 doi:10.20944/preprints201905.0282.v1

Mueller quaternion with matrix coefficients for depolarizing and nondepolarizing media

M. A. Kuntman* and Ahmet Kuntman†

May 21, 2019

Abstract

It is shown that a Mueller quaternion can be formulated as a quaternion with matrix coefficients, and this Mueller quaternion transforms the Stokes quaternion for depolarizing process as well as for nondepolarizing process. Mueller quaternion keeps track of the phase acquired by the Stokes quaternion and, in general, quaternion states of optical media comprises all properties of Jones, Mueller-Jones and Mueller matrices.

1 Introduction

In previous works [1], [2] there were two different representations of Mueller-Jones states: the covariance vector $|h\rangle$ and the $\mathbf{Z}$ matrix.

The dimensionless components of $|h\rangle$ can be parametrized as τ, α, β, γ:

$$|h\rangle = \begin{pmatrix} \tau \\ \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{1}$$

where α, β and γ are generally complex numbers, while τ can always be chosen as real and positive if the global phase is not taken into account.

On the other hand, in the above mentioned works there was another object, $\mathbf{Z}$, that also serves as a Mueller-Jones state in matrix form:

$$\mathbf{Z} = \begin{pmatrix} \tau & \alpha & \beta & \gamma \\ \alpha & \tau & -i\gamma & i\beta \\ \beta & i\gamma & \tau & -i\alpha \\ \gamma & -i\beta & i\alpha & \tau \end{pmatrix}. \tag{2}$$

By direct matrix multiplication it was shown that the Mueller matrix of nondepolarizing optical media can be written as:

$$\mathbf{M} = \mathbf{Z}\mathbf{Z}^* = \mathbf{Z}^*\mathbf{Z}. \tag{3}$$

*Independent researcher, Ordu, Turkey

†Independent researcher, Izmir, Turkey (email: ahkuntman@gmail.com).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2019 doi:10.20944/preprints201905.0282.v1

Explicit form of the Mueller matrix in terms of the parameters τ, α, β and γ can be found in [1].

Z matrices, being 4×4 analogues of the Jones matrices, are suitable to obtain the product state of the combined system associated with a serial combination of optical media:

$$\mathbf{Z} = \mathbf{Z}_N \cdot \mathbf{Z}_{N-1} \cdots \mathbf{Z}_2 \cdot \mathbf{Z}_1. \tag{4}$$

But, similar algebra is not possible with $|h\rangle$ vectors. Therefore, $|h\rangle$ vectors and $\mathbf{Z}$ matrices appear as different entities in their present vector and matrix forms.

In a recent work [3] it was shown that $\mathbf{Z}$ matrices and $|h\rangle$ vectors are actually two different representations of the same quantity which is isomorphic to the $\boldsymbol{h}$ quaternion by observing that the $\mathbf{Z}$ matrix can be written as a linear combination of four basis matrices:

$$\mathbf{Z} = \tau\mathbb{1} + i\alpha I + i\beta J + i\gamma K, \tag{5}$$

where

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

with the following properties:

$$\begin{gathered} I^2 = J^2 = K^2 = IJK = -\mathbb{1} \\ IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J \end{gathered} \tag{7}$$

Obviously, $\mathbb{1}, I, J, K$ basis are isomorphic to basis quaternions, $\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$, defined by Hamilton [5]:

$$\begin{gathered} \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1} \\ \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j} \end{gathered} \tag{8}$$

Therefore, the $\mathbf{Z}$ matrix is isomorphic to the $\boldsymbol{h}$ quaternion:

$$\boldsymbol{h} = \tau\mathbf{1} + i\alpha\boldsymbol{i} + i\beta\boldsymbol{j} + i\gamma\boldsymbol{k}, \tag{9}$$

which is directly related to the covariance vector $|h\rangle$.

It was also shown that the Jones matrix is also isomorphic to the quaternion $\boldsymbol{h}$ by writing the Jones matrix in terms of Pauli matrices [3]:

$$\mathbf{J} = \tau\sigma_0 + \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3. \tag{10}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2019 doi:10.20944/preprints201905.0282.v1

which can be written as

$$\mathbf{J} = \tau\sigma_0 + i\alpha(-i\sigma_1) + i\beta(-i\sigma_2) + i\gamma(-i\sigma_3). \tag{11}$$

The 2×2 matrices σ_0, $-i\sigma_1$, $-i\sigma_2$ and $-i\sigma_3$ are, respectively, isomorphic to the quaternion basis, $\mathbf{1}$, $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. Therefore, the Jones matrix can be written as a Jones quaternion:

$$\boldsymbol{J} = \tau\mathbf{1} + i\alpha\boldsymbol{i} + i\beta\boldsymbol{j} + i\gamma\boldsymbol{k} \equiv \boldsymbol{h}. \tag{12}$$

In short, the vector state $|h\rangle$, the matrix state $\mathbf{Z}$ and the Jones matrix $\mathbf{J}$ are isomorphic to the same quaternion state, $\boldsymbol{h}$.

It was also shown that a Stokes quaternion can be rotated by $\boldsymbol{h}$ [3], [7]:

$$\boldsymbol{s'} = \boldsymbol{h}\boldsymbol{s}\boldsymbol{h}^\dagger, \tag{13}$$

where $\boldsymbol{h}^\dagger$ is the the Hermitian conjugate of the $\boldsymbol{h}$ quaternion:

$$\boldsymbol{h}^\dagger = \tau^*\mathbf{1} + i\alpha^*\boldsymbol{i} + i\beta^*\boldsymbol{j} + i\gamma^*\boldsymbol{k}, \tag{14}$$

$\boldsymbol{s}$ is the Stokes quaternion that corresponds to the Stokes vector $|s\rangle = (s_0, s_1, s_2, s_3)^T$:

$$\boldsymbol{s} = s_0\mathbf{1} + is_1\boldsymbol{i} + is_2\boldsymbol{j} + is_3\boldsymbol{k}, \tag{15}$$

and $\boldsymbol{s'}$ is the transformed (rotated) Stokes quaternion which corresponds to the transformed Stokes vector $|s'\rangle$:

$$|s'\rangle = \mathbf{M}|s\rangle, \tag{16}$$

where $\mathbf{M}$ is a nondepolarizing Mueller matrix.

In order to incorporate the depolarization effects into the formalism the following procedure was followed [3]:

Any linear combination of quaternions is also a quaternion, and a coherent linear combination of quaternion states can be written as,

$$\boldsymbol{h} = a\boldsymbol{h}_1 + b\boldsymbol{h}_2 + c\boldsymbol{h}_3 \cdots \tag{17}$$

where the coefficients $a, b, c, \ldots$ are, in general, complex numbers.

If the process is coherent then the Stokes quaternion is subjected to a rotation by the quaternion state $\boldsymbol{h}$ associated with the combined system as given by Eq. (13). If the process is incoherent we have to consider depolarization effects. In this case, the covariance matrix $\mathbf{H}$ associated with a depolarizing Mueller matrix will be of rank > 1, and depolarizing Mueller matrix can be written as a convex sum of at most four nondepolarizing Mueller matrices [4], [6]:

$$\mathbf{M} = w_1\mathbf{M_1} + w_2\mathbf{M_2} + w_3\mathbf{M_3} + w_4\mathbf{M_4}, \tag{18}$$

where $\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}$ and $\mathbf{M_4}$ are nondepolarizing Mueller matrices; w_1, w_2, w_3, and w_4 are real and positive numbers with the condition,

$$w_1 + w_2 + w_3 + w_4 = 1. \tag{19}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2019 doi:10.20944/preprints201905.0282.v1

Decomposition of a depolarizing Mueller matrix into its nondepolarizing components is not unique. In the spectral (Cloude) decomposition [8], weights w_i are the eigenvalues of the covariance matrix, $\mathbf{H}$, and the component matrices $\mathbf{M}_i$ are the nondepolarizing Mueller matrices corresponding to the associated eigenvectors of $\mathbf{H}$.

For an incoherent combination, from the linearity of the convex summation, we can immediately write a transformation formula for the Stokes quaternion:

$$\boldsymbol{s'} = \sum_{i=1}^{4} w_i \boldsymbol{h}_i \boldsymbol{s} \boldsymbol{h}_i^{\dagger}. \tag{20}$$

The same depolarization scheme given in [2] applies to this quaternion formulation as well. But, in the following it will be shown that there exists a Mueller quaternion with matrix coefficients which can transform the Stokes quaternion for depolarizing and nondepolarizing processes.

2 Muller quaternion: A quaternion with matrix coefficients and transformation of the Stokes quaternion for depolarizing and nondepolarizing processes

Let $\mathbf{M}$ be the Mueller-Jones state of a nondepolarizing optical media. This state can be represented by a covariance vector $|h\rangle$, by a matrix state $\mathbf{Z}$, by a Jones matrix $\mathbf{J}$ or by a quaternion $\boldsymbol{h}$. But, how can it be possible to formulate the Mueller matrix of a depolarizing medium in terms of quaternions?

It is possible to start with $\mathbf{Z}\mathbf{Z}^* = \mathbf{M}$ and show that a nondepolarizing Mueller matrix can be written as follows:

$$\mathbf{M} = T_0 \mathbb{1} + iT_1 I + iT_2 J + iT_3 K. \tag{21}$$

where,

$$T_0 = \begin{pmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{pmatrix}, \quad T_1 = \begin{pmatrix} M_{12} & 0 & 0 & 0 \\ 0 & M_{21} & 0 & 0 \\ 0 & 0 & iM_{34} & 0 \\ 0 & 0 & 0 & -iM_{43} \end{pmatrix}, \tag{22}$$

$$T_2 = \begin{pmatrix} M_{13} & 0 & 0 & 0 \\ 0 & -iM_{24} & 0 & 0 \\ 0 & 0 & M_{31} & 0 \\ 0 & 0 & 0 & iM_{42} \end{pmatrix}, \quad T_3 = \begin{pmatrix} M_{14} & 0 & 0 & 0 \\ 0 & iM_{23} & 0 & 0 \\ 0 & 0 & -iM_{32} & 0 \\ 0 & 0 & 0 & -iM_{41} \end{pmatrix}. \tag{23}$$

Actually, any 4×4 matrix can be written as in Eq.(21) with the basis matrices $\mathbb{1}, I, J$ and K; but, Eq.(21) is still a matrix equation. In order to capture the Mueller quaternion, $\hat{\boldsymbol{m}}$, it is enough to shift to the quaternion basis, $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2019 doi:10.20944/preprints201905.0282.v1

$$\hat{\boldsymbol{m}} = T_0\mathbf{1} + iT_1\boldsymbol{i} + iT_2\boldsymbol{j} + iT_3\boldsymbol{k}, \tag{24}$$

where the hat symbol indicates that the Mueller quaternion is a quaternion with matrix coefficients.

It is worth to note that the Mueller quaternion $\hat{\boldsymbol{m}}$ is based on the elements of the Mueller matrix and it is capable of representing both nondepolarizing and depolarizing processes.

Transformation of the Stokes quaternion by means of the Mueller quaternion comes into play as a quaternion product of two quaternions:

$$\hat{\boldsymbol{s}}' = \hat{\boldsymbol{m}}\boldsymbol{s}, \tag{25}$$

where $\hat{\boldsymbol{s}}'$ is the transformed Stokes quaternion with matrix coefficients and $\hat{\boldsymbol{m}}$ is the Mueller quaternion that can be a quaternion state corresponding to a nondepolarizing or a depolarizing optical medium.

In an explicit form of $\hat{\boldsymbol{s}}'$ takes the form:

$$\begin{aligned}\hat{\boldsymbol{s}}' = \hat{\boldsymbol{m}}\boldsymbol{s} = &(T_0s_0 + T_1s_1 + T_2s_2 + T_3s_3)\mathbf{1} + i(T_0s_1 + T_1s_0 + iT_2s_3 - iT_3s_2)\boldsymbol{i} \\ &+ i(T_0s_2 + T_2s_0 + iT_3s_1 - iT_1s_3)\boldsymbol{j} + i(T_0s_3 + T_3s_0 + iT_1s_2 - iT_2s_1)\boldsymbol{k}.\end{aligned} \tag{26}$$

Note that $\boldsymbol{s}$ is a quaternion with scalar coefficients, but $\hat{\boldsymbol{s}}'$ is a quaternion with matrix coefficients, hence, $\hat{\boldsymbol{s}}'$ contains more information from the Stokes quaternion, $\boldsymbol{s}$. In particular, $\hat{\boldsymbol{s}}'$ contains the phase acquired by the light while passing through the medium. This is because, in contrast to the Mueller matrix, $\mathbf{M}$, the Mueller quaternion, $\hat{\boldsymbol{m}}$, can keep track of the phase.

It is also possible to recover the vector form of the transformed Stokes vector $|s'\rangle$ of Eq.(16) by simply taking the quaternion scalar product of $\hat{\boldsymbol{s}}'$ with the vector quaternion $\vec{\boldsymbol{u}}$:

$$|s'\rangle = \hat{\boldsymbol{s}}' \cdot \vec{\boldsymbol{u}} \tag{27}$$

where $\vec{\boldsymbol{u}}$ is a quaternion with standard basis vector coefficients:

$$\vec{\boldsymbol{u}} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\mathbf{1} + i\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\boldsymbol{i} + i\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\boldsymbol{j} + i\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\boldsymbol{k}. \tag{28}$$

It can also be shown that, in case of a nondepolarizing process, the $\boldsymbol{h}$ quaternion can be easily obtained from the Mueller quaternion.

3 Conclusion

A Mueller quaternion, $\hat{\boldsymbol{m}}$, is introduced. Mueller quaternion is a quaternion with matrix coefficients and it can describe depolarization effects. Mueller quaternion resides on the elements of the Mueller matrix which are directly accessible by measurement. Mueller matrix transformation of the Stokes vector can also be accomplished by the Mueller quaternion for depolarizing and nondepolarizing processes. Furthermore, Mueller quaternion can keep track of the phase acquired by the Stokes quaternion. Hence, the Mueller quaternion comprises all properties of Jones, Mueller-Jones and Mueller matrices and in case of a nondepolarizing process it is possible to recover the basic vector, matrix or quaternion states of the previous formalisms. As a result, quaternion forms of optical media states embraces all views of the Jones-Stokes-Mueller formalism, including coherent linear combination of Mueller-Jones states with phases, and depolarization effects in case of incoherent processes.

References

[1] E. Kuntman, M. Ali Kuntman, and O. Arteaga, J. Opt. So c. Am. A 34, 80 (2017).

[2] E. Kuntman, M. A. Kuntman, J. Sancho-Parramon, and O. Arteaga, Formalism of optical coherence and polarization based on material media states, Phys. Rev. A 95, 063819 (2017). 11.

[3] E. Kuntman, M. A. Kuntman, A. Canillas, O. Arteaga, J. Opt. Soc. Am. A 36, 492-497 (2019).

[4] J. J. Gil, Eur. Phys. J. Appl. Phys. 40, 1 (2007).

[5] W. R. Hamilton. On a new species of imaginary quantities connected with the theory of quaternions. Proceedings of the Royal Irish Academy, 2:424434, 1844.

[6] J. J. Gil, Journal of Applied Remote Sensing 8, 081599 (2014).

[7] Lanlan Liu Chongqing Wu Chao Shang Zhengyong Li Jian Wang, IEEE Photonics Journal (Volume: 7, Issue: 4, Aug. 2015).

[8] S.R. Cloude, Group theory and polarization algebra," Optik 75, 26-36 (1986)