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Article

The sums of a pair of orthogonal frames

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Abstract: A sum of different frames under the action of a bounded linear operator is studied with the help of analysis, synthesis and frame operators. In particular, it is shown that the sum of a pair of orthogonal frames is a frame. This provides an easy construction of a frame where the frame bounds can be computed easily. For a pair of orthogonal frames, the necessary and sufficient condition is presented for their alternate duals to be orthogonal.

Keywords: Frames; Orthogonal frames; Canonical dual; Alternate dual

MSC: 42C15

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1. Introduction

Let $\mathbb{H}$ be a separable Hilbert space and let $\mathbb{N}$ be a countable index set. A sequence $\mathbb{X} = \{x_j\}_j$, $j \in \mathbb{N}$ in $\mathbb{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that for all $f \in \mathbb{H}$,

$$\sum_{j\in\mathbb{N}} |\langle f, x_j\rangle|^2 \leq B||f||^2.$$

$\mathbb{X}$ is said to be a frame if there exists constants $0 < A \leq B$ such that for all $f \in \mathbb{H}$,

$$A||f||^2 \leq \sum_{j\in\mathbb{N}} |\langle f, x_j\rangle|^2 \leq B||f||^2.$$

A and B are called the frame bounds. $\mathbb{X}$ is called a tight frame or an A tight frame if $A = B$. It's called a normalized tight frame if $A = B = 1$. If $\mathbb{X}$ is an orthonormal basis, it is a normalized tight frame. The left inequality in the definition of frame implies that the sequence $\mathbb{X}$ is complete [8].

Note: Throughout this paper the sequence of scalars will be denoted by $(c_j)_j$ and the sequence of vectors will be denoted by $\{x_j\}_j$.

Let $\mathbb{X}$ be a Bessel sequence. The *analysis* and *synthesis* operators, denoted respectively by $T^*_{\mathbb{X}} : \mathbb{H} \to l_2(\mathbb{N})$ and $T_{\mathbb{X}} : l_2(\mathbb{N}) \to \mathbb{H}$, are defined respectively by,

$$T^*_{\mathbb{X}} : f \to (\langle f, x_j\rangle)_j;$$

and

$$T_{\mathbb{X}} : (c_j)_j \to \sum_{j\in\mathbb{N}} c_j x_j.$$

The analysis operator is actually the Hilbert space adjoint operator of synthesis operator. These operators are well defined and bounded because $\mathbb{X}$ is a Bessel sequence [8]. It turns out that $\mathbb{X}$ is a frame if and only if the analysis operator is injective. Also, it is a frame if and only if the synthesis operator is surjective [8].

The *frame operator*, denoted by $S_{\mathbb{X}}$, is defined by $S_{\mathbb{X}} := T_{\mathbb{X}} T^*_{\mathbb{X}} : \mathbb{H} \to \mathbb{H}$, and is given by

$$S_{\mathbb{X}} f = \sum_{j\in\mathbb{N}} \langle f, x_j\rangle x_j. \tag{1}$$

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It is known that if $\mathbb{X}$ is a frame, the series (1) converges unconditionally, the operator $S_{\mathbb{X}}$ is bounded, self adjoint, positive and has bounded inverse [8]. Thus we have the following reconstruction formula,

$$f = \sum_{j\in\mathbb{N}} \langle f, S_{\mathbb{X}}^{-1} x_j \rangle x_j = \sum_{j\in\mathbb{N}} \langle f, x_j \rangle S_{\mathbb{X}}^{-1} x_j. \tag{2}$$

Let $\mathbb{Y} = \{y_j\}_j$ be another Bessel sequences in $\mathbb{H}$. If the operator, $S_{\mathbb{X},\mathbb{Y}} := T_{\mathbb{X}} T_{\mathbb{Y}}^*$ given by,

$$T_{\mathbb{X}} T_{\mathbb{Y}}^* f = \sum_{j\in\mathbb{N}} \langle f, y_j \rangle x_j,$$

is an identity, then the Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ are actually frames and are called dual frames [6]. In this case, the reconstruction formula takes the form

$$f = \sum_{j\in\mathbb{N}} \langle f, y_j \rangle x_j.$$

So from (2), it follows that the sequence $S_{\mathbb{X}}^{-1}(\mathbb{X}) := \{S_{\mathbb{X}}^{-1}(x_j)\}_j, j \in \mathbb{N}$ is a dual to $\mathbb{X}$, called the canonical dual. Besides the canonical dual, a frame has many duals known as the alternate duals.

Two Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ in a Hilbert space $\mathbb{H}$ are said to be orthogonal [3–6] if $ran(T_{\mathbb{X}}^*) \perp ran(T_{\mathbb{Y}}^*)$. This is equivalent to

$$S_{\mathbb{Y},\mathbb{X}} = T_{\mathbb{Y}} T_{\mathbb{X}}^* = 0 \Leftrightarrow S_{\mathbb{X},\mathbb{Y}} = T_{\mathbb{X}} T_{\mathbb{Y}}^* = 0.$$

These are equivalent to

$$\sum_{j\in\mathbb{N}} \langle f, x_j \rangle y_j = 0 \text{ for all } f \in \mathbb{H} \Leftrightarrow \sum_{j\in\mathbb{N}} \langle f, y_j \rangle x_j = 0 \text{ for all } f \in \mathbb{H}.$$

The sum of a pair of frames is not always a frame. Some known results about the sums being the frames are provided in section 2. Section 3 provides conditions under which the sum of a pair of frames is a frame. In particular, it is shown that the sum of an orthogonal pair of frames is a frame. We provide an easy proof of this through the use of analysis and synthesis operators [Theorem 1]. This improves the result presented in [15, Proposition 3.1]. More sums under the action of a surjective operator are also provided. It is known that the canonical duals of a pair of orthogonal frames are orthogonal [12]. The alternate duals of a pair of orthogonal frames need an extra condition to be an orthogonal pair. This condition is provided here [Theorem 2]. This generalizes the results provided in [12, Lemma 2 and 3]. These can be added to get the new frames.

2. Sum of Frames

Frames are considerably more stable than the basis upon the action of operators [8]. For example let $L : \mathbb{H} \to \mathbb{H}$ be an operator, $\mathbb{X} = \{x_i\}_i, i \in \mathbb{N}$ be an orthonormal basis, and let $L(\mathbb{X}) := \{L(x_i)\}_i$. Then $L(\mathbb{X})$ is an orthonormal basis for $\mathbb{H}$ if L is a unitary operator, $L(\mathbb{X})$ is a Riesz basis for $\mathbb{H}$ if L is a bounded bijective operator, $L(\mathbb{X})$ is a Bessel sequence in $\mathbb{H}$ if L is a bounded operator, $L(\mathbb{X})$ is a frame sequence (a frame sequence is a frame of its span) for $\mathbb{H}$ if L is a bounded with closed range, and $L(\mathbb{X})$ is a frame for $\mathbb{H}$ if L is a bounded surjective operator.

The frames have their own desired properties which can be studied from the properties of the operators that can be associated with them [2,3]. Let L_1 and L_2 be two bounded operators on the Hilbert space $\mathbb{H}$. In [15], authors provide condition under which the sum of Gabor frames is a frame. Authors in [14] have provided conditions under which the sequences $\mathbb{X} + L(\mathbb{X})$ and $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ form a Riesz basis for the space $\mathbb{H}$, where $\mathbb{X}$ and $\mathbb{Y}$ are Bessel sequences. In [11], authors study the sums of frame sequences in a Hilbert space that are strongly disjoint, disjoint, complementary pair, and weekly disjoint. The authors in [9] study the sums of frames under the same conditions.

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Let $\mathbb{X}$ be a frame in $\mathbb{H}$ with frame bounds A and B and L be a bounded surjective operator. Then $L(\mathbb{X})$ is a frame for $\mathbb{H}$ with the frame bounds $A||L^{\dagger}||^2$ and $B||L||^2$, where $L^{\dagger}$ is the pseudo-inverse of L [14]. The associated analysis, synthesis and frame operators of $L(\mathbb{X})$ are given by the following lemma.

Lemma 1. *The analysis, synthesis and frame operators for the frame $L(\mathbb{X})$ are given by $T^*_{\mathbb{X}}L^*$, $LT_{\mathbb{X}}$, and $LS_{\mathbb{X}}L^*$ respectively.*

Proof. Simple calculations

$$\begin{aligned} T^*_{L(\mathbb{X})}f &= (\langle f, L(x_i)\rangle)_i = (\langle L^*f, x_i\rangle)_i = T^*_{\mathbb{X}}L^*f, \\ T_{L(\mathbb{X})}(c_k)_k &= \sum_k c_k L(x_k) = \sum_k L(c_k x_k) = LT_{\mathbb{X}}(c_k)_k, \text{ and} \\ S_{L(\mathbb{X})}f &= \sum_k \langle f, L(x_k)\rangle L(x_k) = L\sum_k \langle L^*f x_k\rangle x_k = LS_{\mathbb{X}}L^*f, \end{aligned}$$

establish the lemma.

Since the analysis operator is injective, $T^*_{\mathbb{X}}L^*$ is injective and $LT_{\mathbb{X}}$ is surjective. So it follows that $L(\mathbb{X})$ is frame iff L is surjective as in [14, Proposition 2.3]. However the sequences $L(\mathbb{X})$ and $L^*(\mathbb{X})$ are both frames iff the operator L is invertible. Authors in [15, Proposition 2.1] have shown that the sequence $\mathbb{X} + L(\mathbb{X})$ is a frame iff operator $(I + L)$ is invertible, however it turns out to be the case when the operator is simply surjective. Calculations similar to the ones in Lemma 1 prove the following lemma [14,15].

Lemma 2. *The analysis, synthesis, and the frame operators for the frame $\mathbb{X} + L(\mathbb{X})$ are given by $T^*_{\mathbb{X}}(I + L^*)$, $(I + L)T_{\mathbb{X}}$, and $(I + L)S_{\mathbb{X}}(I + L^*)$ respectively.*

This lemma and the remarks before the previous lemma reveal that the frame bounds are $A||(I + L)^{\dagger}||^{-2}$, and $B||I + L||^2$. A special case of the above lemma is that $\{\mathbb{X} + S_{\mathbb{X}}(\mathbb{X})\}$ is a also a frame. The system $\{\mathbb{X} + S^{-1}_{\mathbb{X}}(\mathbb{X})\}$, where the given frame is being added to its canonical dual and the system $\{\mathbb{X} + S^{-1/2}_{\mathbb{X}}(\mathbb{X})\}$, where the frame is being added to its canonical Parseval frame are all frames. In sections 3, it is shown that a frame added to any of its alternate dual is also a frame.

An extra condition yields a Riesz basis for the space $\mathbb{H}$. The following proposition is taken from [14, Proposition 2.8].

Proposition 1. *Let $L : \mathbb{H} \to \mathbb{H}$ be bounded operator and $\mathbb{X}$ be a Riesz basis for $\mathbb{H}$, where $T^*_{\mathbb{X}}, T_{\mathbb{X}}, S_{\mathbb{X}}$ are respectively the analysis, synthesis and frame operators with Riesz basis bounds $A \leq B$. Then $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis for $\mathbb{H}$ with bounds $A||(I + L)^{-1}||^2$ and $B||I + L||^2$ iff $I + L$ is invertible on $\mathbb{H}$.*

Proof. Since $\mathbb{X}$ is a Riesz basis, $T^*_{\mathbb{X}}$ is invertible operator. If $I + L$ is invertible, then the operator $T^*_{\mathbb{X}}(L^* + I)$ is also invertible. But this is the analysis operator for the sequence $I + L(\mathbb{X})$. Hence the sequence $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis. If $\mathbb{X} + L(\mathbb{X})$ is a Riesz basis, then the analysis operator $T^*_{\mathbb{X}}(L^* + I)$ is invertible. Since $T^*_{\mathbb{X}}$ is invertible, so is the operator $I + L$.

The following proposition, mentioned incorrectly in [15, Proposition 3.1], is corrected in [14, Proposition 2.12].

Proposition 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences in $\mathbb{H}$ with analysis operators $T^*_{\mathbb{X}}$, $T^*_{\mathbb{Y}}$ and frame operators $S_{\mathbb{X}}, S_{\mathbb{Y}}$ respectively. Let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ be bounded operators. Then the following are equivalent.*

(A) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a Riesz basis for $\mathbb{H}$.

(B) $T^*_{\mathbb{X}}L^*_1 + T^*_{\mathbb{Y}}L^*_2$ is an invertible operator on $\mathbb{H}$.

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The sum of two frames is not a frame in general. A sequence being a frame is equivalent to its analysis operator being injective or the synthesis operator being surjective [3]. For the sum in the above proposition to be a frame, we have the following proposition.

Proposition 3. *Let L_1 and L_2 be bounded operators, and $\mathbb{X}$ and $\mathbb{Y}$ be Bessel sequences in a Hilbert space $\mathbb{H}$. Then the following are equivalent.*

(A) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ *is a frame.*
(B) $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$ *is surjective.*

The frame operator is given by $S = L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^* + L_1 T_{\mathbb{X}} T_{\mathbb{Y}}^* L_2^* + L_2 T_{\mathbb{Y}} T_{\mathbb{X}}^* L_1^*$.

Proof. The synthesis operator for the sequence $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$. It therefore follows that (A) and (B) are equivalent.

Corollary 1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a two Bessel sequences. Then the following are equivalent.*

(A) $\mathbb{X} + \mathbb{Y}$ *is a frame.*
(B) $T_{\mathbb{X}} + T_{\mathbb{Y}}$ *is surjective.*

The frame operator is given by $S = S_{\mathbb{X}} + S_{\mathbb{Y}} + T_{\mathbb{X}} T_{\mathbb{Y}}^* + T_{\mathbb{Y}} T_{\mathbb{X}}^*$.

It is still difficult to verify the conditions of the Proposition 3 or its corollary. The sum happens to be a frame if we impose an extra condition of orthogonality on the sequences. Assuming the Bessel sequences to be orthogonal, the following proposition is easily established.

Proposition 4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences such that the frame operator $S_{\mathbb{X},\mathbb{Y}}$ is a zero operator. Let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ be bounded operators, and let $\mathbb{Z} = L_1(\mathbb{X}) + L_2(\mathbb{Y})$. Then the following are equivalent.*

(A) $\mathbb{Z}$ *is a frame for* $\mathbb{H}$.
(B) $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$ *is surjective.*
(C) $L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*$ *is an invertible positive operator on* $\mathbb{H}$.

Proof. Since the frames $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal frames, we have $T_{\mathbb{Y}} T_{\mathbb{X}}^* = 0 = T_{\mathbb{X}} T_{\mathbb{Y}}^*$ i.e. the frame operator $S_{\mathbb{X},\mathbb{Y}} = 0$. The analysis operator for the sequence $\mathbb{Z}$ is $T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*$, and the frame operator $S_{\mathbb{Z}}$ is given by

$$\begin{aligned} S_{\mathbb{Z}} &= (L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}})(T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*) \\ &= L_1 T_{\mathbb{X}} T_{\mathbb{X}}^* L_1^* + L_1 T_{\mathbb{X}} T_{\mathbb{Y}}^* L_2^* + L_2 T_{\mathbb{Y}} T_{\mathbb{X}}^* L_1^* + L_2 T_{\mathbb{Y}} T_{\mathbb{Y}}^* L_2^* \\ &= L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*. \end{aligned}$$

$(A) \Leftrightarrow (C)$. Let $\mathbb{Z}$ be a frame. Then its frame operator $S_{\mathbb{Z}}$ in an invertible and positive operator. So (C) follows. (C) implies (A) is straight forward. $(A) \Leftrightarrow (B)$ because the synthesis operator of a frame is surjective.

In fact, one of operators L_1 or L_2 being surjective is enough, as the following Lemma.

Lemma 3. *If $\mathbb{X}$ and $\mathbb{Y}$ are a pair of orthogonal frames and if either L_1 or L_2 is surjective, then $L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}}$ is surjective.*

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Proof. Since $\mathbb{X}$ is a frame, the operator $T_{\mathbb{X}}T_{\mathbb{X}}^*$ is invertible and since $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal, we have $T_{\mathbb{X}}T_{\mathbb{Y}}^* = T_{\mathbb{Y}}T_{\mathbb{X}}^* = 0$. Let L_1 be surjective. Then for each $f \in \mathbb{H}$ there exists a $g \in \mathbb{H}$ such that $f = L_1 g$. Let $(c_j)_j \in l_2(\mathbb{N})$ be such that $(c_j)_j = T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(g)$. But then,

$$\begin{aligned}(L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}})(c_j)_j &= (L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}})T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(g) \\ &= L_1T_{\mathbb{X}}T_{\mathbb{X}}^*(T_{\mathbb{X}}T_{\mathbb{X}}^*)^{-1}(y) \\ &= L_1(g) \\ &= f.\end{aligned}$$

So the operator $L_1T_{\mathbb{X}} + L_2T_{\mathbb{Y}}$ is surjective.

2.1. Sum of orthogonal frames

Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames. Let A_1 and A_2 are the lower and B_1 and B_2 be the upper frame bounds for the frames $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then the following theorem provides the frame as a sum and also provides the bounds.

Theorem 1. *If the pair $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal and if one of L_1 or L_2 is surjective, then $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a frame whose frame operator is $L_1S_{\mathbb{X}}L_1^* + L_2S_{\mathbb{Y}}L_2^*$ and $B_1||L_1||^2 + B_2||L_2||^2$ being the upper bound. The lower bound is $\frac{A_i}{||L_i^{\dagger *}||^2}$ if $L_i(i = 1, 2)$ is surjective.*

Proof. Proposition 4 and Lemma 3 are enough for the above sum to be a frame. The bounds can be computed too. Let L_1 be surjective. We note that, $T_{\mathbb{X}}(e_i) = (x_i)_i$ and $T_{\mathbb{Y}}(e_i) = (y_i)_i$.

$$\begin{aligned}\sum_{i\in\mathbb{N}} |\langle f, L_1(x_i) + L_2(y_i)\rangle|^2 &= \sum_{i\in\mathbb{N}} |\langle L_1^*f, x_i\rangle + \langle L_2^*f, y_i\rangle|^2 \\ &= \sum_{i\in\mathbb{N}} |\langle L_1^*f, T_{\mathbb{X}}(e_i)\rangle + \langle L_2^*f, T_{\mathbb{Y}}(e_i)\rangle|^2 \\ &= \sum_{i\in\mathbb{N}} |\langle (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f, e_i\rangle|^2 \\ &= ||(T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f||^2 \\ &= \langle (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f, (T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*)f\rangle \\ &= \langle T_{\mathbb{X}}^*L_1^*f, T_{\mathbb{X}}^*L_1^*f\rangle + \langle T_{\mathbb{Y}}^*L_2^*f, T_{\mathbb{Y}}^*L_2^*f\rangle \\ &= \langle T_{\mathbb{X}}T_{\mathbb{X}}^*L_1^*f, L_1^*f\rangle + \langle T_{\mathbb{Y}}T_{\mathbb{Y}}^*L_2^*f, L_2^*f\rangle \\ &= ||S_{\mathbb{X}}L_1^*f||^2 + ||S_{\mathbb{Y}}L_2^*||^2 \\ &\geq A_1||L_1^*f||^2 + A_2||L_2^*f||^2 \\ &\geq A_1||L_1^*f||^2 \\ &\geq A_1\frac{||f||^2}{||L_1^{\dagger *}||^2},\end{aligned}$$

since $L_1L_1^\dagger = I$, $L_1^{\dagger *}L_1^* = I$, we have $||f|| = ||L_1^{\dagger *}L_1^*|| \leq ||L_1^{\dagger *}||||L_1^*f||$. So

$$||L_1^*f|| \geq \frac{||f||}{||L_1^{\dagger *}||}.$$

For the upper bound,

$$\begin{aligned}\sum_{i\in\mathbb{N}} |\langle f, L_1(x_i)+L_2(y_i)\rangle|^2 &= ||S_{\mathbb{X}}L_1^*f||^2+||S_{\mathbb{Y}}L_2^*f||^2\\ &\leq B_1||L_1^*f||^2+B_2||L_2^*f||^2\\ &\leq (B_1||L_1^*||^2+B_2||L_2^*||^2)||f||^2.\end{aligned}$$

So the upper bound is $B_1||L_1||^2+B_2||L_2||^2$.

In particular if $L_1=I$, the sum $\mathbb{X}+L_2(\mathbb{X})$ is a frame iff $(I+L_2)T_{\mathbb{X}}$ is surjective. In addition, if $L_1=L_2=I$ in Theorem 1, then the frame operator is simply $S_{\mathbb{X}}+S_{\mathbb{Y}}$.

Corollary 2. *If $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal Parseval frames, the sum $\mathbb{Z}=\mathbb{X}+\mathbb{Y}$ is Parseval frame if and only if the operators L_1 and L_2 are scaled unitary operators i.e $L_1=\frac{U_1}{\sqrt{2}}$, and $L_2=\frac{U_2}{\sqrt{2}}$ where U_1 and U_2 are unitary operators.*

The frame operator in this case is

$$S_{\mathbb{Z}}=L_1S_{\mathbb{X}}L_1^*+L_2S_{\mathbb{Y}}L_2^*=\frac{U_1U_1^*}{2}+\frac{U_2U_2^*}{2}=\frac{I}{2}+\frac{I}{2}=I.$$

This generalizes to any finite sum.

Corollary 3. *Let $\mathbb{X}_1,\cdots\mathbb{X}_k$ be pairwise orthogonal Parseval frames. Let $U_1,\cdots U_k$ be unitary operators. Then the sum $\mathbb{Z}=L_1(\mathbb{X}_1)+\cdots+L_k(\mathbb{X}_k)$ is a Parseval frame, where $L_i=\frac{U_i}{\sqrt{k}}$.*

The following example takes a pair of orthogonal frames for $l_2(\mathbb{N})$ from [13].

Example 1. *Sum of discrete Gabor frames in $l_2(\mathbb{N})$.*

Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{H}=l_2(\mathbb{N})$. Let $g=\frac{1}{\sqrt{3}}(e_1+e_2)$, and $h=\frac{1}{\sqrt{3}}(e_3+e_4)$, and let

$$g_{k,m}(n):=e^{\frac{2\pi ikn}{3}}g(n-2m),\ \text{and}\ \ h_{k,m}(n):=e^{\frac{2\pi ikn}{3}}h(n-2m).$$

Then the systems

$$\{g_{k,m}:0\leq k\leq 2, m\in\mathbb{Z}\}\ \text{and}\ \{h_{k,m}:0\leq k\leq 2, m\in\mathbb{Z}\}$$

form Parseval frames for the space $\mathbb{H}$, since $\mathbb{H}=\oplus_{m\in\mathbb{Z}}M_m$ is the orthogonal direct sum of $M_m=span\{e_{1+2m},e_{2+2m}\}$ and for each fixed m the system $\{g_{k,m}\}_{k=0}^2$ is a Parseval frame for M_m [13]. Similar is the case for the system $\{h_{k,m}\}_{k=0}^2$. It turns out that the two systems form an orthogonal pair of frames for $\mathbb{H}$ [13, Theorem 1.4]. The above corollary implies that the sum $s=\frac{1}{\sqrt{2}}g+\frac{1}{\sqrt{2}}h=\frac{1}{\sqrt{6}}(e_1+e_2+e_3+e_4)$, provides a Parseval frame for $\mathbb{H}$ as well. i.e the system

$$s_{k,m}(n)=e^{\frac{\pi ikn}{3}}s(n-2m),\ 0\leq k\leq 5,\ m\in\mathbb{Z}$$

forms a Parseval frame for $\mathbb{H}$. This can be verified by the argument from [13, Example 1.3].

Example 2. *Sum of Gabor frames in $L_2(\mathbb{R})$.*

For $x,y\in\mathbb{R}$, let E_x, and T_y be operators defined on $L_2(\mathbb{R})$ by

$$E_x(f(t))=e^{2\pi ixt},\quad \text{and}\ \ T_y(f(t))=f(t-y).$$

Since the polynomial $1 + pz$ doesn't have root on the unit circle for $p \neq 1$, the set $[0,1) \cup [1,2)$ forms a Gabor frame wavelet set [7,10]. Likewise, the set $[-2,-1) \cup [-1,0)$ forms the Gabor frame wavelet set. Let

$$g_1(t) = \kappa_{[0,1)} + p\kappa_{[1,2)}, \text{ and } g_2(t) = \kappa_{[-1,0)} + p\kappa_{[-2,-1)}$$

The families

$$\mathbb{X} = \{E_m T_n g_1(t)\}_{m,n\in\mathbb{Z}}, \text{ and } \mathbb{Y} = \{E_m T_n g_2(t)\}_{m,n\in\mathbb{Z}}$$

form the frames for the space $L_2(\mathbb{R})$. Since the $support(\mathbb{X}) \cap support(\mathbb{Y}) = 0$ for all $m,n \in \mathbb{Z}$, it follows that for all $f \in L_2(\mathbb{R})$, we have

$$\sum_{m,n\in Z} \langle f(t), E_m T_n g_1(t)\rangle \, E_m T_n g_2(t) = 0,$$

So $\mathbb{X}$ and $\mathbb{Y}$ form a pair of orthogonal frames for the space $L_2(\mathbb{R})$. Therefore the sum

$$h(t) = g_1(t) + g_2(t) = \kappa_{[-1,0)} + p\kappa_{[-2,1)} + \kappa_{[0,1)} + p\kappa_{[1,2)}$$

forms a frame for $L_2(\mathbb{R})$.

Lemma 4. *Let $\tilde{\mathbb{X}}$ be the dual frame of $\mathbb{X}$. Then $L_1^{\dagger^*}(\tilde{\mathbb{X}})$ is a dual to $L_1(\mathbb{X})$, where L_1 is a surjective operator.*

Proof. Since L_1 is surjective, the operator $L_1 L_1^*$ is invertible. Let $L_1^\dagger = L_1^*(L_1 L_1^*)^{-1}$. So

$$S_{L_1(\mathbb{X}),L_1^{\dagger *}(\tilde{\mathbb{X}})} = L_1 T_{\mathbb{X}} T_{\tilde{\mathbb{X}}}^* L^\dagger = L_1 L_1^\dagger = I,$$

and

$$S_{L^{\dagger *}(\tilde{\mathbb{X}}),L_1(\mathbb{X})} = L_1^{\dagger *} T_{\tilde{\mathbb{X}}} T_{\mathbb{X}}^* L_1^* = L_1^{\dagger *} L_1^* = I.$$

This completes the proof.

If the operator L is invertible, we have the following result.

Corollary 4. *Let $\tilde{\mathbb{X}}$ be the dual frame of $\mathbb{X}$. Then $L_1^{-1^*}(\tilde{\mathbb{X}})$ is dual to $L_1(\mathbb{X})$.*

As an consequence of Lemma 1, we have the following theorem for a pair of orthogonal frames. The following theorem assumes that the operator L is surjective.

Corollary 5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames for $\mathbb{H}$. Then the frames $L(\mathbb{X})$ and $L(\mathbb{Y})$ are orthogonal too.*

Proof. From Lemma 1, it follows that the frame operator $S_{L(\mathbb{X}),L(\mathbb{Y})}$ is given by

$$S_{L(\mathbb{X}),L(\mathbb{Y})} = L T_{\mathbb{X}} T_{\mathbb{Y}}^* L^* = 0.$$

So $L(\mathbb{X})$ and $L(\mathbb{Y})$ are orthogonal.

The following is proved in [9], but the Lemma 1 provides a very simple proof.

Corollary 6. *Let $\mathbb{X}_1, \mathbb{X}_2 \cdots \mathbb{X}_k$ be a pairwise orthogonal frames. If L_i is surjective, then $L_i^{\dagger *}(\mathbb{X}_i)$ is a dual of $L_1(\mathbb{X}_1) + L_2(\mathbb{X}_2) + \cdots + L_k(\mathbb{X}_k)$.*

Proof. Use of Lemma 1 establishes this. The synthesis operator of the sum is $L_1\mathbb{X}_1 + \cdots L_k\mathbb{X}_k$, and the analysis operator of $L_i^{\dagger *}(\mathbb{X}_i)$ is $T_{\mathbb{X}_i}^* L_i^\dagger$, it turns out that the composition is

$$(L_1\mathbb{X}_1 + \cdots L_k\mathbb{X}_k) T_{\mathbb{X}_i}^* L_i^\dagger = L_i T_{\mathbb{X}_i} T_{\mathbb{X}_i}^* L_i^\dagger = L_i L_i^\dagger = I.$$

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In general, $L_i^{\dagger *}(\mathbb{X}_i)$ is dual $L_i(\mathbb{X}_i) + \sum_{j \neq i} d_j L_j(\mathbb{X}_j)$ where $d_j = 0$ or 1.

2.2. Orthogonality of alternate duals

Alternate dual of a frame $\mathbb{X}$ are given by $\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(x_i) + \psi^*(e_i)\}_i$, where $\psi \in B(\mathbb{H}, l_2(\mathbb{N}))$ such that $T_{\mathbb{X}}\psi = 0$, and $\{e_i\}_{i\in\mathbb{N}}$ is an standard orthonormal basis of $l_2(\mathbb{N})$ [1]. It is also known that $\{\psi^*(e_i)\}_{i\in\mathbb{N}}$ is a Bessel sequence in $\mathbb{H}$ [1]. Authors in [12] have studied the orthogonality of canonical dual of a pair of orthogonal frames. The following theorem establishes the conditions needed for the orthogonality of any alternate dual.

Let $\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(x_i) + \psi^*(e_i)\}_i$, and $\tilde{\mathbb{Y}} = \{S_{\mathbb{Y}}^{-1}(y_i) + \phi^*(e_i)\}_i$, where $\psi, \phi \in B(\mathbb{H}, l_2(\mathbb{N}))$ such that $T_{\mathbb{X}}\psi = 0$, and $T_{\mathbb{X}}\phi = 0$, be the alternate duals of $\mathbb{X}$ and $\mathbb{Y}$ respectively. The following theorem holds.

Theorem 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a pair of orthogonal frames and let $\tilde{\mathbb{X}}$, and $\tilde{\mathbb{Y}}$ be their corresponding duals. Then*

(A) The pair $\mathbb{X}$ and $\tilde{\mathbb{Y}}$ are orthogonal if and only if $T_{\mathbb{X}}\phi = 0$.
(B) The pair $\tilde{\mathbb{X}}$ and $\mathbb{Y}$ are orthogonal if and only if $T_{\mathbb{Y}}\psi = 0$.
(C) The pair $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ are orthogonal if $T_{\mathbb{X}}\phi = 0$, $T_{\mathbb{Y}}\psi = 0$ and $\phi^*\psi = 0$.
(C′) If $\mathbb{X}$ is orthogonal to $\tilde{\mathbb{Y}}$ and $\tilde{\mathbb{X}}$ is orthogonal to $\mathbb{Y}$, then $\tilde{\mathbb{X}}$ is orthogonal to $\tilde{\mathbb{Y}}$ if and only if $\phi^*\psi = 0$.

Proof. (A) Let $\Psi := \{\psi^*(e_i)\}_{i\in\mathbb{N}}$, such that $T_{\mathbb{X}}\psi = 0$ for some $\psi \in B(\mathbb{H}, l_2(\mathbb{N}))$, and $\Phi := \{\phi^*(e_i)\}_{i\in\mathbb{N}}$, for some $\phi \in B(\mathbb{H}, l_2(\mathbb{N}))$, such that $T_{\mathbb{Y}}\phi = 0$. Then $\tilde{\mathbb{X}} = \{S_{\mathbb{X}}^{-1}(\mathbb{X}) + \Psi\}$, and $\tilde{\mathbb{Y}} = \{S_{\mathbb{Y}}^{-1}(\mathbb{Y}) + \Phi\}$. For each $f \in \mathbb{H}$, the sequence Ψ provides

$$T_{\Psi}^*(f) = (\langle f, \psi^*(e_i)\rangle)_i = (\langle \psi(f), e_i\rangle)_i = (\psi(f))_i,$$

and for each $(c_j) \in l_2(\mathbb{N})$,

$$T_{\Psi}(c_j) = \sum_i c_i\psi^*(e_i) = \sum_i \psi^*(c_ie_i) = \psi^*\left(\sum_i c_je_j\right) = \psi^*(c_j).$$

(A) The frame operator $S_{\mathbb{X},\tilde{\mathbb{Y}}}$ is given by

$$S_{\mathbb{X},\tilde{\mathbb{Y}}} = T_{\mathbb{X}}T_{\tilde{\mathbb{Y}}}^* = T_{\mathbb{X}}(T_{\mathbb{Y}}^*S_{\mathbb{Y}}^{-1} + T_{\Phi}) = T_{\mathbb{X}}T_{\mathbb{Y}}^*S_{\mathbb{Y}}^{-1} + T_{\mathbb{X}}T_{\Phi}^* = T_{\mathbb{X}}T_{\Phi}^*,$$

and $T_{\mathbb{X}}T_{\Phi}^*f = T_{\mathbb{X}}\phi(f)$. This establishes (A).
(B) The frame operator $S_{\tilde{\mathbb{X}},\mathbb{Y}}$ is

$$S_{\tilde{\mathbb{X}},\mathbb{Y}} = T_{\tilde{\mathbb{X}}}T_{\mathbb{Y}}^* = (S_{\mathbb{X}}^{-1}T_{\mathbb{X}} + T_{\Psi})T_{\mathbb{Y}}^* = S_{\mathbb{X}}^{-1}T_{\mathbb{X}}T_{\mathbb{Y}}^* + T_{\Psi}T_{\mathbb{Y}}^* = T_{\Psi}T_{\mathbb{X}}^*$$

and $T_{\Psi}T_{\mathbb{X}}^*f = \psi^*T_{\mathbb{X}}^*f$. Since $\langle \psi^*T_{\mathbb{X}}^*f, g\rangle = \langle f, T_{\mathbb{X}}\psi g\rangle$, it follows that $\psi^*T_{\mathbb{X}}^*f = 0$ for all f iff $\mathrm{T}_{\mathbb{X}}\psi = 0$. This establishes (B).
(C) We notice that

$$\begin{aligned} S_{\tilde{\mathbb{Y}},\tilde{\mathbb{X}}} &= (S_{\mathbb{Y}}^{-1}T_{\mathbb{Y}} + T_{\Phi})(T_{\mathbb{X}}^*S_{\mathbb{X}}^{-1} + T_{\Psi}^*) \\ &= S_{\mathbb{Y}}^{-1}T_{\mathbb{Y}}T_{\mathbb{X}}^*S_{\mathbb{X}}^{-1} + S_{\mathbb{Y}}^{-1}T_{\mathbb{Y}}T_{\Psi}^* + T_{\Phi}T_{\mathbb{X}}^*S_{\mathbb{X}}^{-1} + T_{\Phi}T_{\Psi}^* \\ &= S_{\mathbb{Y}}^{-1}T_{\mathbb{Y}}T_{\Psi}^* + T_{\Phi}T_{\mathbb{X}}^*S_{\mathbb{X}}^{-1} + T_{\Phi}T_{\Psi}^* \\ &= S_{\mathbb{Y}}^{-1}T_{\mathbb{Y}}\psi + \phi^*T_{\mathbb{X}}^*S_{\mathbb{X}}^{-1} + \phi^*\psi, \end{aligned}$$

since $T_{\Phi}T_{\Psi}^*f = \phi^*\psi(f)$. So (C) follows using (A) and (B). (C′) follows from (A), (B) and (C).

Corollary 7. *Let $\mathbb{X}$ and $\mathbb{Y}$ be orthogonal frames with canonical dual $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ respectively. Then the sum $\frac{\mathbb{X}+L(\mathbb{Y})}{\sqrt{2}}$ is dual to $\frac{\tilde{\mathbb{X}}+L^{\dagger *}(\tilde{\mathbb{Y}})}{\sqrt{2}}$.*

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Proof. The Lemma 1 and Theorem 2 establish this.

Let $\tilde{\mathbb{X}} = S_{\mathbb{X}}^{-1}(\mathbb{X}) + \Psi$ be an alternate dual to $\mathbb{X}$. We now show that the frame can be added to any of its alternate dual frame to yield a new frame.

Theorem 3. *The sum $\mathbb{Z} = \mathbb{X} + \tilde{\mathbb{X}}$ is a frame.*

Proof. From Proposition 4, it suffices to show that the operator $T_{\mathbb{Z}}$ is surjective. The operator $T_{\mathbb{Z}} = T_{\mathbb{X}} + S_{\mathbb{X}}^{-1}T_{\mathbb{X}} + T_{\Psi}$. Since $S_{\mathbb{X}}$ is a positive operator, the operator $S_{\mathbb{X}} + I$ is invertible. Therefore for each $f \in \mathbb{H}$, there exists $g \in \mathbb{H}$ such that $(S_{\mathbb{X}} + I)(g) = f$. Let $T_{\mathbb{X}}^*(g) = (d_i)$. Now,

$$\begin{aligned}
(T_{\mathbb{X}} + S_{\mathbb{X}}^{-1}T_{\mathbb{X}} + T_{\Psi})(d_i) &= (T_{\mathbb{X}} + S_{\mathbb{X}}^{-1}T_{\mathbb{X}} + T_{\Psi})T_{\mathbb{X}}^*(g) \\
&= T_{\mathbb{X}}T_{\mathbb{X}}^*(g) + S_{\mathbb{X}}^{-1}T_{\mathbb{X}}T_{\mathbb{X}}^*(g) + T_{\Psi}T_{\mathbb{X}}^*(g) \\
&= S_{\mathbb{X}}(g) + g + \psi^* T_{\mathbb{X}}^*(g) \\
&= (S_{\mathbb{X}} + I)(g) \\
&= f.
\end{aligned}$$

3. Conclusion

A construction of frame as a sum of frames is established with the aid of analysis and synthesis operators. The condition for the orthogonality of alternate duals for a pair of orthogonal frames is presented. This enables us to construct more frames as sum of frames from the given ones.

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