap} (A_j, \mathbb{E}_j) \not\cong (\tilde{\emptyset}, E_j)$$

i.e.

$$(F_j, \mathbb{E}_j) \tilde{\cap} (N_j, \mathbb{E}_j) \not\cong (\tilde{\emptyset}, E_j)\,.$$

Thus, by Definition 33, we have that $\Big((x_j)_{\alpha_j}, \mathbb{E}_j\Big)$ is a soft adherent point for the soft set $(F_j, \mathbb{E}_j)$ and so, by Proposition 15, that

$$\Big((x_j)_{\alpha_j}, \mathbb{E}_j\Big) \tilde{\in} \text{s-cl}_{X_j}(F_j, \mathbb{E}_j), \quad \text{for any fixed } j \in I$$

that, by Proposition 5, is equivalent to say that

$$\left(x_\alpha, \prod_{i\in I} \mathbb{E}_i\right) \tilde{\in} \widetilde{\prod}_{i\in I} \text{s-cl}_{X_i}(F_i, \mathbb{E}_i)$$

and, by using Proposition 4, this proves that

$$\text{s-cl}_{\prod_{i\in I} X_i}\Big(\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)\Big) \tilde{\subseteq} \widetilde{\prod}_{i\in I} \text{s-cl}_{X_i}(F_i, \mathbb{E}_i)\,.$$

On the other hand, let $(x_\alpha, \prod_{i\in I} \mathbb{E}_i) \tilde{\in} \widetilde{\prod}_{i\in I} \text{s-cl}_{X_i}(F_i, \mathbb{E}_i)$. By Proposition 5, we have that $((x_i)_{\alpha_i}, \mathbb{E}_i) \tilde{\in} \text{s-cl}_{X_i}(F_i, \mathbb{E}_i)$ for every $i \in I$. Let us consider a soft open set $(N, \prod_{i\in I} \mathbb{E}_i)$ of $\prod_{i\in I} X_i$

such that $(x_\alpha, \prod_{i\in I}\mathbb{E}_i) \tilde{\in} (N, \prod_{i\in I}\mathbb{E}_i)$. By Propositions 12 and 27 and Definition 30, we have that there exists a finite family of soft open sets $(N_{i_k}, \mathbb{E}_{i_k}) \in \mathcal{T}_{i_k}$ with $k = 1, \dots n$ and $n \in \mathbb{N}^*$ such that

$$\left(x_\alpha, \prod_{i\in I}\mathbb{E}_i\right) \tilde{\in} \langle (N_{i_1}, \mathbb{E}_{i_1}), \dots (N_{i_n}, \mathbb{E}_{i_n})\rangle \tilde{\subseteq} \left(N, \prod_{i\in I}\mathbb{E}_i\right).$$

Since, by Proposition 28, we have that

$$\langle (N_{i_1}, \mathbb{E}_{i_1}), \dots (N_{i_n}, \mathbb{E}_{i_n})\rangle \tilde{=} \widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i)$$

where

$$(A_i, \mathbb{E}_i) = \begin{cases} (N_{i_k}, \mathbb{E}_{i_k}) & \text{if } i = i_k \text{ for some } k = 1, \dots n \\ (\tilde{X}_i, \mathbb{E}_i) & \text{otherwise} \end{cases},$$

it follows that

$$\left(x_\alpha, \prod_{i\in I}\mathbb{E}_i\right) \tilde{\in} \widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \tilde{\subseteq} \left(N, \prod_{i\in I}\mathbb{E}_i\right).$$

Now, we claim that

$$\widetilde{\prod}_{i\in I}(A_i, \mathbb{E}_i) \tilde{\cap} \widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i) \tilde{\neq} \left(\tilde{\emptyset}, \prod_{i\in I}\mathbb{E}_i\right).$$

In fact, for every $k = 1, \dots n$, we have that $\left((x_{i_k})_{\alpha_{i_k}}, \mathbb{E}_{i_k}\right) \in \mathcal{T}_{i_k}$ and so, being $\left((x_{i_k})_{\alpha_{i_k}}, \mathbb{E}_{i_k}\right) \tilde{\in}\, \text{s-cl}_{X_{i_k}}(F_{i_k}, \mathbb{E}_{i_k})$, by Proposition 15 and Definition 33, it follows that

$$(A_{i_k}, \mathbb{E}_{i_k}) \tilde{\cap} (F_{i_k}, \mathbb{E}_{i_k}) \tilde{=} (N_{i_k}, \mathbb{E}_{i_k}) \tilde{\cap} (F_{i_k}, \mathbb{E}_{i_k}) \tilde{\neq} (\tilde{\emptyset}, E_{i_k})$$

while, for every $i \in I \setminus \{i_1, \dots i_n\}$, by Proposition 1(6), it trivially results

$$(A_i, \mathbb{E}_i) \tilde{\cap} (F_i, \mathbb{E}_i) \tilde{=} (\tilde{X}_i, \mathbb{E}_i) \tilde{\cap} (F_i, \mathbb{E}_i) \tilde{=} (F_i, \mathbb{E}_i) \tilde{\neq} (\tilde{\emptyset}, E_i)$$

and so the previous assertion follows from Proposition 1.

Thus, a fortiori, we have that

$$\left(N, \prod_{i\in I}\mathbb{E}_i\right) \tilde{\cap} \widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i) \tilde{\neq} \left(\tilde{\emptyset}, \prod_{i\in I}\mathbb{E}_i\right)$$

which, by Definition 33 and Proposition 15, means that

$$\left(x_\alpha, \prod_{i\in I}\mathbb{E}_i\right) \tilde{\in}\, \text{s-cl}_{\prod_{i\in I}X_i}\left(\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)\right)$$

and hence, by Proposition 4, we have

$$\widetilde{\prod}_{i\in I}\text{s-cl}_{X_i}(F_i, \mathbb{E}_i) \tilde{\subseteq}\, \text{s-cl}_{\prod_{i\in I}X_i}\left(\widetilde{\prod}_{i\in I}(F_i, \mathbb{E}_i)\right)$$

that concludes our proof. □

3. Soft Embedding Lemma

Definition 44. [62] *Let $(X, \mathcal{T}, \mathbb{E})$ and $(X', \mathcal{T}', \mathbb{E}')$ be two soft topological spaces over the universe sets X and X' with respect to the sets of parameters $\mathbb{E}$ and $\mathbb{E}'$, respectively. We say that a soft mapping $\varphi_\psi : \mathcal{SS}(X)_\mathbb{E} \to \mathcal{SS}(X')_{\mathbb{E}'}$ is a **soft homeomorphism** if it is soft continuous, bijective and its soft inverse* *mapping* $\varphi_\psi^{-1} : \mathcal{SS}(X')_{\mathbb{E}'} \to \mathcal{SS}(X)_{\mathbb{E}}$ *is soft continuous too. In such a case, the soft topological spaces* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$ *are said* ***soft homeomorphic*** *and we write that* $(X, \mathcal{T}, \mathbb{E}) \widetilde{\approx} (X', \mathcal{T}', \mathbb{E}')$.

Definition 45. *Let* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$ *be two soft topological spaces. We say that a soft mapping* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *is a* ***soft embedding*** *if its corestriction* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \varphi_\psi\left(\mathcal{SS}(X)_{\mathbb{E}}\right)$ *is a soft homeomorphism.*

Definition 46. [62] *Let* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$ *be two soft topological spaces. We say that a soft mapping* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *is a* ***soft closed mapping*** *if the soft image of every soft closed set of* $(X, \mathcal{T}, \mathbb{E})$ *is a soft closed set of* $(X', \mathcal{T}', \mathbb{E}')$*, that is if for any* $(C, \mathbb{E}) \in \sigma(X, \mathbb{E})$*, we have* $\varphi_\psi(C, \mathbb{E}) \in \sigma(X', \mathbb{E}')$.

Proposition 31. *Let* $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X')_{\mathbb{E}'}$ *be a soft mapping between two soft topological spaces* $(X, \mathcal{T}, \mathbb{E})$ *and* $(X', \mathcal{T}', \mathbb{E}')$*. If* φ_ψ *is a soft continuous, injective and soft closed mapping then it is a soft embedding.*

Proof. If we consider the soft mapping $\varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \varphi_\psi\left(\mathcal{SS}(X)_{\mathbb{E}}\right)$, by hypothesis and Proposition 22, it immediately follows that it is a soft continuous bijective mapping and so we have only to prove that its soft inverse mapping $\varphi_\psi^{-1} = \left(\varphi^{-1}\right)_{\psi^{-1}} : \varphi_\psi\left(\mathcal{SS}(X)_{\mathbb{E}}\right) \to \mathcal{SS}(X)_{\mathbb{E}}$ is continuous too. In fact, because the bijectiveness of the corestriction and Remark 6, for every soft closed set $(C, \mathbb{E}) \in \sigma(X, \mathbb{E})$, the soft inverse image of the $(C, \mathbb{E})$ under the soft inverse mapping φ_ψ^{-1} coincides with the soft image of the same soft set under the soft mapping φ_ψ, that is $\left(\varphi_\psi^{-1}\right)^{-1}(C, \mathbb{E}) \widetilde{=} \varphi_\psi(C, \mathbb{E})$ and since by hypothesis φ_ψ is soft closed, it follows that $\left(\varphi_\psi^{-1}\right)^{-1}(C, \mathbb{E}) \in \sigma(X', \mathbb{E}')$ which, by Proposition 20, proves that $\varphi_\psi^{-1} : \mathcal{SS}(X')_{\mathbb{E}'} \to \mathcal{SS}(X)_{\mathbb{E}}$ is a soft continuous mapping, and so, by Proposition 21, we finally have that $\varphi_\psi^{-1} : \varphi_\psi\left(\mathcal{SS}(X)_{\mathbb{E}}\right) \to \mathcal{SS}(X)_{\mathbb{E}}$ is a soft continuous mapping. □

Definition 47. *Let* $(X, \mathcal{T}, \mathbb{E})$ *be a soft topological space over a universe set* X *with respect to a set of parameter* $\mathbb{E}$*, let* $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i \in I}$ *be a family of soft topological spaces over a universe set* X_i *with respect to a set of parameters* $\mathbb{E}_i$*, respectively and consider a family* $\left\{(\varphi_\psi)_i\right\}_{i \in I}$ *of soft mappings* $(\varphi_\psi)_i = (\varphi_i)_{\psi_i} : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *induced by the mappings* $\varphi_i : X \to X_i$ *and* $\psi_i : \mathbb{E} \to \mathbb{E}_i$ *(with* $i \in I$*). Then the soft mapping* $\Delta = \varphi_\psi : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i}$ *induced by the diagonal mappings (in the classical meaning)* $\varphi = \Delta_{i \in I} \varphi_i : X \to \prod_{i \in I} X_i$ *on the universes sets and* $\psi = \Delta_{i \in I} \psi_i : \mathbb{E} \to \prod_{i \in I} \mathbb{E}_i$ *on the sets of parameters (respectively defined by* $\varphi(x) = \langle \varphi_i(x) \rangle_{i \in I}$ *for every* $x \in X$ *and by* $\psi(e) = \langle \psi_i(e) \rangle_{i \in I}$ *for every* $e \in \mathbb{E}$*) is called the* ***soft diagonal mapping*** *of the soft mappings* $(\varphi_\psi)_i$ *(with* $i \in I$*) and it is denoted by* $\Delta = \Delta_{i \in I}(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i}$.

The following proposition establishes a useful relation about the soft image of a soft diagonal mapping.

Proposition 32. [60] *Let* $(X, \mathcal{T}, \mathbb{E})$ *be a soft topological space over a universe set* X *with respect to a set of parameter* $\mathbb{E}$*, let* $(F, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ *be a soft set of* X*, let* $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i \in I}$ *be a family of soft topological spaces over a universe set* X_i *with respect to a set of parameters* $\mathbb{E}_i$*, respectively and let* $\Delta = \Delta_{i \in I}(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i \in I} X_i)_{\prod_{i \in I} \mathbb{E}_i}$ *be the soft diagonal mapping of the soft mappings* $(\varphi_\psi)_i$*, with* $i \in I$*. Then the soft image of the soft set* $(F, \mathbb{E})$ *under the soft diagonal mapping* Δ *is soft contained in the soft product of the soft images of the same soft set under the soft mappings* $(\varphi_\psi)_i$*, that is*

$$\Delta(F, \mathbb{E}) \widetilde{\subseteq} \widetilde{\prod}_{i \in I} (\varphi_\psi)_i(F, \mathbb{E}).$$

Proof. Set $\varphi = \Delta_{i\in I}\varphi_i : X \to \prod_{i\in I} X_i$ and $\psi = \Delta_{i\in I}\psi_i : \mathbb{E} \to \prod_{i\in I} \mathbb{E}_i$, by Definition 47, we know that $\Delta = \Delta_{i\in I}(\varphi_\psi)_i = \varphi_\psi$. Suppose, by contradiction, that there exists some soft point $(x_\alpha, \mathbb{E})\tilde{\in}(F, \mathbb{E})$ such that

$$\Delta(x_\alpha, \mathbb{E}) \tilde{\notin} \widetilde{\prod}_{i\in I}(\varphi_\psi)_i(F, \mathbb{E}).$$

Set $\left(y_\beta, \prod_{i\in I} \mathbb{E}_i\right) \tilde{=} \Delta(x_\alpha, \mathbb{E}) \tilde{=} \varphi_\psi(x_\alpha, \mathbb{E})$, by Proposition 8, it follows that

$$\left(y_\beta, \prod_{i\in I} \mathbb{E}_i\right) \tilde{=} \left(\varphi(x)_{\psi(\alpha)}, \prod_{i\in I} \mathbb{E}_i\right)$$

where

$$y = \langle y_i \rangle_{i\in I} = \varphi(x) = \left(\Delta_{i\in I}\varphi_i\right)(x) = \langle \varphi_i(x) \rangle_{i\in I}$$

and

$$\beta = \langle \beta_i \rangle_{i\in I} = \psi(\alpha) = \left(\Delta_{i\in I}\psi_i\right)(\alpha) = \langle \psi_i(\alpha) \rangle_{i\in I}.$$

So, set $(G_i, \mathbb{E}_i) \tilde{=} (\varphi_\psi)_i(F, \mathbb{E})$ for every $i \in I$, we have that

$$\left(y_\beta, \prod_{i\in I} \mathbb{E}_i\right) \tilde{\notin} \widetilde{\prod}_{i\in I}(G_i, \mathbb{E}_i)$$

hence, by Proposition 5, it follows that there exists some $j \in I$ such that

$$\left((y_j)_{\beta_j}, \mathbb{E}_j\right) \tilde{\notin} (G_j, \mathbb{E}_j)$$

that, by Definition 17, means

$$y_j \notin G_j(\beta_j)$$

i.e.

$$\varphi_j(x) \notin G_j\left(\psi_j(\alpha)\right)$$

and so, by using again Definition 17, we have

$$\left(\varphi_j(x)_{\psi_j(\alpha)}, \mathbb{E}_j\right) \tilde{\notin} (G_j, \mathbb{E}_j)$$

that, by Proposition 8, is equivalent to

$$(\varphi_\psi)_j(x_\alpha, \mathbb{E}) \tilde{\notin} (G_j, \mathbb{E}_j)$$

which is a contradiction because we know that $(x_\alpha, \mathbb{E})\tilde{\in}(F, \mathbb{E})$ and by Corollary 3(1) it follows $(\varphi_\psi)_j(x_\alpha, \mathbb{E}) \tilde{\in} (\varphi_\psi)_j(F, \mathbb{E}) \tilde{=} (G_j, \mathbb{E}_j)$. □

Definition 48. *Let $\{(\varphi_\psi)_i\}_{i\in I}$ be a family of soft mappings $(\varphi_\psi)_i : \mathcal{SS}(X)_\mathbb{E} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ between a soft topological space $(X, \mathcal{T}, \mathbb{E})$ and the members of a family of soft topological spaces $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i\in I}$. We say that the family $\{(\varphi_\psi)_i\}_{i\in I}$* ***soft separates soft points*** *of $(X, \mathcal{T}, \mathbb{E})$ if for every $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_\mathbb{E}$ such that $(x_\alpha, \mathbb{E})\tilde{\neq}(y_\alpha, \mathbb{E})$ there exists some $j \in I$ such that $(\varphi_\psi)_j(x_\alpha, \mathbb{E})\tilde{\neq}(\varphi_\psi)_j(y_\beta, \mathbb{E})$.*

Definition 49. *Let $\{(\varphi_\psi)_i\}_{i\in I}$ be a family of soft mappings $(\varphi_\psi)_i : \mathcal{SS}(X)_\mathbb{E} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ between a soft topological space $(X, \mathcal{T}, \mathbb{E})$ and the members of a family of soft topological spaces $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i\in I}$. We say that the family $\{(\varphi_\psi)_i\}_{i\in I}$* ***soft separates soft points from soft closed sets*** *of $(X, \mathcal{T}, \mathbb{E})$ if for every $(C, \mathbb{E}) \in \sigma(X, \mathbb{E})$ and every $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_\mathbb{E}$ such that $(x_\alpha, \mathbb{E})\tilde{\in}\left(\tilde{X}, \mathbb{E}\right)\widetilde{\setminus}(C, \mathbb{E})$ there exists some $j \in I$ such that $(\varphi_\psi)_j(x_\alpha, \mathbb{E})\tilde{\notin}\,\text{s-cl}_{X_j}((\varphi_\psi)_j(C, \mathbb{E}))$.*

Proposition 33 (Soft Embedding Lemma). *Let* $(X, \mathcal{T}, \mathbb{E})$ *be a soft topological space,* $\{(X_i, \mathcal{T}_i, \mathbb{E}_i)\}_{i\in I}$ *be a family of soft topological spaces and* $\{(\varphi_\psi)_i\}_{i\in I}$ *be a family of soft continuous mappings* $(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(X_i)_{\mathbb{E}_i}$ *that separates both the soft points and the soft points from the soft closed sets of* $(X, \mathcal{T}, \mathbb{E})$*. Then the soft diagonal mapping* $\Delta = \Delta_{i\in I}(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ *of the soft mappings* $(\varphi_\psi)_i$ *is a soft embedding.*

Proof. Let $\varphi = \Delta_{i\in I}\varphi_i$, $\psi = \Delta_{i\in I}\psi_i$ and $\Delta = \Delta_{i\in I}(\varphi_\psi)_i = \varphi_\psi$ as in Definition 47, for every $i \in I$, by using Definition 25, we have that every corresponding soft composition is given by

$$(\pi_\rho)_i \,\tilde{\circ}\, \Delta = \left((\pi_i)_{\rho_i}\right) \tilde{\circ}\, \varphi_\psi = (\pi_i \circ \varphi)_{\rho_i \circ \psi} = (\varphi_i)_{\psi_i} = (\varphi_\psi)_i$$

which, by hypothesis, is a soft continuous mapping. Hence, by Proposition 29, it follows that the soft diagonal mapping $\Delta : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ is a soft continuous mapping.

Now, let $(x_\alpha, \mathbb{E})$ and $(y_\beta, \mathbb{E})$ be two distinct soft points of $\mathcal{SP}(X)_{\mathbb{E}}$. Since, by hypothesis, the family $\{(\varphi_\psi)_i\}_{i\in I}$ of soft mappings soft separates soft points, by Definition 48, we have that there exists some $j \in I$ such that $(\varphi_\psi)_j(x_\alpha, \mathbb{E}) \tilde{\neq} (\varphi_\psi)_j(y_\beta, \mathbb{E})$, that is

$$(\varphi_j)_{\psi_j}(x_\alpha, \mathbb{E}) \tilde{\neq} (\varphi_j)_{\psi_j}(y_\beta, \mathbb{E}).$$

Hence, by Proposition 8, we have that:

$$\left(\varphi_j(x)_{\psi_i(\alpha)}, \mathbb{E}_j\right) \tilde{\neq} \left(\varphi_j(y)_{\psi_i(\beta)}, \mathbb{E}_j\right)$$

and so, by the Definition 19 of distinct soft points, it necessarily follows that:

$$\varphi_j(x) \neq \varphi_j(y) \quad \text{or} \quad \psi_j(\alpha) \neq \psi_j(\beta).$$

Since $\varphi = \Delta_{i\in I}\varphi_i : X \to \prod_{i\in I} X_i$ and $\psi = \Delta_{i\in I}\psi_i : \mathbb{E} \to \prod_{i\in I}\mathbb{E}_i$ are usual diagonal mappings, we have that:

$$\varphi(x) \neq \varphi(y) \quad \text{or} \quad \psi(\alpha) \neq \psi(\beta)$$

and, by Definition 19, it follows that:

$$\left(\varphi(x)_{\psi(\alpha)}, \prod_{i\in I}\mathbb{E}_i\right) \tilde{\neq} \left(\varphi(y)_{\psi(\beta)}, \prod_{i\in I}\mathbb{E}_i\right)$$

hence, applying again Proposition 8, we get:

$$\varphi_\psi(x_\alpha, \mathbb{E}) \tilde{\neq} \varphi_\psi(y_\beta, \mathbb{E})$$

that is:

$$\Delta_{i\in I}(\varphi_\psi)_i(x_\alpha, \mathbb{E}) \tilde{\neq} \Delta_{i\in I}(\varphi_\psi)_i(y_\beta, \mathbb{E})$$

i.e. that $\Delta(x_\alpha, \mathbb{E}) \tilde{\neq} \Delta(y_\beta, \mathbb{E})$ which, by Corollary 2, proves the injectivity of the soft diagonal mapping $\Delta : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$.

Finally, let $(C, \mathbb{E}) \in \sigma(X, \mathbb{E})$ be a soft closed set in X and, in order to prove that the soft image $\Delta(C, \mathbb{E})$ is a soft closed set of $\sigma(\prod_{i\in I} X_i, \prod_{i\in I}\mathbb{E}_i)$, consider a soft point $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ such that $\Delta(x_\alpha, \mathbb{E}) \tilde{\notin} \Delta(C, \mathbb{E})$ and, hence, by Corollary 3(1), such that $(x_\alpha, \mathbb{E}) \tilde{\notin} (C, \mathbb{E})$. Since, by hypothesis, the family $\{(\varphi_\psi)_i\}_{i\in I}$ of soft mappings soft separates soft points from soft closed sets, by Definition 49, we have that there exists some $j \in I$ such that $(\varphi_\psi)_j(x_\alpha, \mathbb{E}) \tilde{\notin}$ s-cl$_{X_j}\left((\varphi_\psi)_j(C, \mathbb{E})\right)$, that is:

$$(\varphi_j)_{\psi_j}(x_\alpha, \mathbb{E}) \tilde{\notin} \text{s-cl}_{X_j}\left((\varphi_\psi)_j(C, \mathbb{E})\right)$$

that, by Proposition 8, corresponds to:

$$\left(\varphi_j(x)_{\psi_j(\alpha)}, \mathbb{E}_j\right) \tilde{\notin} \text{s-cl}_{X_j}\big((\varphi_\psi)_j(C, \mathbb{E})\big) \ .$$

So, set $(C_i, \mathbb{E}_i) \tilde{=} \text{s-cl}_{X_i}\big((\varphi_\psi)_i(C, \mathbb{E})\big)$ for every $i \in I$, we have in particular for $i = j$ that

$$\left(\varphi_j(x)_{\psi_j(\alpha)}, \mathbb{E}_j\right) \tilde{\notin} (C_j, \mathbb{E}_j)$$

which, by Definition 17, is equivalent to say that:

$$\varphi_j(x) \notin C_j\left(\psi_j(\alpha)\right)$$

and since the diagonal mapping $\varphi = \Delta_{i \in I} \varphi_i : X \to \prod_{i \in I} X_i$ on the universes sets is defined by $\varphi(x) = \langle \varphi_i(x) \rangle_{i \in I}$, it follows that:

$$\varphi(x) \notin \prod_{i \in I} C_i\left(\psi_i(\alpha)\right).$$

Now, since the diagonal mapping $\psi = \Delta_{i \in I} \psi_i : X \to \prod_{i \in I} X_i$ on the sets of parameters is defined by $\psi(\alpha) = \Delta_{i \in I} \psi_i(\alpha) = \langle \psi_i(\alpha) \rangle_{i \in I}$, using Definition 21, we obtain:

$$\prod_{i \in I} C_i\left(\psi_i(\alpha)\right) = \left(\prod_{i \in I} C_i\right)\left(\psi(\alpha)\right)$$

and hence that

$$\varphi(x) \notin \left(\prod_{i \in I} C_i\right)\left(\psi(\alpha)\right)$$

which, by Definitions 17 and 21, is equivalent to say that:

$$\left(\varphi(x)_{\psi(\alpha)}, \prod_{i \in I} \mathbb{E}_i\right) \tilde{\notin} \widetilde{\prod}_{i \in I} (C_i, \mathbb{E}_i)$$

that, by Proposition 8, means:

$$\varphi_\psi(x_\alpha, \mathbb{E}) \tilde{\notin} \widetilde{\prod}_{i \in I} (C_i, \mathbb{E}_i)$$

i.e.

$$\Delta(x_\alpha, \mathbb{E}) \tilde{\notin} \widetilde{\prod}_{i \in I} \text{s-cl}_{X_i}\big((\varphi_\psi)_i(C, \mathbb{E})\big) \ .$$

So, recalling, by Proposition 30, that

$$\text{s-cl}_{\prod_{i \in I} X_i}\left(\widetilde{\prod}_{i \in I} (\varphi_\psi)_i(C, \mathbb{E})\right) \tilde{=} \widetilde{\prod}_{i \in I} \text{s-cl}_{X_i}\big((\varphi_\psi)_i(C, \mathbb{E})\big)$$

it follows that:

$$\Delta(x_\alpha, \mathbb{E}) \tilde{\notin} \text{s-cl}_{\prod_{i \in I} X_i}\left(\widetilde{\prod}_{i \in I} (\varphi_\psi)_i(C, \mathbb{E})\right) .$$

Since, by Propositions 32 and 13(3) we have

$$\Delta(C, \mathbb{E}) \tilde{\subseteq} \widetilde{\prod}_{i \in I} (\varphi_\psi)_i(C, \mathbb{E}) \tilde{\subseteq} \text{s-cl}_{\prod_{i \in I} X_i}\left(\widetilde{\prod}_{i \in I} (\varphi_\psi)_i(C, \mathbb{E})\right)$$

and, by applying Propositions 14(1) and 13(5), we obtain

$$\text{s-cl}_{\prod_{i \in I} X_i}(\Delta(C, \mathbb{E})) \tilde{\subseteq} \text{s-cl}_{\prod_{i \in I} X_i}\left(\widetilde{\prod}_{i \in I} (\varphi_\psi)_i(C, \mathbb{E})\right)$$

it follows, a fortiori, that

$$\Delta(x_\alpha, \mathbb{E}) \tilde{\notin} \text{s-cl}_{\prod_{i\in I} X_i}(\Delta(C, \mathbb{E}))\,.$$

So, it is proved by contradiction that s-cl$_{\prod_{i\in I} X_i}(\Delta(C,\mathbb{E})) \tilde{\subseteq} \Delta(C,\mathbb{E})$ and hence, by Proposition 13(4) and Definition 46, that $\Delta : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ is a soft closed mapping.

Thus, we finally have that the soft diagonal mapping $\Delta = \Delta_{i\in I}(\varphi_\psi)_i : \mathcal{SS}(X)_{\mathbb{E}} \to \mathcal{SS}(\prod_{i\in I} X_i)_{\prod_{i\in I}\mathbb{E}_i}$ is a soft continuous, injective and soft closed mapping and so, by Proposition 31, it is a soft embedding. □

4. Conclusion

In this paper we have introduced the notions of family of soft mappings separating points and points from closed sets and that of soft diagonal mapping and we have proved a generalization to soft topological spaces of the well-known Embedding Lemma for classical (crisp) topological spaces. Such a result could be the start point for extending and investigating other important topics such as extension and compactifications theorems, metrization theorems etc. in the context of soft topology.

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