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Approximation of fixed points for Suzuki's generalized non-expansive mappings

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Abstract

In this paper, we study a three step iterative scheme to approximate fixed points of Suzuki's generalized non-expansive mappings. We establish some weak and strong convergence results for such mappings in uniformly convex Banach spaces. Further, we show numerically that iterative scheme (1.8) converges faster than some other known iterations for Suzuki's generalized non-expansive mappings. To support our claim, we give an illustrative example and approximate fixed points of such mappings using Matlab program. Our results are new and generalize several relevant results in the literature.

Key Words: Suzuki's generalized non-expansive mappings; iterative schemes; fixed points; weak and strong convergence results; uniformly convex Banach spaces.

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1. Introduction

Throughout this paper, we assume that $\mathbb{N}$ is the set of all positive integers. We consider that C is nonempty subset of a Banach space X and $F(T)$, the set of all fixed points of the mapping T on C. A mapping $T : C \to C$ is said to be non-expansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. It is called quasi non-expansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and for all $p \in F(T)$.
We know that $F(T)$ is nonempty when X is uniformly convex, C is bounded closed convex subset of X and T is non-expansive mapping, (cf. [2]).
In 2008, Suzuki [18] introduced the concept of generalized non-expansive mappings which is also called condition (C) and defined as:
Let C be a nonempty subset of Banach space X. A mapping $T : C \to C$ is said to satisfy condition (C) if,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in C.$$

Suzuki obtained existence of fixed point and convergence theorems for such mappings. Suzuki also showed that the notion of mappings satisfying condition (C) is weaker than non-expansiveness and stronger than quasi non-expansiveness.

The following example in support of above claim.

Example 1.1. [18] Define a self mapping T on $[0,3]$ by

$$T(x) = \begin{cases} 0, & if \ x \neq 3, \\ 1, & if \ x = 3. \end{cases}$$

Here T satisfies Suzuki's condition (C), but T is not non-expansive.

On the other hand, Banach contraction principle states that fixed point of contraction mappings can be approximated by Picard iteration where the sequence $\{x_n\}$ is generated from an arbitrary guess $x_1 \in C$ as follows:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = Tx_n, \ \ n \in \mathbb{N}. \end{cases} \tag{1.1}$$

Unlike contraction mappings, Picard iteration for non-expansive mappings need not converge to a fixed point, even map has a fixed point.

Therefore, in 1953, Mann [3] introduced an iterative scheme, which has been extensively used to approximate fixed points of non-expansive mappings. In this iterative scheme the sequence $\{x_n\}$ is generated from an arbitrary guess $x_1 \in C$, in the following manner:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_nTx_n, \ \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ is a sequence in $(0,1)$, satisfying appropriate conditions. It is also known that Mann iteration fail to converge to fixed points of pseudo-contractive mappings.

So in 1974, Ishikawa [4] introduced a two step Mann iterative scheme to approximate fixed points of pseudo-contractive mappings, where the sequence $\{x_n\}$ is defined by

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTx_n \ , n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$, satisfying appropriate conditions. Rhoades [5] made an interesting remark on the rate of convergence of these iteration processes that: Mann iteration for decreasing functions converges faster than Ishikawa iteration. For increasing functions Ishikawa iteration process is better than Mann iteration process, also Mann iteration process appears to be independent of the initial guess (see also [6]).

In 2000, Noor [7] introduced the following iterative scheme for general variational inequalities, in this scheme $\{x_n\}$ is defined by

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

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where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$, satisfying satisfactory conditions. He also studied the convergence criteria of this scheme. After that, in 2007, Agrawal et al. [8] introduced the following two step iterative scheme for nearly asymptotically non-expansive mappings, in this scheme $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \\ y_n = (1 - b_n)x_n + b_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$, satisfying appropriate conditions. They claimed that, this process converges at a rate same as Picard iteration and faster than Mann iteration for contractions.

In 2014, Abbas and Nazir [1] introduced the following three step iterative scheme for non-expansive mappings in uniformly convex Banach space. The sequence $\{x_n\}$ starting at initial guess $x_1 \in C$ is defined as:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)Ty_n + a_nTz_n, \\ y_n = (1 - b_n)Tx_n + b_nTz_n, \\ z_n = (1 - c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$. Authors showed that this process converges faster than all of Picard, Mann and Agarwal et al. processes for contractions, by giving a numerical example in support of their claim.

In 2014, Thakur et al. [9] introduced the following iterative scheme for non-expansive mappings, in this scheme the sequence $\{x_n\}$ is defined as:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \\ y_n = (1 - b_n)z_n + b_nTz_n, \\ z_n = (1 - c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$. Authors claimed that this process converges faster than all of Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas and Nazir iteration processes for contractions in the sense of Berinde [21].

Recently, Sahu et al. [10] and Thakur et al. [11] introduced the following same iterative scheme for non-expansive mappings in uniformly convex Banach space:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)Tz_n + a_nTy_n, \\ y_n = (1 - b_n)z_n + b_nTz_n, \\ z_n = (1 - c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$. They claimed that this process converges faster than all the known iteration processes for contractions in the sense of Berinde [21] and they verified an example to support this claim.

In 2011, Phuengrattana [15] proved convergence theorems for mappings satisfying condition (C) using the Ishikawa iteration in uniformly convex Banach spaces. Recently, fixed point theorems for Suzuki's generalized non expansive mappings and nonlinear mappings have been studied by a large number of researchers, e.g. see [14, 16, 17, 20].

Motivated by the above, we prove some weak and strong convergence results using iterative scheme (1.8) for Suzuki's generalized non-expansive mappings in uniformly convex Banach space. Our results generalize and extend the corresponding results of Sahu et al. [10], Thakur et al. [11] and many others in the literature.

2. Preliminaries

We now recall some definitions, propositions and lemmas to be use in the main results.

Definition 2.1. Let C be a nonempty, closed and convex subset of a Banach space X. A mapping $T : C \to X$ is called demiclosed with respect to $y \in X$, if for each sequence $\{x_n\}$ in C and each $x \in C$, $\{x_n\}$ converges weakly at x and $\{Tx_n\}$ converges strongly at y imply that $Tx = y$.

Definition 2.2. A Banach space X is said to satisfy Opial's property [12] if for each weakly convergent sequence $\{x_n\}$ in X with weak limit x,

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|$$

holds, for all $y \in X$, with $y \neq x$.

Definition 2.3. Let C be a nonempty, closed and convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, $A(C, \{x_n\})$ consists exactly one point, in the case when X is uniformly convex Banach space.

Proposition 2.4. [18] *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping.*
(i) If T is non-expansive then T satisfies the condition (C).
(ii) If T satisfies condition (C) and has a fixed point, then T is quasi non-expansive mapping.
(iii) If T satisfies condition (C), then

$$\|x - Ty\| \leq 3\|Tx - y\| + \|x - y\|, \ \forall\ x, y \in C.$$

Lemma 2.5. [18] *Let T be a self mapping on a subset C of a Banach space X with the Opial's property. Assume that T satisfies condition (C). If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Lemma 2.6. [18] *Let C be a weakly compact convex subset of a uniformly convex Banach space X and T be a self mapping on C. Assume that T satisfies condition (C), then T has a fixed point.*

Lemma 2.7. [19] *Suppose X is uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|y_n\| \leq d$ and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d$ holds, for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main Results

In this section, we prove some weak and strong convergence theorems using iterative scheme (1.8) for Suzuki's generalized non-expansive mappings in uniformly convex Banach spaces. First, we obtain following useful lemmas to be use in our main results:

Lemma 3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping with $F(T) \neq \emptyset$ and $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (1.8), then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T satisfies condition (C), therefore by Proposition 2.4, T is quasi non-expansive mapping. That is,

$$\|Tx - p\| \leq \|x - p\|, \; for \ all \ x \in C \ and \ for \ all \ p \in F(T).$$

Now from iterative scheme (1.8), we get

$$\begin{aligned} \|z_n - p\| &= \|(1-c_n)x_n + c_n Tx_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.1}$$

And

$$\begin{aligned} \|y_n - p\| &= \|(1-b_n)z_n + b_n Tz_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|Tz_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|z_n - p\| \\ &= \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-a_n)Tz_n + a_nTy_n - p\| \\ &\leq (1-a_n)\|Tz_n - p\| + a_n\|Ty_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.3}$$

$\Longrightarrow$ The sequence $\{\|x_n - p\|\}$ is non-increasing and bounded below for all $p \in F(T)$. Hence, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (1.8). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|x_n - p\| = \alpha. \tag{3.4}$$

From (3.1), (3.2) and (3.4), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \tag{3.5}$$

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \tag{3.6}$$

Since T satisfies condition (C), we have

$$\begin{aligned} & \|Tx_n - p\| = \|Tx_n - Tp\| \leq \|x_n - p\| \\ \Longrightarrow \quad & \limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \end{aligned} \tag{3.7}$$

Similarly,

$$\limsup_{n\to\infty} \|Ty_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq \alpha. \tag{3.8}$$

$$\limsup_{n\to\infty} \|Tz_n - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq \alpha. \tag{3.9}$$

Again,

$$\begin{aligned}\alpha &= \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1-a_n)Tz_n + a_nTy_n - p\| \\ &= \lim_{n\to\infty} \|(1-a_n)(Tz_n - p) + a_n(Ty_n - p)\|. \end{aligned} \tag{3.10}$$

From (3.8), (3.9), (3.10) and using Lemma 2.7, we have

$$\lim_{n\to\infty} \|Tz_n - Ty_n\| = 0. \tag{3.11}$$

Now,

$$\|x_{n+1} - p\| = \|(1-a_n)Tz_n + a_nTy_n - p\| \leq \|Tz_n - p\| + a_n\|Ty_n - Tz_n\|.$$

Taking the $\liminf$ on both sides, we get

$$\begin{aligned} \alpha &= \liminf_{n\to\infty} \|x_{n+1} - p\| \leq \liminf_{n\to\infty} \|Tz_n - p\| \\ \Longrightarrow \alpha &\leq \liminf_{n\to\infty} \|z_n - p\|. \end{aligned} \tag{3.12}$$

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So that, (3.5) and (3.12) give,

$$\lim_{n\to\infty} \|z_n - p\| = \alpha.$$

Thus,

$$\begin{aligned} \alpha &= \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|(1-c_n)x_n + c_n T x_n - p\| \\ &= \lim_{n\to\infty} \|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|. \end{aligned} \tag{3.13}$$

From (3.4), (3.7), (3.13) and using Lemma 2.7, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(C, \{x_n\})$, by Proposition 2.4, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \lim_{n\to\infty} \sup \|x_n - Tp\| \\ &\leq \lim_{n\to\infty} \sup(3\|Tx_n - x_n)\| + \|x_n - p\|) \\ &= \lim_{n\to\infty} \sup \|x_n - p\| \\ &= r(p, \{x_n\}) = r(C, \{x_n\}). \end{aligned}$$

$\Longrightarrow\ Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is singleton, hence we have $Tp = p$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping with $F(T) \neq \emptyset$ and $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by iterative scheme (1.8). Assume that X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a point of $F(T)$.*

Proof. Let $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 3.1. Now we prove that $\{x_n\}$ has unique weak sub-sequential limit in $F(T)$. Let x and y be weak limits of the subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ respectively. From Lemma 3.2, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero by Lemma 2.5. This implies that $(I-T)x = 0 \implies x = Tx$, similarly $Ty = y$.
Next we show that uniqueness. If $x \neq y$, then by using Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x\| &= \lim_{n_j\to\infty} \|x_{n_j} - x\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - y\| \\ &= \lim_{n\to\infty} \|x_n - y\| \\ &= \lim_{n_k\to\infty} \|x_{n_k} - y\| \\ &< \lim_{n_k\to\infty} \|x_{n_k} - x\| \\ &= \lim_{n\to\infty} \|x_n - x\|. \end{aligned}$$

This is a contradiction, so $x = y$. Consequently, $\{x_n\}$ converges weakly to a point of $F(T)$. This completes the proof. □

Theorem 3.4. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (1.8). Suppose $F(T) \neq \emptyset$ and $p \in F(T)$. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ or $\lim_{n\to\infty} \sup d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{\|x_n - p\| : p \in F(T)\}$.*

Proof. Necessity is obvious.
Conversely, assume that $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ and $p \in F(T)$. From Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in F(T)$ therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$ by assumption. We show that $\{x_n\}$ is a Cauchy sequence in C. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$,

$$\begin{aligned} d(x_n, F(T)) &< \frac{\epsilon}{2} \\ \implies \inf\{\|x_n - p\| : p \in F(T)\} &< \frac{\epsilon}{2}. \end{aligned}$$

In particular, $\inf\{\|x_{m_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Therefore there exists $p \in F(T)$ such that

$$\|x_{m_0} - p\| < \frac{\epsilon}{2}.$$

Now for $m, n \geq m_0$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \|x_{m_0} - p\| + \|x_{m_0} - p\| \\ &= 2\|x_{m_0} - p\| < \epsilon. \end{aligned}$$

$\implies \{x_n\}$ is a Cauchy sequence in C. As C is closed subset of a Banach space X, so that there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0 \implies q \in F(T)$. □

Theorem 3.5. *Let C be a nonempty, compact and convex subset of a uniformly convex Banach space X, and let T and $\{x_n\}$ be as in Lemma 3.2, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.6, $F(T) \neq \emptyset$, so by Lemma 3.2, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ strongly for some $p \in C$. By Proposition 2.4, we have

$$\|x_{n_j} - Tp\| \leq 3\|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - p\|, \ \forall\ n \geq 1.$$

Letting $j \to \infty$, we get that $x_{n_j} \to Tp$. This implies that $Tp = p$ *i.e.* $p \in F(T)$. Also, $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 3.1. Thus p is the strong limit of the sequence $\{x_n\}$ itself. □

Senter and Dotson [13] introduced the notion of mapping satisfying condition (I). A mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in C$, where $d(x, F(T)) = \inf\{d(x, p) :$

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$p \in F(T)\}$.
Now we also prove a strong convergence result using condition (I).

Theorem 3.6. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping satisfying condition (I). Let $\{x_n\}$ be a sequence defined by iterative scheme (1.8), where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$, for $\epsilon \leq \frac{1}{2}$ and for all $n \geq 1$. Suppose $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Lemma 3.2 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.14}$$

From condition (I) and (3.14), we get

$$\begin{aligned} 0 \leq \lim_{n\to\infty} f(d(x_n, F(T))) &\leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \\ \implies \lim_{n\to\infty} f(d(x_n, F(T))) &= 0. \end{aligned}$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0, \forall\ r > 0$, hence we have

$$\lim_{n\to\infty} (d(x_n, F(T))) = 0.$$

Now all the conditions of Theorem 3.4 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of T. □

Remark 3.7. All the results in this paper generalize the corresponding results of Sahu et al. [10], Thakur et al. [11] and many others due to following reason:
(1) Mappings are generalized non-expansive.

Now we give the following example for comparison of above iteration process with proposed algorithm for Suzuki's generalized non-expansive mapping.

Example 3.8. Define a self mapping T on $[1, 2]$ by

$$T(x) = \begin{cases} 3 - x, & if\ x \in [1, \frac{10}{9}), \\ \frac{x+16}{9}, & if\ x \in [\frac{10}{9}, 2]. \end{cases}$$

Here T is Suzuki's generalized non-expansive mapping, but T is not non-expansive.

Verification. Take $x = \frac{111}{100}$ and $y = \frac{10}{9}$, then

$$\|x - y\| = \|\frac{111}{100} - \frac{10}{9}\| = \frac{1}{900}.$$

And.

$$\begin{aligned} \|Tx - Ty\| &= \|3 - \frac{111}{100} - \frac{154}{81}\| \\ &= \frac{91}{8100} > \frac{1}{900} = \|x - y\|. \end{aligned}$$

Hence T is not non-expansive mapping.
Now we verify that T is Suzuki's generalized non-expansive mapping.

Here following cases arise:
Case I. If either $x, y \in [1, \frac{10}{9})$ or $x, y \in [\frac{10}{9}, 2]$. Then in both the cases T is non-expansive mapping and hence T is Suzuki's generalized non-expansive mapping.

Case II. Let $x \in [1, \frac{10}{9})$. Then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|x-(3-x)\| = \frac{1}{2}\|2x-3\| \in (\frac{7}{18}, \frac{1}{2}]$. For $\frac{1}{2}\|x - Tx\| \leq \frac{1}{2}\|x - y\|$, we must have $\frac{2x-3}{2} \leq x - y \implies y \geq \frac{3}{2}$ and hence $y \in [\frac{3}{2}, 2]$. We have,

$$\|Tx - Ty\| = \|\frac{y+16}{9} - 3 + x\| = \|\frac{y+9x-11}{9}\| < \frac{1}{9}.$$

And.

$$\|x - y\| = |x - y| > |\frac{10}{9} - \frac{3}{2}| = |\frac{20-27}{18}| = \frac{7}{18} > \frac{1}{9}.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Case III. Let $x \in [\frac{10}{9}, 2]$. Then $\frac{1}{2}\|x-Tx\| = \frac{1}{2}\|\frac{x+16}{9} - x\| = \|\frac{16-8x}{18}\| \in [0, \frac{64}{162}]$. For $\frac{1}{2}\|x-Tx\| \leq \frac{1}{2}\|x-y\|$, we must have $\frac{16-8x}{18} \leq |x-y|$, which gives two possibilities:

(a) Let $x < y$, then $\frac{16-8x}{18} \leq y - x$, *i.e.* $\frac{10x+16}{18} \leq y \implies y \in [\frac{244}{162}, 2] \subset [\frac{10}{9}, 2]$. So

$$\|Tx - Ty\| = \|\frac{x+16}{9} - \frac{y+16}{9}\| = \frac{1}{9}\|x - y\| \leq \|x - y\|.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

(b) Let $x > y$, then $\frac{16-8x}{18} \leq x - y$, *i.e.* $y \leq \frac{26x-16}{18} \implies y \leq \frac{116}{162}$ and $y \leq 2$ $\implies y \in [1, 2]$. Since $y \in [1, 2]$ and $y \leq \frac{26x-16}{18} \implies \frac{18y+16}{26} \leq x \implies x \in [\frac{34}{26}, 2]$. Since $x \in [\frac{34}{26}, 2]$ and $y \in [\frac{10}{9}, 2]$ is already included in Case I. Therefore consider, $x \in [\frac{34}{26}, 2]$ and $y \in [1, \frac{10}{9})$. Then

$$\|Tx - Ty\| = \|\frac{x+16}{9} - 3 + y\| = \|\frac{x+9y-11}{9}\| < \frac{1}{9}.$$

And.

$$\|x - y\| = |x - y| > |\frac{34}{6} - \frac{10}{9}| = |\frac{306-260}{234}| = \frac{46}{234} > \frac{1}{9}.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.
Thus T is Suzuki's generalized non-expansive mapping.

Now for this Example, we construct the following comparison table and graph for various iterations with control sequences $a_n = 0.85$, $b_n = 0.65$, $c_n = 0.45$ and initial guess $x_1 = 1.2$ with the help of Matlab program 2015a.

Remark 3.9. The iterative scheme (1.8) converges faster than the Picard, Mann, Ishikawa, Noor, Agarwal, Abbas and Thakur iterative schemes for Suzuki's generalized non-expansive mappings as shown in the following table and figure, which is wider class of the non-expansive mappings as shown in above numerical example.

Item	Picard	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	Sahu, Thakur
1	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000
2	1.911111	1.804444	1.848099	1.850281	1.954765	1.953570	1.967526	1.972859
3	1.990123	1.952198	1.971158	1.971980	1.997442	1.997305	1.998682	1.999079
4	1.998903	1.988315	1.994523	1.994756	1.999855	1.999844	1.999946	1.999969
5	1.999878	1.997144	1.998960	1.999019	1.999992	1.999991	1.999998	1.999999
6	1.999986	1.999302	1.999803	1.999816	2.000000	1.999999	2.000000	2.000000
7	1.999998	1.999829	1.999963	1.999966	2.000000	2.000000	2.000000	2.000000
8	2.000000	1.999958	1.999993	1.999994	2.000000	2.000000	2.000000	2.000000
9	2.000000	1.999990	1.999999	1.999999	2.000000	2.000000	2.000000	2.000000
10	2.000000	1.999998	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
11	2.000000	1.999999	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
12	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

Table 1. A comparison table of iterative schemes.

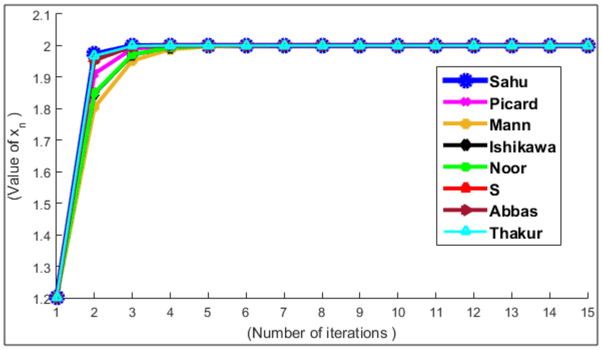

Figure 1. Convergence behavior of Sahu iterative scheme with other iterative schemes.

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