Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at *Mathematical and Cmputational Applications* **2019**, *24*, 53; doi:10.3390/mca24020053

A STUDY ON NOVEL EXTENSIONS FOR THE p-ADIC GAMMA AND p-ADIC BETA FUNCTIONS

UGUR DURAN AND MEHMET ACIKGOZ

Abstract. In this paper, we introduce the (ρ, q)-analogue of the p-adic factorial function. By utilizing some properties of (ρ, q)-numbers, we obtain several new and interesting identities and formulas. We then construct the p-adic (ρ, q)-gamma function by means of the mentioned factorial function. We investigate several properties and relationships belonging to the foregoing gamma function, some of which are given for the case $p = 2$. We also derive more representations of the p-adic (ρ, q)-gamma function in general case. Moreover, we consider the p-adic (ρ, q)-Euler constant derived from the derivation of p-adic (ρ, q)-gamma function at $x = 1$. Furthermore, we provide a limit representation of aforementioned Euler constant based on (ρ, q)-numbers. Finally, we consider (ρ, q)-extension of the p-adic beta function via the p-adic (ρ, q)-gamma function and we then investigate various formulas and identities.

1. Introduction

The p-adic numbers are a counterintuitive arithmetic system, which were firstly introduced by the Kummer in 1850. Then, the German mathematician, Kurt Hensel (1861-1941) developed the p-adic numbers in a paper concerned with the development of algebraic numbers in power series in circa 1897, *cf.* [25]. There are numbers of all kinds such as natural, rational, real, complex, p-adic, quantum numbers. The p-adic numbers are less well known than the others, however these numbers play a main role in number theory and the related topics in mathematics. Whereas, mentioned p-adic numbers have penetrated some mathematical areas, among algebraic number theory, algebraic geometry, algebraic topolgy and analysis, the foregoing numbers are now well-established in mathematical field and are used also by physicists. In conjunction with the introduction of these numbers, some mathematicians and physicists started to investigate new scientific tools utilizing their useful and positive properties. Some effects of these new researches have emerged in mathematics and physics such as p-adic analysis, string theory, p-adic quantum mechanics, quantum field theory, representation theory, algebraic geometry, complex systems, dynamical systems, genetic codes and so on (*cf.* [1-3, 6-16, 18-24, 25]). The one of the most important tool of these investigations is p-adic gamma function which is firstly described by Yasou Morita in about 1975 (*cf.* [18]). Intense research activities in such an area as p-adic gamma function is principally motivated by their importance in p-adic analysis. Therefore, in recent fourty years, p-adic gamma function and its generalizations have been investigated and studied extensively by many mathematicians (*cf.* [6, 8-11, 13-16, 18, 20, 21, 25]).

Here, we give some basic notations, definitions and properties belonging to the p-adic analysis which are taken from the books: [13], [21] and [25].

Let $p \in \{2, 3, 5, 7, 11, 13, 17, \cdots\}$ be a prime number. For any nonzero integer a, let $ord_p a$ be the highest power of p that divides a, i.e., the greatest m such that $a \equiv 0 \,(\mathrm{mod}\, p^m)$ where we used the notation $a \equiv b \,(\mathrm{mod}\, c)$ meant c divides $a - b$.

Note that $ord_p 0 = \infty$. The following properties hold true for $x = ab$ and $y = \frac{c}{d}$:

$$ord_p x = ord_p a + ord_p b \text{ and } ord_p y = ord_p c - ord_p d.$$

1991 *Mathematics Subject Classification.* Primary 05A10, 05A30; Secondary 11B65, 11S80, 33B15.

Key words and phrases. p-adic numbers, p-adic factorial function, p-adic gamma function, p-adic beta function, p-adic Euler constant, (ρ, q)-numbers.

Note: p and q (or (p, q)) were used as parameters of (p, q)-calculus in [2, 3, 5, 7, 17, 22-24], but in this paper we use the notation (ρ, q) in order to avoid confusions with p-adic q-gamma function.

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

The p-adic absolute value (norm) of x is given by

$$|x|_p = \begin{cases} p^{-ord_p x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{1.1}$$

The p-adic norm provides the so called strong triangle inequality

$$|x+y|_p \leqq \max\left\{|x|_p, |y|_p\right\},$$

which is also known as non-Archimedean norm.

Now we provide some basic notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of all natural numbers, $\mathbb{Z} = \{\cdots,, -1, 0, 1, \cdots\}$ denotes the ring of all integers, $\mathbb{Q} = \left\{\frac{a}{b} \,|a, b \in \mathbb{Z}, b \neq 0\right\}$ denotes the field of all rational numbers, $\mathbb{C}$ denotes the field of all complex numbers, $\mathbb{Q}_p = \left\{x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leqq a_i \leqq p-1\right\}$ denotes the field of all p-adic numbers, $\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p : |x|_p \leqq 1\right\}$ denotes the ring of all p-adic integers and $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For more information about p-adic analysis, see, e.g., [1-3, 6-16, 18-24, 25].

The notations ρ and q can be variously considered as indeterminates, complex numbers ρ and $q \in \mathbb{C}$ with $0 < |q| < |\rho| \leqq 1$, or p-adic numbers ρ and $q \in \mathbb{C}_p$ with $|\rho - 1|_p < p^{-\frac{1}{p-1}}$ and $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $\rho^x = \exp(x \log \rho)$ and $q^x = \exp(x \log q)$ for $|x|_p \leqq 1$.

The classical gamma function is firstly introduced by Leonard Euler (1707-1783) as

$$\Gamma(x) = \int_0^1 (-\log t)^{x-1} dt \quad (x > 0).$$

In 1964, the common form of the gamma function is given by Artin [4] with appropriate variable change:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0).$$

The classical gamma function is closely related with the factorial function $n!$ as $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

By inspiring the beautiful and interesting relation between gamma function and factorial function above, the p-adic gamma function is also introduced by means of the p-adic factorial function $(n!)_p$ as follows

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n (n!)_p, \tag{1.2}$$

where the factorial function $(n!)_p$ in $\mathbb{Q}_p$ is defined by

$$(n!)_p = \prod_{\substack{j<n \\ (p,j)=1}} j \tag{1.3}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. For detailed statement of these issue, see [9, 10, 13, 18, 21, 25].

The q-extension of the p-adic gamma function is defined as follows (see [20])

$$\Gamma_{p,q}(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} [j]_q \text{ where } [j]_q = \frac{1-q^j}{1-q}. \tag{1.4}$$

These functions have been studied and investigated by many mathematicians, see [8-11, 13-15, 18, 20].

The (ρ, q)-numbers are defined by

$$[n]_{\rho,q} := \frac{\rho^n - q^n}{\rho - q} \tag{1.5}$$

which reduce to the q-numbers when $\rho = 1$ as $[n]_{1,q} \to [n]_q$.

It is clear that $[n]_{\rho,q} = \rho^{n-1} [n]_{q/\rho}$, which means that q-numbers and (ρ, q)-numbers are different, that is, (ρ, q)-numbers can not be obtained just by substituting q by q/ρ in the definition of q-numbers (see [2, 3, 5, 7, 17, 22-24] for details). But, when $\rho = 1$, q-numbers becomes a special case of (ρ, q)-numbers as shown above.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at *Mathematical and Cmputational Applications* **2019**, *24*, 53; doi:10.3390/mca24020053

In conjuction with the introduction of these (ρ, q)-numbers (see [5]), (ρ, q)-calculus has been investigated and studied extensively by many mathematicians and also physicists since 1991. For example, Araci *et al.* [2] introduced an analogue of Haar distribution based on (ρ, q)-numbers. By means of this distribution, they derived (ρ, q)-analogue of Volkenborn integral (p-adic integral) and obtained some properties. Then, they constructed (ρ, q)-Bernoulli polynomials arising from (ρ, q)-Volkenborn integral. Aral *et al.* [3] defined a (ρ, q)-analogue of Gamma function and as an application, they proposed (ρ, q)-Szasz–Durrmeyer operators, estimated moments and established some direct results. Chakrabarti *et al.* [5] investigated the necessary elements of the (ρ, q)-calculus involving (ρ, q)-exponential, (ρ, q)-integration, and the (ρ, q)-differentiation. Duran *et al.* [7] considered a generalization of the fermionic p-adic measure based on (ρ, q)-integers and set the corresponding integral to this measure. They also defined Carlitz's-type (ρ, q)-Euler polynomials and numbers in terms of this corresponding integral and acquired some of their identities and properties. Milovanovic *et al.* [17] provide a novel extension of beta functions based on (ρ, q)-numbers and committed the integral modification of the generalized Bernstein polynomials. Sadjang [22] introduced new generalizations of the gamma and the beta functions and investigated their properties. Sadjang [23] investigated some properties of the (ρ, q)-derivative and the (ρ, q)-integration and provided two appropriate polynomial bases for the (ρ, q)-derivative, and then he obtained various properties of these bases. As an application, he gave two (ρ, q)-Taylor formulas for polynomials. Furthermore, he gave the fundamental theorem of (ρ, q)-calculus and proved the formula of (ρ, q)-integration by part. Sahai *et al.* [24] developed the connection between (ρ, q)-analogue of special functions and representations of certain two parameter quantum algebras.

The paper is organized as follows. The first part is introduction which provides the required information, notations, definitions and motivation. In the second part, we are interested in constructing the p-adic (ρ, q)-gamma function $\Gamma_p^{[\rho,q]}(x)$ by means of p-adic (ρ, q)-factorial function $(x!)_p^{[\rho,q]}$. We investigate some properties and relationships of the mentioned gamma function. In Part 3, the p-adic (ρ, q)-Euler constant is derived from the derivation of p-adic (ρ, q)-gamma function at $x = 1$ and limit representation of this constant are shown. In the third part, we examine the results derived in this paper and give some further remarks of our results. The last part provides the (ρ, q)-extension of the p-adic beta function via the p-adic (ρ, q)-gamma function and includes multifarious formulas and identities.

2. The p-adic (ρ, q)-Gamma Function

This section provides a new definition of p-adic (ρ, q)-gamma function and gives some properties, identities and relations for the mentioned gamma function.

We firstly introduce (ρ, q)-extension of the p-adic factorial function as follows.

Definition 1. *Let ρ and $q \in \mathbb{C}_p$ with $|\rho - 1|_p < 1$ and $|q-1|_p < 1$, $\rho \neq 1$ and $q \neq 1$. We introduce the p-adic (ρ, q)-factorial function $(x!)_p^{[\rho,q]}$ in $\mathbb{Q}_p$ as*

$$(x!)_p^{[\rho,q]} = \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} \frac{\rho^j - q^j}{\rho - q} = \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\rho,q} \tag{2.1}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers.

Note that for $n \in \mathbb{N}$, the p-adic (ρ, q)-factorial function can be written as

$$(n!)_p^{[\rho,q]} = \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\rho,q}\,. \tag{2.2}$$

Proposition 1. *For $n \in \mathbb{N}$, we have*

$$(1!)_p^{[\rho,q]} = 1,\ (2!)_p^{[\rho,q]} = 1 \text{ and } \left|(n!)_p^{[\rho,q]}\right|_p = 1.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Example 1. *We provide some examples of the foregoing function:*

$(3!)_2^{[\rho,q]}=1$	$(3!)_3^{[\rho,q]}=[2]_{\rho,q}$	$(3!)_5^{[\rho,q]}=[2]_{\rho,q}$
$(6!)_2^{[\rho,q]}=[3]_{\rho,q}[5]_{\rho,q}$	$(6!)_3^{[\rho,q]}=[2]_{\rho,q}[4]_{\rho,q}[5]_{\rho,q}$	$(6!)_5^{[\rho,q]}=[2]_{\rho,q}[3]_{\rho,q}[4]_{\rho,q}$
$(7!)_2^{[\rho,q]}=[3]_{\rho,q}[5]_{\rho,q}$	$(7!)_3^{[\rho,q]}=[2]_{\rho,q}[4]_{\rho,q}[5]_{\rho,q}$	$(7!)_5^{[\rho,q]}=[2]_{\rho,q}[3]_{\rho,q}[4]_{\rho,q}[6]_{\rho,q}$

By (1.5), we note that

$$[n+m]_{\rho,q}=\rho^n[m]_{\rho,q}+q^m[n]_{\rho,q}=\rho^m[n]_{\rho,q}+q^n[m]_{\rho,q}. \tag{2.3}$$

Using the addition property (2.3) of the (ρ,q)-integers, we give the following theorem.

Theorem 1. *For $n,m\in\mathbb{N}$, we have*

$$((n+m)!)_p^{[\rho,q]}=(n!)_p^{[\rho,q]}\cdot\begin{cases}\left(\rho^n(m!)_p^{[\rho,q]}+[n]_{\rho,q}q^{\binom{m-1}{2}-p\left(1+2+\lfloor\frac{m-1}{p}\rfloor\right)}\right) & \text{if } d=0\\ \rho^n\prod\limits_{\substack{j<m\\(p,d+j)=1}}[j]_{\rho,q}+[n]_{\rho,q}\prod\limits_{\substack{j<m\\(p,d+j)=1}}q^j & \text{if } d\in A\end{cases}, \tag{2.4}$$

where $n=pk+d$ and $A=\{1,2,\ldots,p-1\}$ and $\lfloor\cdot\rfloor$ is the greatest integer function.

Proof. In view of (2.7) and (2.3), we get

$$\begin{aligned}((n+m)!)_p^{[\rho,q]} &= \prod_{\substack{j<n+m\\(p,j)=1}}[j]_{\rho,q}=\prod_{\substack{j<n\\(p,j)=1}}[j]_{\rho,q}\prod_{\substack{j<m\\(p,n+j)=1}}[n+j]_{\rho,q}\\ &= \prod_{\substack{j<n\\(p,j)=1}}[j]_{\rho,q}\prod_{\substack{j<m\\(p,n+j)=1}}\left(\rho^n[j]_{\rho,q}+q^j[n]_{\rho,q}\right)\\ &= (n!)_p^{[\rho,q]}\left(\rho^n\prod_{\substack{j<m\\(p,n+j)=1}}[j]_{\rho,q}+[n]_{\rho,q}\prod_{\substack{j<m\\(p,n+j)=1}}q^j\right)\\ &= (n!)_p^{[\rho,q]}\begin{cases}\left(\rho^n(m!)_p^{[\rho,q]}+[n]_{\rho,q}q^{\binom{m-1}{2}-\left(p+2p+\lfloor\frac{m-1}{p}\rfloor p\right)}\right) & \text{if } d=0\\ \rho^n\prod\limits_{\substack{j<m\\(p,d+j)=1}}[j]_{\rho,q}+[n]_{\rho,q}\prod\limits_{\substack{j<m\\(p,d+j)=1}}q^j & \text{if } d\in A\end{cases},\end{aligned}$$

where $n=pk+d$ and $A=\{1,2,\ldots,p-1\}$. Thus, we attain the asserted result (2.4). □

We give the following interesting result.

Theorem 2. *For $m\in\mathbb{N}_0$, we have*

$$(\varphi_p(m)!)_p^{[\rho,q]}=(a_0!)_p^{[\rho,q]}\prod_{t=1}^{m}\prod_{\substack{j<a_tp^t\\(p,j)=1}}[\varphi_{t-1}+j]_{\rho,q}, \tag{2.5}$$

where $\varphi_p(m)=a_0+a_1p+a_2p^2+\cdots+a_mp^m$ with $a_0,a_1,\ldots a_m\in\{1,2,\ldots,p-1\}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at *Mathematical and Cmputational Applications* **2019**, *24*, 53; doi:10.3390/mca24020053

Proof. Indeed,

$$\begin{aligned}\left(\varphi_p(m)!\right)_p^{[\rho,q]} &= \left(\left(\varphi_p(m-1)\right)!\right)_p^{[\rho,q]} \prod_{\substack{j<a_m p^m\\(p,j)=1}} \left[\varphi_p(m-1)+j\right]_{\rho,q} \\ &= \left(\left(\varphi_p(m-2)\right)!\right)_p^{[\rho,q]} \prod_{\substack{j<a_{m-1} p^{m-1}\\(p,j)=1}} \left[\varphi_p(m-2)+j\right]_{\rho,q} \prod_{\substack{j<a_m p^m\\(p,j)=1}} \left[\varphi_p(m-1)+j\right]_{\rho,q} \\ &\vdots \\ &= (a_0!)_p^{[\rho,q]} \prod_{t=1}^{m} \prod_{\substack{j<a_t p^t\\(p,j)=1}} \left[\varphi_p(t-1)+j\right]_{\rho,q},\end{aligned}$$

which completes the proof of this theorem. □

The following definition is new and plays an important role in deriving the main results of this paper. Now we are ready to state the following Definition 2.

Definition 2. *Let ρ and $q \in \mathbb{C}_p$ with $|\rho-1|_p < 1$ and $|q-1|_p < 1$, $\rho \neq 1$ and $q \neq 1$. We define the p-adic (ρ,q)-gamma function as follows*

$$\Gamma_p^{[\rho,q]}(x) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n\\(p,j)=1}} \frac{\rho^j - q^j}{\rho - q} = \lim_{n\to x} (-1)^n \prod_{\substack{j<n\\(p,j)=1}} [j]_{\rho,q} \tag{2.6}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers.

Note that for $n \in \mathbb{N}$, the p-adic (ρ,q)-gamma function can be written as

$$\Gamma_p^{[\rho,q]}(n) = (-1)^n \prod_{\substack{j<n\\(p,j)=1}} [j]_{\rho,q}.$$

Example 2. *We give some examples of the aforementioned function:*

$\Gamma_2^{[\rho,q]}(3) = -1$	$\Gamma_3^{[\rho,q]}(3) = -[2]_{\rho,q}$	$\Gamma_5^{[\rho,q]}(3) = -[2]_{\rho,q}$
$\Gamma_2^{[\rho,q]}(6) = [3]_{\rho,q}$	$\Gamma_3^{[\rho,q]}(6) = [2]_{\rho,q}[4]_{\rho,q}$	$\Gamma_5^{[\rho,q]}(6) = [2]_{\rho,q}[3]_{\rho,q}[4]_{\rho,q}$
$\Gamma_2^{[\rho,q]}(7) = -[3]_{\rho,q}[5]_{\rho,q}$	$\Gamma_3^{[\rho,q]}(7) = -[2]_{\rho,q}[4]_{\rho,q}[5]_{\rho,q}$	$\Gamma_5^{[\rho,q]}(7) = -[2]_{\rho,q}[3]_{\rho,q}[4]_{\rho,q}[6]_{\rho,q}$

Remark 1. *Upon setting $\rho = 1$ in Def. 2, p-adic (ρ,q)-gamma function reduces to the p-adic q-gamma function in (1.4).*

Remark 2. *When $q \to \rho = 1$ in Def. 2, Eq. (2.6) yields to the p-adic gamma function in (1.2).*

We now investigate some properties and relations of the aforementioned function.

Lemma 1. *For $n \in \mathbb{N}$, we have*

$$\Gamma_p^{[\rho,q]}(0) = 1,\ \Gamma_p^{[\rho,q]}(1) = -1,\ \Gamma_p^{[\rho,q]}(2) = 1 \text{ and } \left|\Gamma_p^{[\rho,q]}(n)\right|_p = 1.$$

Proof. The proof of this lemma just follows from the Definition 2. So we omit the proof. □

Taking into account Theorem 1, we obtain the following relation.

Corollary 1. *For $n, m \in \mathbb{N}$, we have*

$$\Gamma_p^{[\rho,q]}(n+m) = (-1)^{n+m} \Gamma_p^{[\rho,q]}(n) \cdot \begin{cases} \left(\rho^n \Gamma_p^{[\rho,q]}(m) + [n]_{\rho,q}\, q^{\left(\binom{m}{2}-1\right) - p\left(1+2+\left\lfloor \frac{m-1}{p}\right\rfloor\right)}\right) & \text{if } d = 0 \\ \rho^n \prod\limits_{\substack{j<m\\(p,d+j)=1}} [j]_{\rho,q} + [n]_{\rho,q} \prod\limits_{\substack{j<m\\(p,d+j)=1}} q^j & \text{if } d \in A \end{cases},$$

where $n = pk + d$ and $A = \{1, 2, \ldots, p-1\}$ and $\lfloor\cdot\rfloor$ is the greatest integer function.

Considering that Theorem 2, we have the following identity.

Corollary 2. *For $m \in \mathbb{N}_0$, we have*

$$\Gamma_p^{[\rho,q]}\left(\varphi_p(m)\right) = (-1)^{\varphi_m} (a_0!)_p^{[\rho,q]} \prod_{t=1}^{m} \prod_{\substack{j<a_t p^t \\ (p,j)=1}} \left[\varphi_p(t-1)+j\right]_{\rho,q},$$

where $\varphi_p(m) = a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^m$ with $a_0, a_1, \ldots a_m \in \{1, 2, \ldots, p-1\}$.

Here is a recurrence relation for $\Gamma_p^{[\rho,q]}(n)$ by the following theorem.

Theorem 3. *The following recurrence formula holds true for all $x \in \mathbb{Z}_p$:*

$$\Gamma_p^{[\rho,q]}(x+1) = \epsilon_p^{[\rho,q]}(x)\, \Gamma_p^{[\rho,q]}(x), \tag{2.7}$$

where

$$\epsilon_p^{[\rho,q]}(x) = \begin{cases} -[x]_{\rho,q} & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1. \end{cases} \tag{2.8}$$

Proof. Using Definition 2 and Eq. (1.1), we easily get

$$\Gamma_p^{[\rho,q]}(x+1) = \lim_{n\to x} (-1)^n \prod_{\substack{j<n+1 \\ (p,j)=1}} [j]_{\rho,q} = \left(\lim_{n\to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\rho,q}\right) \cdot \begin{cases} -[x]_{\rho,q} & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p = 1, \end{cases}$$

which gives the desired result (2.7). □

The result obtained in the Theorem 3 seems to be p-adic (ρ, q)-analogue of the well known result for classical gamma function $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

We now give an explicit formula for $\Gamma_p^{[\rho,q]}(n)$ as follows.

Theorem 4. *The following recurrence formula holds true for all $n \in \mathbb{N}$:*

$$\Gamma_p^{[\rho,q]}(n+1) = (-1)^{n+1} \frac{[n]_{\rho,q}!}{[p]_{\rho,q}^{\lfloor \frac{n}{p} \rfloor} \left[\left\lfloor \frac{n}{p} \right\rfloor\right]_{\rho^p,q^p}!}, \tag{2.9}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function.

Proof. From Definition 2, we observe that

$$\begin{aligned} \Gamma_p^{[\rho,q]}(n+1) &= (-1)^{n+1} \prod_{\substack{j<n \\ (p,j)=1}} [j]_{\rho,q} \\ &= (-1)^{n+1} \frac{[1]_{\rho,q} [2]_{\rho,q} \cdots [n]_{\rho,q}}{[p]_{\rho,q} [2p]_{\rho,q} \cdots \left[\left\lfloor \frac{n}{p} \right\rfloor p\right]_{\rho,q}}. \end{aligned}$$

Using the product rule $[kp]_{\rho,q} = [k]_{\rho^p,q^p} [p]_{\rho,q}$ for (ρ, q)-numbers, we acquire

$$\Gamma_p^{[\rho,q]}(n+1) = (-1)^{n+1} \frac{[n]_{\rho,q}!}{[p]_{\rho,q}^{\lfloor \frac{n}{p} \rfloor} [1]_{\rho^p,q^p} [2]_{\rho^p,q^p} \cdots \left[\left\lfloor \frac{n}{p} \right\rfloor\right]_{\rho^p,q^p}},$$

which yields to the asserted result (2.9). □

Particularly, we derive the following result.

Corollary 3. *We have*

$$\Gamma_p^{[\rho,q]}(p^n) = (-1)^p \frac{[p^n - 1]_{\rho,q}!}{[p]_{\rho,q}^{p^{n-1}-1} [p^{n-1} - 1]_{\rho^p,q^p}!}. \tag{2.10}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Here are two relations for $\Gamma_p^{[\rho,q]}(x)$ and the latter provides a representation of (ρ,q)-factorial function associated with p-adic (ρ,q)-gamma function.

Theorem 5. *For $n\in\mathbb{N}$, let m_n be the sum of digits of $n=\sum_{j=0}^{m}a_jp^j$ $(a_m\neq 0)$ in base p. We then derive*

$$\left[\left\lfloor \frac{n}{p^m}\right\rfloor\right]_{\rho,q}! = (-1)^{n+1-m}\left(-[p]_{\rho,q}\right)^{(n-m_n)/(p-1)}\prod_{j=0}^{m-1}\frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^{j}}\right\rfloor\right]_{\rho,q}!}\prod_{i=0}^{m}\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right) \tag{2.11}$$

and

$$[n]_{\rho,q}! = (-1)^{n+1-m}\left(-[p]_{\rho,q}\right)^{(n-m_n)/(p-1)}\left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\rho^p,q^p}!\prod_{j=1}^{m}\frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^{j}}\right\rfloor\right]_{\rho,q}!}\prod_{i=0}^{m}\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right). \tag{2.12}$$

Proof. By Eq. (2.9), we have

$$[n]_{\rho,q}! = (-1)^{n+1}[p]_{\rho,q}^{\lfloor \frac{n}{p}\rfloor}\left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\rho^p,q^p}!\Gamma_p^{[\rho,q]}(n+1).$$

Then if we put $\left\lfloor \frac{n}{p^j}\right\rfloor$ where j lies in $\{0,1,\cdots,m\}$ instead of n, respectively, we observe that

$$\left[\left\lfloor \frac{n}{p^0}\right\rfloor\right]_{\rho,q}! = (-1)^{\lfloor \frac{n}{p^0}\rfloor+1}[p]_{\rho,q}^{\lfloor \frac{n}{p^1}\rfloor}\left[\left\lfloor \frac{n}{p^1}\right\rfloor\right]_{\rho^p,q^p}!\,\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^0}\right\rfloor+1\right)$$

$$\left[\left\lfloor \frac{n}{p^1}\right\rfloor\right]_{\rho,q}! = (-1)^{\lfloor \frac{n}{p^1}\rfloor+1}[p]_{\rho,q}^{\lfloor \frac{n}{p^2}\rfloor}\left[\left\lfloor \frac{n}{p^2}\right\rfloor\right]_{\rho^p,q^p}!\,\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^1}\right\rfloor+1\right)$$

$$\vdots$$

$$\left[\left\lfloor \frac{n}{p^m}\right\rfloor\right]_{\rho,q}! = (-1)^{\lfloor \frac{n}{p^m}\rfloor+1}[p]_{\rho,q}^{\lfloor \frac{n}{p^{m+1}}\rfloor}\left[\left\lfloor \frac{n}{p^{m+1}}\right\rfloor\right]_{\rho^p,q^p}!\,\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^m}\right\rfloor+1\right).$$

Multiplying the both sides above, one can acquire with ease that

$$\begin{aligned}\left[\left\lfloor \frac{n}{p^m}\right\rfloor\right]_{\rho,q}! &= (-1)^{\lfloor \frac{n}{p^0}\rfloor+\lfloor \frac{n}{p^1}\rfloor+\cdots+\lfloor \frac{n}{p^m}\rfloor+m+1}[p]_{\rho,q}^{\lfloor \frac{n}{p^1}\rfloor+\lfloor \frac{n}{p^2}\rfloor+\cdots+\lfloor \frac{n}{p^{m+1}}\rfloor}\\ &\quad\cdot\left[\left\lfloor \frac{n}{p^{m+1}}\right\rfloor\right]_{\rho^p,q^p}!\prod_{j=0}^{m-1}\frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^{j}}\right\rfloor\right]_{\rho,q}!}\prod_{i=0}^{m}\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right)\\ &= (-1)^{(n-m_n)/(p-1)}(-1)^{n+1-m}[p]_{\rho,q}^{(n-m_n)(p-1)}\left[\left\lfloor \frac{n}{p^{m+1}}\right\rfloor\right]_{\rho^p,q^p}!\\ &\quad\cdot\prod_{j=0}^{m-1}\frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^{j}}\right\rfloor\right]_{\rho,q}!}\prod_{i=0}^{m}\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right).\end{aligned}$$

So, we get the asserted result (2.11):

$$\begin{aligned}\left[\left\lfloor \frac{n}{p^m}\right\rfloor\right]_{\rho,q}! &= (-1)^{n+1-m}\left(-[p]_{\rho,q}\right)^{(n-m_n)(p-1)}\\ &\quad\cdot\prod_{j=0}^{m-1}\frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^{j}}\right\rfloor\right]_{\rho,q}!}\prod_{i=0}^{m}\Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right).\end{aligned}$$

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Also, from the applications above,

$$\begin{aligned}
[n]_{\rho,q}! &= (-1)^{\left\lfloor \frac{n}{p^0}\right\rfloor+\left\lfloor \frac{n}{p^1}\right\rfloor+\cdots+\left\lfloor \frac{n}{p^m}\right\rfloor+m+1}[p]_{\rho,q}^{\left\lfloor \frac{n}{p^1}\right\rfloor+\left\lfloor \frac{n}{p^2}\right\rfloor+\cdots+\left\lfloor \frac{n}{p^{m+1}}\right\rfloor} \\
&\cdot \left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\rho^p,q^p}! \prod_{j=1}^{m} \frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^j}\right\rfloor\right]_{\rho,q}!} \prod_{i=0}^{m} \Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right) \\
&= (-1)^{(n-m_n)(p-1)}(-1)^{n+1-m}[p]_{\rho,q}^{(n-m_n)(p-1)}\left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\rho^p,q^p}! \\
&\cdot \prod_{j=1}^{m} \frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^j}\right\rfloor\right]_{\rho,q}!} \prod_{i=0}^{m} \Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right).
\end{aligned}$$

Thus we obtain the Eq. (2.12):

$$\begin{aligned}
[n]_{\rho,q}! &= (-1)^{n+1-m}\left(-[p]_{\rho,q}\right)^{(n-m_n)/(p-1)}\left[\left\lfloor \frac{n}{p}\right\rfloor\right]_{\rho^p,q^p}! \\
&\cdot \prod_{j=1}^{m} \frac{\left[\left\lfloor \frac{n}{p^{j+1}}\right\rfloor\right]_{\rho^p,q^p}!}{\left[\left\lfloor \frac{n}{p^j}\right\rfloor\right]_{\rho,q}!} \prod_{i=0}^{m} \Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i}\right\rfloor+1\right).
\end{aligned}$$

□

We give the following theorem.

Theorem 6. *The following relation holds true for any prime p and $n \in \mathbb{N}$:*

$$\begin{aligned}
[p^n-1]_{\rho,q}! &= (-1)^p\left(-[p]_{\rho,q}\right)^{(p^n-1)/(p-1)}[p]_{\rho,q}^{-n}\left[p^{n-1}-1\right]_{\rho^p,q^p}! \\
&\cdot \prod_{j=0}^{n-2} \frac{\left[p^j-1\right]_{\rho^p,q^p}}{\left[p^{j+1}-1\right]_{\rho,q}} \prod_{j=0}^{n} \Gamma_p^{[\rho,q]}\left(p^j\right).
\end{aligned} \tag{2.13}$$

Proof. In view of Eq. (2.10), we have

$$\left[p^k-1\right]_{\rho,q}! = (-1)^p\,\Gamma_p^{[\rho,q]}\left(p^k\right)[p]_{\rho,q}^{p^{k-1}-1}\left[p^{k-1}-1\right]_{\rho^p,q^p}!.$$

If we put $0,1,2,\ldots,n$ instead of k, respectively, we then get

$$\begin{aligned}
\left[p^0-1\right]_{\rho,q}! &= 1 = (-1)\,\Gamma_p^{[\rho,q]}\left(p^0\right), \\
\left[p^1-1\right]_{\rho,q}! &= (-1)^p\,\Gamma_p^{[\rho,q]}\left(p^1\right)[p]_{\rho,q}^{p^{1-1}-1}\left[p^{1-1}-1\right]_{\rho^p,q^p}!, \\
&\vdots \\
\left[p^n-1\right]_{\rho,q}! &= (-1)^p\,\Gamma_p^{[\rho,q]}\left(p^n\right)[p]_{\rho,q}^{p^{n-1}-1}\left[p^{n-1}-1\right]_{\rho^p,q^p}!.
\end{aligned}$$

If we multiply to the both sides above, we attain

$$\left[p^n-1\right]_{\rho,q}! = (-1)^{np+1}[p]_{\rho,q}^{p^0+p^1+\cdots+p^{n-1}-n}\left[p^{n-1}-1\right]_{\rho^p,q^p}!\prod_{j=0}^{n-2}\frac{\left[p^j-1\right]_{\rho^p,q^p}!}{\left[p^{j+1}-1\right]_{\rho,q}!}\prod_{j=0}^{n}\Gamma_p^{[\rho,q]}\left(p^j\right),$$

which gives to the asserted result (2.13). □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Theorem 7. *For $n \in \mathbb{N}$, let p be a prime number and m_n be the sum of digits of $n = \sum_{j=0}^{m} a_j p^j$ $(a_m \neq 0)$ in base p. The following identity holds true for $j = 0, 1, \ldots m$:*

$$\frac{\left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho,q}!}{[p]_{\rho,q}^{\left\lfloor \frac{n}{p^j} \right\rfloor} \left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho^p,q^p}!} = \prod_{k=1}^{\left\lfloor \frac{n}{p^j} \right\rfloor} \frac{\rho^k - q^k}{\rho^{kp} - q^{kp}} \qquad (0 \leqq k \leqq m)\,. \tag{2.14}$$

Proof. For $0 \leqq j \leqq m$, we get

$$\begin{aligned}
\frac{\left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho,q}!}{[p]_{\rho,q}^{\left\lfloor \frac{n}{p^j} \right\rfloor} \left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho^p,q^p}!} &= \frac{[1]_{\rho,q}[2]_{\rho,q}\cdots\left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho,q}}{[p]_{\rho,q}^{\left\lfloor \frac{n}{p^j} \right\rfloor}[1]_{\rho^p,q^p}[2]_{\rho^p,q^p}\cdots\left[\left\lfloor \frac{n}{p^j} \right\rfloor\right]_{\rho^p,q^p}} \\
&= \frac{\frac{\rho-q}{\rho-q}\frac{\rho^2-q^2}{\rho-q}\cdots\frac{\rho^{\left\lfloor \frac{n}{p^j} \right\rfloor}-q^{\left\lfloor \frac{n}{p^j} \right\rfloor}}{\rho-q}}{\left(\frac{\rho^p-q^p}{\rho-q}\right)^{\left\lfloor \frac{n}{p^j} \right\rfloor}\frac{\rho^p-q^p}{\rho^p-q^p}\frac{\rho^{2p}-q^{2p}}{\rho^p-q^p}\cdots\frac{\rho^{\left\lfloor \frac{n}{p^j} \right\rfloor p}-q^{\left\lfloor \frac{n}{p^j} \right\rfloor p}}{\rho^p-q^p}} \\
&= \frac{(\rho-q)\left(\rho^2-q^2\right)\cdots\left(\rho^{\left\lfloor \frac{n}{p^j} \right\rfloor}-q^{\left\lfloor \frac{n}{p^j} \right\rfloor}\right)}{\left(\rho^p-q^p\right)\left(\rho^{2p}-q^{2p}\right)\cdots\left(\rho^{\left\lfloor \frac{n}{p^j} \right\rfloor p}-q^{\left\lfloor \frac{n}{p^j} \right\rfloor p}\right)},
\end{aligned}$$

which completes the proof of this theorem. □

The following result can be easily derived from Theorem 5 and Theorem 7.

Corollary 4. *For $n \in N$, let p be a prime number and m_n be the sum of digits of $n = \sum_{j=0}^{m} a_j p^j$ $(a_m \neq 0)$ in base p. We then get*

$$[n]_{\rho,q}! = (-1)^{(n-m_n)/(p-1)+n+1-m} \prod_{k=1}^{\left\lfloor \frac{n}{p^1} \right\rfloor} \frac{\left(\rho^{kp}-q^{kp}\right)}{\left(\rho^k-q^k\right)} \cdots \prod_{k=1}^{\left\lfloor \frac{n}{p^m} \right\rfloor} \frac{\left(\rho^{kp}-q^{kp}\right)}{\left(\rho^k-q^k\right)} \prod_{i=0}^{m} \Gamma_p^{[\rho,q]}\left(\left\lfloor \frac{n}{p^i} \right\rfloor + 1\right).$$

We here provide a representation for $\Gamma_p^{[\rho,q]}(-n)$ via the following theorem.

Theorem 8. *The following relation holds true for any prime p and for any $n \in \mathbb{N}$:*

$$\Gamma_p^{[\rho,q]}(-n) = (-1)^{n+1-\left\lfloor \frac{n}{p} \right\rfloor} \prod_{\substack{j<n+1 \\ (p,j)=1}} (\rho q)^j \left(\Gamma_p^{[\rho,q]}(n+1)\right)^{-1}.$$

Proof. In view of Lemma 1 and Theorem 3, we can write

$$\begin{aligned}
1 &= \Gamma_p^{[\rho,q]}(0) = \Gamma_p^{[\rho,q]}(1+(-1)) = \epsilon_p^{[\rho,q]}(-1)\,\Gamma_p^{[\rho,q]}(-1) \\
&= \epsilon_p^{[\rho,q]}(-1)\,\epsilon_p^{[\rho,q]}(-2)\,\Gamma_p^{[\rho,q]}(-2) = \cdots = \prod_{j=1}^{n} \epsilon_p^{[\rho,q]}(-j)\,\Gamma_p^{[\rho,q]}(-n),
\end{aligned}$$

therefore, we get

$$\left(\Gamma_p^{[\rho,q]}(-n)\right)^{-1} = \prod_{j=1}^{n} \epsilon_p^{[\rho,q]}(-j)\,.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

By utilizing the definitons of (ρ,q)-numbers and $\epsilon_p^{[\rho,q]}$, we have

$$\begin{aligned}\left(\Gamma_p^{[\rho,q]}(-n)\right)^{-1} &= (-1)^{\lfloor \frac{n}{p}\rfloor}\prod_{\substack{j<n+1\\(p,j)=1}}(\rho q)^{-j}[j]_{\rho,q}\\ &= (-1)^{\lfloor \frac{n}{p}\rfloor-n-1}\prod_{\substack{j<n+1\\(p,j)=1}}(\rho q)^{-j}(-1)^{n+1}\prod_{\substack{j<n+1\\(p,j)=1}}[j]_{\rho,q}\\ &= (-1)^{\lfloor \frac{n}{p}\rfloor-n-1}\prod_{\substack{j<n+1\\(p,j)=1}}(\rho q)^{-j}\,\Gamma_p^{[\rho,q]}(n+1).\end{aligned}$$

Thereby, the proof of this theorem is completed. □

Corollary 5. *Substituting $n-1$ by n in Theorem 8, one can readily write that*

$$\Gamma_p^{[\rho,q]}(n)\,\Gamma_p^{[\rho,q]}(1-n) = (-1)^{n-\lfloor \frac{n-1}{p}\rfloor}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^{j}. \tag{2.15}$$

Now, we introduce $l:\mathbb{Z}_p\to\{1,2,\cdots,p\}$ by assigning to $x\in\mathbb{Z}_p$ its residue modulo $p\mathbb{Z}_p$. Let $n = a_0+a_1p+a_2p^2+\cdots$ be a positive in base p. If $a_0\neq 0$, then $\left\lfloor\frac{n-1}{p}\right\rfloor = a_1+a_2p+\cdots$. So we obtain $n-p\left\lfloor\frac{n-1}{p}\right\rfloor = a_0 = l(n)$. If $a_0=0$, then $n-1 = p-1+b_1p+b_2p^2+\cdots$. Thus $\left\lfloor\frac{n-1}{p}\right\rfloor = b_1+b_2p+\cdots$. So we get $n-p\left\lfloor\frac{n-1}{p}\right\rfloor = 1+(p-1) = p = l(n)$.

Hence, we give the following theorem.

Theorem 9. *For $p\neq 2$ and all $x\in\mathbb{Z}_p$, we have*

$$\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(1-x) = (-1)^{l(x)}\lim_{n\to x}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^{j}. \tag{2.16}$$

Letting $x=\frac{1}{2}$ in Theorem 9 yields to the following result

$$\begin{aligned}\left(\Gamma_p^{[\rho,q]}\left(\frac{1}{2}\right)\right)^2 &= (-1)^{l(\frac{1}{2})}\lim_{n\to\frac{1}{2}}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^{j}\\ &= \begin{cases}-\lim_{n\to\frac{1}{2}}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^{j} & \text{if } p\equiv 1\,(\operatorname{mod} 4),\\ \lim_{n\to\frac{1}{2}}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^{j} & \text{if } p\equiv 3\,(\operatorname{mod} 4),\end{cases}\end{aligned}$$

where we used the equality $l\left(\frac{1}{2}\right) = l\left(\frac{p+1}{2}\right) = \frac{p+1}{2}$ by definition.

Corollary 6. *We have for $p=2$ in Theorem 8,*

$$\Gamma_2^{[\rho,q]}(n+1)\,\Gamma_2^{[\rho,q]}(-n) = (-1)^{n+1-\lfloor\frac{n}{2}\rfloor}\prod_{\substack{j<n+1\\(2,j)=1}}(\rho q)^{j} = (-1)^{n+1-\lfloor\frac{n}{2}\rfloor}(\rho q)^{\left(n-\lfloor\frac{n}{2}\rfloor\right)^2}. \tag{2.17}$$

We give an identity for special case $p=2$.

Theorem 10. *For all $x\in\mathbb{Z}_2$, we obtain*

$$\Gamma_2^{[\rho,q]}(x)\,\Gamma_2^{[\rho,q]}(1-x) = (-1)^{1+\eta_1(x)}\lim_{n\to x}\prod_{\substack{j<n\\(2,j)=1}}(\rho q)^{j}, \tag{2.18}$$

where $\eta_1\left(\sum_{j=0}^{\infty}a_j2^j\right) = a_1$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Proof. For $n \in \mathbb{N}$, by Eq. (2.15), we have

$$\Gamma_2^{[\rho,q]}(n)\,\Gamma_2^{[\rho,q]}(1-n) = (-1)^{n-\left\lfloor \frac{n-1}{2} \right\rfloor} \prod_{\substack{j<n \\ (2,j)=1}} (\rho q)^j .$$

Let $n = a_0 + a_1 2 + a_2 2^2 + \cdots$ in base 2. If $a_0 \neq 0$, thereby $a_0 = 1$ in base 2 and $\left\lfloor \frac{n-1}{2} \right\rfloor = a_1 \,(\mathrm{mod}\, 2)$. Hence, we obtain $(-1)^{n-\left\lfloor \frac{n-1}{2} \right\rfloor} = (-1)^{a_0-a_1} = (-1)^{1+a_1} = (-1)^{1+\eta_1(n)}$. If $a_0 = 0$, then we see $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{-1+a_1 2+a_2 2^2+\cdots}{2} \right\rfloor = \left\lfloor \frac{1+(a_1-1)2+a_2 2^2+\cdots}{2} \right\rfloor = a_1 - 1 \,(\mathrm{mod}\, 2)$. Therefore, we get $(-1)^{n-\left\lfloor \frac{n-1}{2} \right\rfloor} = (-1)^{2-(a_1-1)} = (-1)^{1+a_1} = (-1)^{1+\eta_1(n)}$. Consequently, we derive the following identity

$$\Gamma_2^{[\rho,q]}(n)\,\Gamma_2^{[\rho,q]}(1-n) = (-1)^{1+\eta_1(n)} \prod_{\substack{j<n \\ (2,j)=1}} (\rho q)^j ,$$

which provides the claimed result (2.18). □

3. The p-adic (ρ, q)-Euler Constant

The p-adic Euler constant $\gamma_p \in \mathbb{Q}_p$ is firstly given by Diamond [6] in 1977 as follows:

$$\gamma_p = \frac{-\Gamma_p'(1)}{\Gamma_p(1)}.$$

In this section, we explore the (ρ, q)-analogue of the p-adic Euler constant. We can readily consider that $\Gamma_p^{[\rho,q]}$ is locally analytic function thanks to Lemma 1.

Then, we derive the following theorem.

Theorem 11. *For $n \in \mathbb{N}$, we have*

$$\frac{\left(\Gamma_p^{[\rho,q]}\right)'(n)}{\Gamma_p^{[\rho,q]}(n)} = \frac{\left(\Gamma_p^{[\rho,q]}\right)'(1)}{\Gamma_p^{[\rho,q]}(1)} + \frac{1}{(\rho-q)} \sum_{j=1}^{n-1} \frac{\rho^j \log\rho - q^j \log q}{[j]_{\rho,q}}. \tag{3.1}$$

Proof. From Theorem 3, we know that

$$\log\left(\Gamma_p^{[\rho,q]}(n)\right) = \log\left(\Gamma_p^{[\rho,q]}(n-1)\right) + \log\left(\epsilon_p^{[\rho,q]}(n-1)\right).$$

Then,

$$\begin{aligned} \frac{\left(\Gamma_p^{[\rho,q]}\right)'(n)}{\Gamma_p^{[\rho,q]}(n)} &= \frac{\left(\Gamma_p^{[\rho,q]}\right)'(n-1)}{\Gamma_p^{[\rho,q]}(n-1)} + \frac{\left(\epsilon_p^{[\rho,q]}\right)'(n-1)}{\epsilon_p^{[\rho,q]}(n-1)} \\ &= \frac{\left(\Gamma_p^{[\rho,q]}\right)'(1)}{\Gamma_p^{[\rho,q]}(1)} + \sum_{j=1}^{n-1} \frac{\left(\epsilon_p^{[\rho,q]}\right)'(j)}{\epsilon_p^{[\rho,q]}(j)}, \end{aligned}$$

which implies the desired result (3.1). □

Remark 3. *The formula (3.1) can be called (ρ, q)-generalization of the known formula for p-adic gamma function*

$$\frac{\Gamma_p'(n)}{\Gamma_p(n)} = \frac{\Gamma_p'(1)}{\Gamma_p(1)} + \sum_{\substack{j<n \\ (p,j)=1}} \frac{1}{j},$$

or (ρ, q)-generalization of p-adic analogue of the formula for classical gamma function

$$\frac{\Gamma'(n)}{\Gamma(n)} = \frac{\Gamma'(1)}{\Gamma(1)} + \sum_{j<n} \frac{1}{j}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Thereby, we are ready to define (ρ,q)-analogue of the p-adic Euler constant $\gamma_p^{[\rho,q]}$ as follows

$$\gamma_p^{[\rho,q]} := \frac{\left(\Gamma_p^{[\rho,q]}\right)'(1)}{\Gamma_p^{[\rho,q]}(1)} = \left(\Gamma_p^{[\rho,q]}\right)'(1) = -\left(\Gamma_p^{[\rho,q]}\right)'(0). \tag{3.2}$$

The p-adic (ρ,q)-Euler constant has a limit representation by the following theorem.

Theorem 12. *We have*

$$\gamma_p^{[\rho,q]} = \lim_{n\to\infty} p^{-n}\left\{1-(-1)^p \frac{[p^n-1]_{\rho,q}!}{[p]_{\rho,q}^{p^{n-1}-1}\left[\left[\frac{n}{p}\right]\right]_{\rho^p,q^p}!}\right\}.$$

Proof. In conjunction with the Eq. (2.10), we have

$$\Gamma_p^{[\rho,q]}(p^n) = (-1)^p \frac{[p^n-1]_{\rho,q}!}{[p]_{\rho,q}^{p^{n-1}-1}\left[\left[\frac{n}{p}\right]\right]_{\rho^p,q^p}!}.$$

Then, we investigate

$$\begin{aligned}\lim_{n\to\infty} p^{-n}\left\{1-(-1)^p \frac{[p^n-1]_{\rho,q}!}{[p]_{\rho,q}^{p^{n-1}-1}\left[\left[\frac{n}{p}\right]\right]_{\rho^p,q^p}!}\right\} &= \lim_{n\to\infty}\frac{1-\Gamma_p^{[\rho,q]}(p^n)}{p^n} \\ &= -\left(\Gamma_p^{[\rho,q]}\right)'(0) = \gamma_p^{[\rho,q]}.\end{aligned}$$

□

Corollary 7. *By means of the Lemma 1, we deduce that* $\left|\gamma_p^{[\rho,q]}\right|_p = \left|\left(\Gamma_p^{[\rho,q]}\right)'(1)\right|_p \leqq 1.$

4. The p-adic (ρ,q)-Beta Function

In this part, we define (ρ,q)-extension p-adic beta function by means of the p-adic (ρ,q)-gamma function discussed in the second section. Then, we present several properties, identities and relations for the mentioned beta function.

The classical beta function $B(x,y)$ is defined by means of the classical gamma functions as follows:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad (x,y\in\mathbb{N})$$

which also have the following subsequent properties (*cf.* [16]):

$$\begin{gathered} B(x,y) = B(y,x) \\ B(x,y) = B(x,y+1) + B(x+1,y) \\ B(x+1,y) = B(x,y)\frac{x}{x+y} \\ B(x,y+1) = B(x,y)\frac{y}{x+y} \\ B(x+1,y) = \frac{x}{y}B(x,y+1). \end{gathered}$$

The p-adic beta function is defined by means of the p-adic gamma functions as follows:

$$B_p(x,y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}, \qquad (x,y\in\mathbb{Z}_p)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

which also have the following subsequent properties (*cf.* [10] and [16]):

$$B_p(x,y) = B_p(y,x)$$

$$B_p(x,y) = \frac{h_p(x+y)}{h_p(x)+h_p(y)}\left(B_p(x,y+1)+B_p(x+1,y)\right)$$

$$B_p(x+1,y) = B_p(x,y)\frac{x}{x+y}$$

$$B_p(x,y+1) = B_p(x,y)\frac{y}{x+y}$$

$$B_p(x+1,y) = \frac{h_p(x)}{h_p(y)}B_p(x,y+1).$$

Definition 3. *Let ρ and $q\in\mathbb{C}_p$ with $|\rho-1|_p<1$ and $|q-1|_p<1$, $\rho\neq 1$ and $q\neq 1$. We define the p-adic (ρ,q)-beta function via the p-adic (ρ,q)-gamma functions as follows:*

$$B_p^{[\rho,q]}(x,y) = \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y)}, \tag{4.1}$$

for $x,y\in\mathbb{Z}_p$.

Remark 4. *In the case $\rho=1$, the p-adic (ρ,q)-beta function reduces to the the p-adic q-beta function, cf.* [10].

Remark 5. *When $q\to\rho=1$, the p-adic (ρ,q)-beta function reduces to the usual p-adic beta function, cf.* [16].

We now ready to investigate the properties of the p-adic (ρ,q)-beta function.

Theorem 13. *For $x,y\in\mathbb{Z}_p$, the p-adic (ρ,q)-beta function is symmetric about x and y:*

$$B_p^{[\rho,q]}(x,y) = B_p^{[\rho,q]}(y,x). \tag{4.2}$$

Proof. By (4.1), we readily get

$$B_p^{[\rho,q]}(x,y) = \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y)} = \frac{\Gamma_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(x)}{\Gamma_p^{[\rho,q]}(y+x)} = B_p^{[\rho,q]}(x,y),$$

which is the asserted result (4.2). □

Theorem 14. *For $x,y\in\mathbb{Z}_p$, the p-adic (ρ,q)-beta function has the following formula:*

$$B_p^{[\rho,q]}(x+1,y) = \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y). \tag{4.3}$$

Proof. In view of (2.7) and (4.1), we readily get

$$\begin{aligned} B_p^{[\rho,q]}(x+1,y) &= \frac{\Gamma_p^{[\rho,q]}(x+1)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y+1)} \\ &= \frac{\epsilon_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)\,\Gamma_p^{[\rho,q]}(x+y)} \\ &= \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(x+y)}\frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y)} = \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y), \end{aligned}$$

which is the desired result (4.3). □

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Theorem 15. *For $x,y\in\mathbb{Z}_p$, the p-adic (ρ,q)-beta function satisfies the following identity:*

$$B_p^{[\rho,q]}(x,y+1)=\frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y). \tag{4.4}$$

Proof. By (4.1), we readily get

$$\begin{aligned}B_p^{[\rho,q]}(x,y+1) &= \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y+1)}{\Gamma_p^{[\rho,q]}(x+y+1)}\\ &= \frac{\Gamma_p^{[\rho,q]}(x)\,\epsilon_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)\,\Gamma_p^{[\rho,q]}(x+y)}\\ &= \frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}\frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y)}=\frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y),\end{aligned}$$

which is the claimed result (4.4). $\square$

By Theorems 14 and 15, we see that

$$\begin{aligned}B_p^{[\rho,q]}(x+1,y)+B_p^{[\rho,q]}(x,y+1) &= \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)+\frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)\\ &= \frac{\epsilon_p^{[\rho,q]}(x)+\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)\end{aligned}$$

and

$$\begin{aligned}B_p^{[\rho,q]}(x+1,y) &= \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)\\ &= \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(y)}\frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)\\ &= \frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(y)}B_p^{[\rho,q]}(x,y+1),\end{aligned}$$

which implies the following results.

Corollary 8. *For $x,y\in\mathbb{Z}_p$, the following formulas are valid:*

$$B_p^{[\rho,q]}(x,y)=\frac{\epsilon_p^{[\rho,q]}(x+y)}{\epsilon_p^{[\rho,q]}(x)+\epsilon_p^{[\rho,q]}(y)}\left(B_p^{[\rho,q]}(x+1,y)+B_p^{[\rho,q]}(x,y+1)\right)$$

and

$$B_p^{[\rho,q]}(x+1,y)=\frac{\epsilon_p^{[\rho,q]}(x)}{\epsilon_p^{[\rho,q]}(y)}B_p^{[\rho,q]}(x,y+1).$$

We give the following theorem.

Theorem 16. *Let $x,y\in\mathbb{Z}_p$. For $p=2$, we get*

$$B_p^{[\rho,q]}(x,y)\,B_p^{[\rho,q]}(x+y,1-y)=\frac{(-1)^{1+\eta_1(x)}}{\epsilon_p^{[\rho,q]}(x)}\lim_{n\to y}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^j \tag{4.5}$$

and for $p\neq 2$, we have

$$B_p^{[\rho,q]}(x,y)\,B_p^{[\rho,q]}(x+y,1-y)=\frac{(-1)^{l(y)}}{\epsilon_p^{[\rho,q]}(x)}\lim_{n\to y}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^j. \tag{4.6}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

Proof. From Definition 3, we easily compute that

$$\begin{aligned} B_p^{[\rho,q]}(x,y)\,B_p^{[\rho,q]}(x+y,1-y) &= \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)}{\Gamma_p^{[\rho,q]}(x+y)}\frac{\Gamma_p^{[\rho,q]}(x+y)\,\Gamma_p^{[\rho,q]}(1-y)}{\Gamma_p^{[\rho,q]}(x+1)} \\ &= \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(1-y)}{\epsilon_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(x)} \\ &= \frac{\Gamma_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(1-y)}{\epsilon_p^{[\rho,q]}(x)}. \end{aligned}$$

It just remains to use the formulas (2.16) and (2.18) in order to obtain desired result (4.5) and (4.6). □

We provide the following theorem.

Theorem 17. *Let $x,y\in\mathbb{Z}_p$. We then obtain*

$$B_p^{[\rho,q]}(x+1,y+1) = \frac{\epsilon_p^{[\rho,q]}(x)\,\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y+1)\,\epsilon_p^{[\rho,q]}(x+y)}B_p^{[\rho,q]}(x,y)\,. \tag{4.7}$$

Proof. In view of the Definition 3 and using (2.7), we readily see that

$$\begin{aligned} B_p^{[\rho,q]}(x+1,y+1) &= \frac{\Gamma_p^{[\rho,q]}(x+1)\,\Gamma_p^{[\rho,q]}(y+1)}{\Gamma_p^{[\rho,q]}(x+1+y+1)} \\ &= \frac{\Gamma_p^{[\rho,q]}(x+1)\,\epsilon_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y+1)\,\Gamma_p^{[\rho,q]}(x+y+1)} \\ &= \frac{\epsilon_p^{[\rho,q]}(y)}{\epsilon_p^{[\rho,q]}(x+y+1)}B_p^{[\rho,q]}(x+1,y)\,, \end{aligned}$$

which implies the asserted formula (4.7) thanks to (4.3). □

For $x,y,\xi,\gamma\in\mathbb{Z}_p$, we note that

$$B_p^{[\rho,q]}(x,y)\,B_p^{[\rho,q]}(x+y,\xi)\,B_p^{[\rho,q]}(x+y+\xi,\gamma) = \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(y)\,\Gamma_p^{[\rho,q]}(\xi)\,\Gamma_p^{[\rho,q]}(\gamma)}{\Gamma_p^{[\rho,q]}(x+y+\xi+\gamma)}.$$

We give the following theorem.

Theorem 18. *Let $x\in\mathbb{Z}_p$. For $p=2$, we obtain*

$$B_p^{[\rho,q]}(x,1-x) = (-1)^{2+\eta_1(x)}\lim_{n\to y}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^j \tag{4.8}$$

and for $p\neq 2$, we attain

$$B_p^{[\rho,q]}(x,1-x) = (-1)^{l(y)+1}\lim_{n\to y}\prod_{\substack{j<n\\(p,j)=1}}(\rho q)^j\,. \tag{4.9}$$

Proof. From Definition 3, we easily compute that

$$B_p^{[\rho,q]}(x,1-x) = \frac{\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(1-x)}{\Gamma_p^{[\rho,q]}(1)} = -\Gamma_p^{[\rho,q]}(x)\,\Gamma_p^{[\rho,q]}(1-x)\,,$$

which implies the claimed result (4.8) and (4.9) in conjunction with (2.16) and (2.18). □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 May 2019 doi:10.20944/preprints201905.0261.v1

Peer-reviewed version available at Mathematical and Cmputational Applications 2019, 24, 53; doi:10.3390/mca24020053

By the motivation for usual binomial coefficient, for $n,k\in\mathbb{N}$ with $n\geq k$, we consider the p-adic (ρ,q)-binomial coefficients $\binom{n}{k}_p^{[\rho,q]}$ by means of the p-adic (ρ,q)-factorial (2.1) as follows:

$$\binom{n}{k}_p^{[\rho,q]}=\frac{(n!)_p^{[\rho,q]}\left((n-k)!\right)_p^{[\rho,q]}}{(k!)_p^{[\rho,q]}}. \tag{4.10}$$

Thus, we give the following theorem.

Theorem 19. *Let $n,k\in\mathbb{N}$ with $n\geq k$. We have*

$$\binom{n}{k}_p^{[\rho,q]}B_p^{[\rho,q]}\left(n-k+1,k+1\right)=-\frac{1}{\epsilon_p^{[\rho,q]}\left(n+1\right)}.$$

Proof. The proof just follows from (4.1) and (4.10) with the formula (2.7). □

We provide the following theorem.

Theorem 20. *Let $n,k\in\mathbb{N}$. We have*

$$B_p^{[\rho,q]}\left(-n,-k\right)=(-1)^{1+\left\lfloor\frac{n+k}{p}\right\rfloor-\left\lfloor\frac{n}{p}\right\rfloor-\left\lfloor\frac{k}{p}\right\rfloor}\frac{\epsilon_p^{[\rho,q]}\left(n+k\right)}{\epsilon_p^{[\rho,q]}\left(n\right)\epsilon_p^{[\rho,q]}\left(k\right)}\frac{1}{B_p^{[\rho,q]}\left(n,k\right)}\frac{\prod_{\substack{j<k+1\\(p,j)=1}}(\rho q)^j}{\prod_{\substack{j<n+1\\(p,j)=1}}(\rho q)^j}.$$

Proof. The proof of this theorem just follows from (4.1), (2.7) and Theorem 8 with some basic computations. □

Finally, we present the following theorem.

Theorem 21. *Let $x\in\mathbb{Z}_p$. For $p=2$, we obtain*

$$B_p^{[\rho,q]}\left(x,1-x\right)=(-1)^{2+\eta_1(x)}\lim_{n\rightarrow y}\prod_{\substack{j<n\\(p,j)=1}}q^j \tag{4.11}$$

and for $p\neq 2$, we attain

$$B_p^{[\rho,q]}\left(x,1-x\right)=(-1)^{l(y)+1}\lim_{n\rightarrow y}\prod_{\substack{j<n\\(p,j)=1}}q^j. \tag{4.12}$$

Proof. From Definition 3, we easily compute that

$$B_p^{[\rho,q]}\left(x,1-x\right)=\frac{\Gamma_p^{[\rho,q]}\left(x\right)\Gamma_p^{[\rho,q]}\left(1-x\right)}{\Gamma_p^{[\rho,q]}\left(1\right)}=-\Gamma_p^{[\rho,q]}\left(x\right)\Gamma_p^{[\rho,q]}\left(1-x\right),$$

which implies the claimed result (4.8) and (4.9) in conjunction with (2.16) and (2.18). □

Remark 6. *The results derived in this part are generalizations of the results obtained in* [10] *and* [16]

5. Conclusion

In this paper, we have firstly generalized p-adic factorial function and p-adic gamma fuction based on (ρ,q)-numbers. Utilizing this generalizations, we have constructed some recurrence relations and identities. By using some properties of (ρ,q)-numbers, we have derived several new and interesting identities and formulas for $(n!)_p^{[\rho,q]}$ and $\Gamma_p^{[\rho,q]}(x)$. As an application, we have derived the p-adic (ρ,q)-Euler constant by means of the p-adic (ρ,q)-gamma function and have given a limit representation for the foregoing constant. Moreover, we have considered (ρ,q)-extension of the p-adic beta function via the p-adic (ρ,q)-gamma function and then we have acquired several formulas and identities.

References

[1] M. Acikgoz, N. Aslan, N. Köskeroğlu, and S. Araci, p-adic approach to linear 2-normed spaces, Math. Morav. 13 (2), 2009, 7-22.
[2] S. Araci, U. Duran and M. Acikgoz, (ρ, q)-Volkenborn integration, J. Number Theory, 171 (2017) 18-30.
[3] A. Aral and V. Gupta, Application of (p, q)-gamma function to Szász Durrmeyer operator, Publications de l'Institut Mathematique 2017 Volume 102, Issue 116, Pages: 211-220.
[4] E. Artin, The gamma function (New York, Holt, Rinehart and Winston, 1964).
[5] R. Chakrabarti and R. Jagannathan, A (p, q)-oscillator realization of two-parameter quantum algebras, J. Phys. A: Math. Gen. 24 (1991) L711.
[6] J. Diamond, The p-adic log gamma function and p-adic Euler constant, Trans. Amer. Math. Soc., 233, 1977, 321-337.
[7] U. Duran and M. Acikgoz, On (ρ, q)-Euler numbers and polynomials associated with (ρ, q)-Volkenborn integrals, Int. J. Number Theory, Vol. 14, No. 01, pp. 241-253 (2018).
[8] Ö. Ç. Havara and H. Menken, On the Volkenborn integral of the q-extension of the p-adic gamma function, J. Math. Anal., 8 (2), 2017, 64-72.
[9] Ö. Ç. Havare and H. Menken, Some properties of the p-adic log gamma function, J. Math. Anal.,7 (2), 2016, 26-32.
[10] Ö. Ç. Havare, H. Menken, Some properties of the q-extension of the p-adic beta function, Neural, Parallel, and Sci. Comput., 24, (2014), 409-418.
[11] Y. S. Kim, q-analogues of p-adic gamma functions and p-adic Euler constants, Comm. Korean Math. Soc., 13 (4), 1998, 735–741.
[12] N. Koblitz, On Carlitz's q-Bernoulli numbers, J. Number Theory, 14 (1982) 332-339.
[13] N. Koblitz, p-adic numbers, p-adic analysis, and Zeta functions (Springer-Verlag, New York Inc, 1977).
[14] N. Koblitz, q-extension of the p-adic gamma function, Trans. Amer. Math. Soc., 260 (2), 1980, 449-457.
[15] H. Menken and A. Körükçü, Some properties of the q-extension of the p-adic gamma function, Abst. Appl. Anal., Volume 2013, Article ID 176470, 4 pages.
[16] H. Menken, Ö. Çolakoğlu, Some properties of the p-adic beta function, European J. Pure Appl. Math., V. 8, N. 2, 2015, 214-231.
[17] G. V. Milovanovic, V. Gupta and N. Malik, (p, q)-Beta functions and applications in approximation, Bol. Soc. Mat. Mex. (2016), doi:10.1007/s40590-016-0139-1.
[18] Y. Morita, A p-adic analogue of the Γ-function, J. Fac. Sci., 22 (2), 1975, 255-266.
[19] M.-S. Kim and J. W. Son, Analytic properties of the q-Volkenborn integral on the ring of p-adic integers, Bull. Korean Math. Soc., 44(1) (2007) 1-12.
[20] H. Nakazako, The q-analogue of the p-adic gamma function, Kodai Math. J., 11, 1988, 141-153.
[21] A. M. Robert, A course in p-adic analysis (Springer-Verlag New York, Inc., 2000).
[22] P. N. Sadjang, On the (p, q)-gamma and the (p, q)-beta functions, arXiv:1506.07394 [math.NT] (2015).
[23] P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, arXiv 1309.3934v1 [math.QA] (2013).
[24] V. Sahai and S. Yadav, Representations of two parameter quantum algebras and $p; q$-special functions, J. Math. Anal. Appl., 335 (2007) 268-279.
[25] W. H. Schikhof, Ultrametric Calculus: An introduction to p-adic analysis (Cambridge Studies in Advanced Mathematics 4, Cambridge University Press Cambridge, 1984).

Department of the Basic Concepts of Engineering, Faculty of Engineering and Natural Sciences, İskenderun Technical University, TR-31200 Hatay, Turkey
E-mail address: mtdrnugur@gmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, TR-27310 Gaziantep, TURKEY
E-mail address: acikgoz@gantep.edu.tr