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ON SOME CLASSES OF SPIRALLIKE FUNCTIONS DEFINED BY THE SALAGEAN OPERATOR

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Abstract. In this paper, we introduce two subclasses of analytic and Spirallike functions and investigate convolution properties, the necessary and sufficient condition, coefficient estimates and inclusion properties for these classes.

1. Preface

In recent times, the study of analytic functions has been useful in solving many problems in mechanics, Laplace equation, electrostatics, etc. An analytic function is said to be univalent in a domain if it does not take the same value twice in that domain. Let us denote the family of all meromorphic functions f with no poles in the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

by $\mathcal{A}$. Clearly, functions in $\mathcal{A}$ are analytic in $\mathbb{U}$ and the set of all univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. Functions in $\mathcal{S}$ are of interest because they appear in the Riemann mapping theorem and several other situation in many different contexts. In 1983, Salagean [?] introduced differential operator $\mathcal{D}^k : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = \mathcal{D}f(z) = zf'(z),$$

$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)) = z(\mathcal{D}^{n-1} f(z))', \quad n \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

In this way

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}. \tag{2}$$

For functions f given by (1) and g given by

$$g(z) = z + b_2z^2 + b_3z^3 + \cdots = z + \sum_{k=2}^{\infty} b_k z^k,$$

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the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In this paper, we investigate convolution properties of $\mathcal{S}_n^\alpha[A,B]$ and $\mathcal{K}_n^\alpha[A,B]$ associated with Salagean differential operator. Using convolution properties, we find the necessary and sufficient condition, coefficient estimates and inclusion properties for these classes. More recent works can be found on [2, 5, 10, 12, 16].

2. Preliminaries

We start with some useful definitions, theorems and lemmas.

Definition 2.1. A function $f \in \mathcal{S}$ is said to be starlike in $\mathbb{U}$ if the image $f(\mathbb{U})$ is starlike with respect to 0. It is well known that a function f is starlike if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the class of all functions in $\mathcal{S}$ which are starlike in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if the image $f(\mathbb{U})$ is convex.

Lemma 2.2. *The function f is convex in $\mathbb{U}$ if and only if*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{K}$ the class of all functions in $\mathcal{S}$ which are convex in $\mathbb{U}$. It is easy to see that, $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$.

Definition 2.3. For analytic functions g and h in $\mathbb{U}$, g is said to be subordinate to h if there exists an analytic function w such that

$$w(0) = 0, \quad |w(z)| < 1, \quad and \quad g(z) = h(w(z)), \quad z \in \mathbb{U}.$$

This subordination will be denoted here by

$$g \prec h,$$

or, conventionally, by

$$g(z) \prec h(z),$$

In particular, when h is univalent in $\mathbb{U}$,

$$g \prec h \quad \Longleftrightarrow \quad g(0) = h(0), \quad and \quad g(\mathbb{U}) \subset h(\mathbb{U}), \quad z \in \mathbb{U}.$$

Making use of the principal of subordination and Salagean differential operator between analytic functions, we introduce the following classes of analytic functions for $n \in \mathbb{N}_0$, $|\alpha| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$:

$$\mathcal{S}_n^\alpha[A,B] = \left\{ f \in \mathcal{S} : e^{i\alpha} \frac{z(\mathcal{D}^n f(z))'}{\mathcal{D}^n f(z)} \prec \cos\alpha \left(\frac{1+Az}{1+Bz}\right) + i \sin\alpha, z \in \mathbb{U} \right\}, \tag{3}$$

and

$$\mathcal{K}_n^\alpha[A,B] = \left\{ f \in \mathcal{S} : e^{i\alpha}\left(1 + \frac{z(\mathcal{D}^n f(z))''}{(\mathcal{D}^n f(z))'}\right) \prec \cos\alpha \left(\frac{1+Az}{1+Bz}\right) + i \sin\alpha, z \in \mathbb{U} \right\},$$

Note that

$$f(z) \in \mathcal{K}_n^\alpha[A,B] \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}_n^\alpha[A,B]. \tag{4}$$

By specializing the parameters, we have the following know subclasses studied by various researhers.

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I $\mathcal{S}_0^0[A,B] = \mathcal{S}[A,B]$ and $\mathcal{K}_0^0[A,B] = \mathcal{K}[A,B]$, where the classes $\mathcal{S}[A,B]$ and $\mathcal{K}[A,B]$ are introduced and studied by many authors in [1, 6, 8, 9, 14].

II $\mathcal{S}_0^0[1-2\beta,-1] = \mathcal{S}^*(\beta)$ and $\mathcal{K}_0^0[1-2\beta,-1] = \mathcal{K}(\beta)$, where the classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ are introduced and stadied in [7].

III $\mathcal{S}_0^0[\frac{b^2-a^2+b}{b}, \frac{1-b}{a}] = \mathcal{S}(a,b)$ and $\mathcal{K}_0^0[\frac{b^2-a^2+b}{b}, \frac{1-b}{a}] = \mathcal{K}(a,b)$, where the classes $\mathcal{S}(a,b), \mathcal{K}(a,b)$ are introduced and stadied by in [14, 15].

IV $\mathcal{S}_0^\alpha[A,B] = \mathcal{S}^\alpha[A,B]$ and $\mathcal{K}_0^\alpha[A,B] = \mathcal{K}^\alpha[A,B] = \mathcal{S}_1^\alpha[A,B]$, where the classes $\mathcal{S}^\alpha[A,B]$, $\mathcal{K}^\alpha[A,B]$ are introduced and stadied in [3, 4, 11].

V $\mathcal{S}_0^0[A,B] = \mathcal{R}_0[A,B]$ and $\mathcal{K}_0^0[A,B] = \mathcal{R}_1[A,B]$, where the classes $\mathcal{R}_0[A,B], \mathcal{R}_1[A,B]$ are introduced and stadied in [1].

3. Convolution Properties

In this section, we study some of the properties of foresaid convolution. Unless otherwise mentioned, we assume throughout this paper that $-1 \le B < A \le 1$, $|\alpha| < \frac{\pi}{2}$, $|\xi| = 1$ and $\mathcal{D}^n f(z)$ is defined by (2). To prove our convolution properties, we shall need the following lemmas due to Silverman and Silvia [14].

Lemma 3.1. [14] *The function $f(z)$ defined by (1) is in the class $\mathcal{S}^*[A,B]$ if and only if for all z in $\mathbb{U}$ and all ξ, $|\xi| = 1$,*

$$\frac{1}{z}\Big[f * \frac{z + \frac{\xi - A}{A-B}z^2}{(1-z)^2}\Big] \neq 0. \tag{5}$$

Lemma 3.2. *The function $f(z)$ defined by (1) is in the class $\mathcal{S}_n^\alpha[A,B]$ if and only if for all z in $\mathbb{U}$ and all ξ, $|\xi| = 1$,*

$$\frac{1}{z}\Big[f * \big(z + \sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^n z^k\big)\Big] \neq 0 \tag{6}$$

where

$$\psi = \frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\zeta}{e^{i\alpha}(1+B\zeta)} \tag{7}$$

Proof. An application of lemma 3.1 exhibits that $f \in \mathcal{S}_n^\alpha[A,B]$ if and only if

$$e^{i\alpha}\frac{z(\mathcal{D}^n f(z))'}{\mathcal{D}^n f(z)} \neq \cos\alpha\Big(\frac{1+A\zeta}{1+B\zeta}\Big) + i\sin\alpha \quad (z \in \mathbb{U}, \quad |\xi| = 1)$$

$$\Longleftrightarrow z(\mathcal{D}^n f(z))' - \mathcal{D}^n f(z)\Big(\frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\zeta}{e^{i\alpha}(1+B\zeta)}\Big) \neq 0 \quad z \in \mathbb{U}, |\xi| = 1. \tag{8}$$

Since $zf' = f * \frac{z}{(1-z)^2}$ and $f = f * \frac{z}{1-z}$, we can write $\mathcal{D}^n f(z) = f(z) * h(z) * \frac{z}{1-z}$ and $z(\mathcal{D}^n f(z))' = f(z) * h(z) * \frac{z}{(1-z)^2}$, where $h(z) = z + \sum_{k=2}^{\infty} k_n z^k$ and Substituting $\psi := \frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\zeta}{e^{i\alpha}(1+B\zeta)}$, relation (8) is equivalent to

$$f(z) * h(z) * \Big(\frac{z}{(1-z)^2} - \frac{\psi z}{1-z}\Big) \neq 0 \quad z \in \mathbb{U}. \tag{9}$$

On the other hand, by extention $\frac{z}{(1-z)^2}$ and $\frac{z}{1-z}$, we have

$$\frac{z}{(1-z)^2} - \frac{\psi z}{1-z} = z + \sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})z^k. \tag{10}$$

By substituting (10) in (9), the proof is complete. □

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Theorem 3.3. *A necessary and sufficient condition for the function f defined by (1) to be in the class of $\mathcal{S}_n^\alpha[A,B]$ is that*

$$1-\sum_{k=2}^{\infty}\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}k^n a_k z^{k-1}\neq 0. \tag{11}$$

Proof. Notice that

$$\frac{k-\psi}{1-\psi}=-\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}, \tag{12}$$

where ψ was defined in (7). Using (12), we can write (6) as

$$\frac{1}{z}\Big[z-\sum_{k=2}^{\infty}\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}k^n a_k z^k\Big]\neq 0 \tag{13}$$

Simplifying relation (13), we obtain (11) and the proof is complete. □

Lemma 3.4. *The function $f(z)$ defined by (1) is in the class of $\mathcal{K}_n^\alpha[A,B]$ if and only if for all z in $\mathbb{U}$ and all ξ, $|\xi|=1$,*

$$\frac{1}{z}\Big[f*(z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^{n+1}z^k)\Big]\neq 0, \tag{14}$$

where ψ was defined in (7).

Proof. Set

$$g\prec h \iff g(0)=h(0), \quad \textit{and} \quad g(\mathbb{U})\subset h(\mathbb{U}), \quad z\in\mathbb{U}.$$

Note that

$$zg'(z)=z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^{n+1}z^k. \tag{15}$$

From the identity $zf*g=f*zg'$ and the fact that $f\in\mathcal{K}_n^\alpha[A,B]$ if and only if $zf'\in\mathcal{S}_n^\alpha[A,B]$, from lemma 3.1 we have

$$\frac{1}{z}\Big[zf'(z)*g(z)\Big]\neq 0 \iff \frac{1}{z}\Big[f(z)*zg'(z)\Big]\neq 0. \tag{16}$$

By substituting relation (15) in (16), we have obtain (14) and the proof is complete. □

In a similar way of theorem 3.3 and using lemma 3.4, we can prove the following theorem.

Theorem 3.5. *A necessary and sufficient condition for the function f defined by (1) to be in the class of $\mathcal{K}_n^\alpha[A,B]$ is that*

$$1-\sum_{k=2}^{\infty}\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}k^{n+1} a_k z^{k-1}\neq 0.$$

4. Coefficient Estimates

In the following, as an applications of Theorems 3.3 and 3.5, we determine coefficient estimates and inclusion properties for a function of the form (1) to be in the classes $\mathcal{S}_n^\alpha[A,B]$ and $\mathcal{K}_n^\alpha[A,B]$.

Theorem 4.1. *If the function $f(z)$ defined by (1) belongs to $\mathcal{S}_n^\alpha[A,B]$, then*

$$\sum_{k=2}^{\infty}\Big(|k(B+1)-1|+|\cos\alpha+iB\sin\alpha|\Big)k^n|a_k|\leq (A-B)\cos\alpha.$$

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Proof. Since we have

$$\begin{aligned}&\Big|1-\sum_{k=2}^{\infty}\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}k^n a_k\Big|\\&\geq 1-\sum_{k=2}^{\infty}\Big|\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}\Big|k^n|a_k|,\end{aligned}$$

and

$$\begin{aligned}&\Big|\frac{(k-1)(e^{i\alpha}+iB\zeta\sin\alpha)-(A-kB)\zeta\cos\alpha}{(A-B)\zeta\cos\alpha}\Big|\\&=\frac{|(k-1)(e^{i\alpha}+iB\sin\alpha)-(A-kB)\cos\alpha|}{(A-B)\cos\alpha}\\&\leq\frac{|k(B+1)-1|+|A\cos\alpha+iB\sin\alpha|}{(A-B)\cos\alpha},\end{aligned}$$

the result follows from Theorem 3.3. □

Similarly, we can prove the following theorem.

Theorem 4.2. *If the function $f(z)$ defined by (1) belongs to $\mathcal{K}_n^{\alpha}[A,B]$, then*

$$\sum_{k=2}^{\infty}\Big(|k(B+1)-1|+|\cos\alpha+iB\sin\alpha|\Big)k^{n+1}|a_k|\leq(A-B)\cos\alpha.$$

5. Containment Properties

In this section, we study the containment properties of the mentioned classes.

Theorem 5.1. $\mathcal{S}_{n+1}^{\alpha}[A,B]\subset\mathcal{S}_n^{\alpha}[A,B]$ *for all* $n\in\mathbb{N}$.

Proof. If $f\in\mathcal{S}_{n+1}^{\alpha}[A,B]$, By the lemma 3.2, we have

$$\frac{1}{z}\Big[f*(z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^{n+1}z^k)\Big]\neq 0,\quad z\in\mathbb{U},$$

where ψ is given by (7). Note that we can write

$$z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^{n+1}z^k=\Big(z+\sum_{k=2}^{\infty}kz^k\Big)*\Big(z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^n z^k\Big).\tag{17}$$

But

$$\frac{1}{z}\Big[(z+\sum_{k=2}^{\infty}kz^k)*(z+\sum_{k=2}^{\infty}k^{-1}z^k)\Big]=1+\sum_{k=2}^{\infty}z^{k-1}=\frac{1}{1-z}\neq 0,\quad z\in\mathbb{U}.$$

Thus it follows from (17) that

$$f*\Big(z+\sum_{k=2}^{\infty}(\frac{k-\psi}{1-\psi})k^n z^k\Big)\neq 0,\quad z\in\mathbb{U},$$

and we conclude that $f\in\mathcal{S}_n^{\alpha}[A,B]$. □

Similarly, we can prove the following theorem and corollaries.

Theorem 5.2. $\mathcal{K}_{n+1}^{\alpha}[A,B]\subset\mathcal{K}_n^{\alpha}[A,B]$ *for all* $n\in\mathbb{N}$.

corollary 5.3. $\mathcal{S}_n^{\alpha}[A,B]\subset\mathcal{S}^{\alpha}[A,B]$ *for all* $n\in\mathbb{N}$.

corollary 5.4. $\mathcal{K}_n^{\alpha}[A,B]\subset\mathcal{K}^{\alpha}[A,B]$ *for all* $n\in\mathbb{N}$.

Remark 5.5. In particular, it follow from corollary 5.3 and 5.4 that, $\mathcal{K}^{\alpha}[A,B]\subset\mathcal{S}^{\alpha}[A,B]$.

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Theorem 5.6. *If $f \in \mathcal{S}_n^\alpha[A,B]$ and $\varphi \in \mathcal{K}$, then $f * \varphi \in \mathcal{S}_n^\alpha[A,B]$ for all $n \in \mathbb{N}$.*

Proof. Let .

$$F := \frac{z(\mathcal{D}^n f(z))'}{\mathcal{D}^n f(z)}.$$

If $f \in \mathcal{S}_n^\alpha[A,B]$, then $e^{i\alpha}F \prec h$, where $h(z) = \cos\alpha\left(\frac{1+Az}{1+Bz}\right) + i\sin\alpha$. Now

$$G(z) = \frac{z(\varphi * \mathcal{D}^n f(z))'}{\varphi * \mathcal{D}^n f(z)} = \frac{z(\varphi * (\mathcal{D}^n f(z))')}{\varphi * \mathcal{D}^n f(z)} = \frac{\varphi * (z(\mathcal{D}^n f(z))')}{\varphi * \mathcal{D}^n f(z)} = \frac{\varphi * (F \cdot (\mathcal{D}^n f(z)))}{\varphi * \mathcal{D}^n f(z)}$$

On the other hand, $f \in \mathcal{S}_n^\alpha[A,B]$, $\mathcal{D}^n f(z) \in \mathcal{S}^*$. It follows from Theorem 2.1 [13], that $\frac{\varphi * (F \cdot (\mathcal{D}^n f(z)))}{\varphi * \mathcal{D}^n f(z)}$ lie in convex hall of $F(\mathbb{U})$. But $e^{i\alpha}F \prec h$ and h is convex, so the convex hall of $e^{i\alpha}F(\mathbb{U})$ is subset of $h(\mathbb{U})$, thus $e^{i\alpha}G(\mathbb{U}) \subset h(\mathbb{U})$, also $e^{i\alpha}G(0) = h(0)$, therefore $e^{i\alpha}G(z) \prec h(z)$ and this completes the proof. □

Theorem 5.7. *If $f, g \in \mathcal{S}_n^\alpha[A,B]$. then $f * g \in \mathcal{S}_n^\alpha[A,B]$ for all $n \geq 1$.*

Proof. If $\varphi \in \mathcal{S}_n^\alpha[A,B]$, then theorem 5.1 provides $\mathcal{S}_n^\alpha[A,B] \subset \mathcal{S}_1^\alpha[A,B] = \mathcal{K}_0^\alpha[A,B]$. On the other hand $\mathcal{K}_0^\alpha[A,B] \subset \mathcal{K}^\alpha[1,-1] \subset \mathcal{K}$. Therefore $\varphi \in \mathcal{K}$. Now by theorem 5.6, $f * \varphi \in \mathcal{S}_n^\alpha[A,B]$ whenever $f \in \mathcal{S}_n^\alpha[A,B]$. This completes the proof. □

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