Dynkin game under g-expectation in continuous time

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Abstract

In this paper, we investigate some kind of Dynkin game under g-expectation induced by backward stochastic differential equation (shortly for BSDE). We define the lower and upper value functions $\underline{V}_t = ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} \mathcal{E}_t^g[R(\tau,\sigma)]$ and $\overline{V}_t = ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} \mathcal{E}_t^g[R(\tau,\sigma)]$, respectively. Under some regular assumptions, a pair of saddle point is obtained and the value function of Dynkin game $V(t) = \underline{V}_t = \overline{V}_t$ follows. Furthermore, the constrained case of Dynkin game is also considered.

Keywords: Dynkin game, Ambiguity, Backward stochastic differential equation (BSDE), Reflected backward stochastic differential equation (Reflected BSDE), Constraint
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1 Introduction

Dynkin game was first introduced and studied by Dynkin in [3]. Later, in Neveu [12], EIbakidze [4], Kifer [9] and Ohtsubo [13, 14], the authors considered Dynkin game in discrete parameter case with or without a finite constraint. The version of continuous time was also studied in many literatures (for examples, see Morimoto [11], Stettner [19], Krylov [10] and the references therein).

A general formulation of Dynkin game states as follows. We define the lower and upper value functions

$$\underline{V}_t := ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[R_t(\tau,\sigma)|\mathcal{F}_t], \tag{1}$$

and

$$\overline{V}_t := ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[R_t(\tau,\sigma)|\mathcal{F}_t], \tag{2}$$

respectively, where $R_t(\tau,\sigma)$ is a function of two stopping times τ and σ satisfying some suitable assumptions. Then one often tries to find a sufficient condition such that $\overline{V}_t = \underline{V}_t$ holds. It is easy to see that $\overline{V}_t \geq \underline{V}_t$. In order to get the reverse inequality, one often looks for a pair of saddle point (τ_t^*, σ_t^*), such that

$$E[R_t(\tau,\sigma_t^*)|\mathcal{F}_t] \leq E[R_t(\tau_t^*,\sigma_t^*)|\mathcal{F}_t] \leq E[R_t(\tau_t^*,\sigma)|\mathcal{F}_t] \tag{3}$$

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holds for any σ and τ taken values in $[t, T]$. If (3) is true, denote $V(t) := \overline{V}_t = \underline{V}_t$ and $V(t)$ is called the value function of the Dynkin game.

There are many ways to solve this kind of problem. Since this stopping game problem is an extension of the optimal stopping problem, the martingale approach has been used to find a pair of saddle point, and then the value function is obtained by solving this double optimal stopping problem (for example, see Dynkin [3], Krylov [10] and the references therein). In Friedman [5] and Bensoussan and Friedman [1], the authors developed the analytical theory of stochastic differential games with stopping times in Markov setting. They studied the value function and found the saddle point of Dynkin game by using the theories of partial differential equations, variational inequalities and free-boundary problems. Later, reflected backward stochastic differential equation (shortly for reflected BSDE) with one lower obstacle has been found useful to solve the optimal stopping problem. Since then, many researchers investigated Dynkin game by solving reflected BSDE with lower and upper obstacles (for example, see Cvitanic and Karatzas [2], Hamadène and Lepeltier [6] and the references therein). In addition , there are some other ways to solve this game such as by the pathwise approach (see Karatzas [7]) and by the connection with bounded variation problem (see Karatzas and Wang [8]).

Inspired by Cvitanic and Karatzas [2], in this paper we study Dynkin game in the stochastic environment with ambiguity and evaluate the reward process by g-expectations induced by BSDEs. More explicitly, the problem that we study can be formulated as follows. Define the lower and upper value functions

$$\underline{V}_t := ess \sup_{\tau \in \mathcal{T}_t} ess \inf_{\sigma \in \mathcal{T}_t} \mathcal{E}_t^g[R(\tau, \sigma)] \tag{4}$$

and

$$\overline{V}_t = ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} \mathcal{E}_t^g[R(\tau, \sigma)], \tag{5}$$

respectively, where $R(\tau, \sigma) := L(\tau)1_{\{\tau \leq \sigma\}} + U(\sigma)1_{\{\sigma < \tau\}}$ and $\mathcal{T}_t$ is the set of all stopping times taking values in $[t, T]$. Under some suitable assumptions on two processes $L(t)$ and $U(t)$, we want to find a pair of saddle point (τ_t^*, σ_t^*) such that

$$\mathcal{E}_t^g[R(\tau, \sigma_t^*)] \leq \mathcal{E}_t^g[R(\tau_t^*, \sigma_t^*)] \leq \mathcal{E}_t^g[R(\tau_t^*, \sigma)] \tag{6}$$

holds for any $\tau, \sigma \in \mathcal{T}_t$, and then the game has a value function $V(t) := \overline{V}_t = \underline{V}_t$.

This problem looks very similar to the problem stated and solved in Cvitanic and Karatzas [2] , but there is some differences between them, although the value function is as same as that we obtained. In Section 3, we will point out the main difference between them. Furthermore, the more complicated case with a constraint will be considered and the reward process will be evaluated by g_Γ-expectation. The notion of g_Γ-expectation will be given in Section 2.

This paper is organized as follows. In Section 2, we introduce some notations, assumptions, notions, propositions about BSDE, reflected BSDE and BSDE with a constraint that are used in this paper. The main results and proofs are stated in Section 3.

2 BSDE, reflected BSDE and BSDE with a constraint

In this section, we shall present some notations, assumptions, notions and propositions about BSDE, reflected BSDE and BSDE with a constraint that are used in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by Brownian motion and all P-null subsets, i.e.,

$$\mathcal{F}_t = \sigma\{W_s; s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$.

We define the following usual spaces of processes (random variables):
$L^2(\mathcal{F}_T) = \{\xi : \xi$ is $\mathcal{F}_T$-measurable random variable such that $E[|\xi|^2] < \infty\}$;
$H_T^2(R^d) = \{V : V_t$ is R^d-valued and $\mathcal{F}_t$-adapted process such that $E[\int_0^T |V_s|^2 \mathrm{d}s] < \infty\}$;
$S_T^2(R) = \{V : V_t$ is R-valued and RCLL, $\mathcal{F}_t$-adapted process such that $E[\sup_{0\leq t\leq T} |V_t|^2] < \infty\}$;
$S_{c,i;T}^2(R) = \{A : A_t$ is R-valued and continuous, increasing, $\mathcal{F}_t$-adapted process with $A_0 = 0$ such that $E[A_T^2] < \infty\}$;
$S_{i;T}^2(R) = \{A : A_t$ is R-valued and increasing, RCLL, $\mathcal{F}_t$-adapted process with $A_0 = 0$ such that $E[A_T^2] < \infty\}$.

Now we consider the following 1-dimensional BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}W_s, \quad t \in [0, T]. \tag{7}$$

We make the following assumptions:
(A.0) Let $g : \Omega \times [0, T] \times R \times R^d \mapsto R$ such that for any $(y, z) \in R \times R^d$, $g(t, y, z)$ is $\mathcal{F}_t$-progressively measurable;
(A.1) $g(\cdot, 0, 0) \in H_T^2(R)$;
(A.2) There exists a positive constant μ such that for all $y_1, y_2 \in R$, $z_1, z_2 \in R^d$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \mu\left(|y_1 - y_2| + |z_1 - z_2|\right);$$

(A.3) $g(\cdot, y, 0) \equiv 0, \quad \forall y \in R$.

From Pardoux and Peng [15], we know that: Suppose that g satisfies (A.0)-(A2). Then for any $\xi \in L^2(\mathcal{F}_T)$, BSDE (7) has a unique pair of adapted processes $(Y, Z) \in S_T^2(R) \times H_T^2(R^d)$.

Now we present some notions and propositions of g-expectation induced by BSDE. For more details, we can see Peng [16, 17].

Definition 2.1. *Suppose that g satisfies (A.0)-(A.2). For any $\xi \in L^2(\mathcal{F}_T)$, let (Y, Z) be the solution of BSDE (7) with terminal value ξ. Consider the mapping $\mathcal{E}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)$, denoted by $\mathcal{E}_{t,T}^g[\xi] = Y_t$. We call $\mathcal{E}_{t,T}^g[\xi]$ the g-expectation of ξ. Furthermore, if g also satisfies (A.3). We write $\mathcal{E}_t^g[\xi] := \mathcal{E}_{t,T}^g[\xi]$.*

Proposition 2.1. *Suppose that g satisfies (A.0)-(A.2), and σ, τ are two stopping times satisfying $\tau \leq \sigma \leq T$. Let $\zeta \in L^2(\mathcal{F}_\sigma)$ and (Y, Z) be the solution of the following BSDE*

$$Y_\tau = \zeta + \int_\tau^\sigma g(s, Y_s, Z_s)\mathrm{d}s - \int_\tau^\sigma Z_s \cdot \mathrm{d}W_s.$$

Then

(i) *If ξ, $\eta \in L^2(\mathcal{F}_\sigma)$ and $\xi \geq \eta$ a.s., we have $\mathcal{E}_{\tau,\sigma}^g[\xi] \geq \mathcal{E}_{\tau,\sigma}^g[\eta]$ a.s.*

(ii) *If g satisfies (A.3), we have $\mathcal{E}_{\tau,\sigma}^g[\zeta] = \mathcal{E}_{\tau,T}^g[\zeta]$ a.s. So we write $\mathcal{E}_\tau^g[\zeta] := \mathcal{E}_{\tau,\sigma}^g[\zeta]$.*

(iii) *If g satisfies (A.3), we have $\mathcal{E}_\tau^g[\mathcal{E}_\sigma^g[\zeta]] = \mathcal{E}_\tau^g[\zeta]$ a.s.*

(iv) *Let $\xi_n, \xi, \eta \in L^2(\mathcal{F}_\sigma)$, $n = 1, 2, \cdots$. If $\lim_{n\to\infty} \xi_n = \xi$ a.s. and $|\xi_n| \leq \eta$ a.s., we have $\lim_{n\to\infty} \mathcal{E}_{\tau,\sigma}^g[\xi_n] = \mathcal{E}_{\tau,\sigma}^g[\xi]$ a.s.*

Definition 2.2. *(g-martingale, g-supermartingale, g-submartingale) Suppose that g satisfies (A.0)-(A.2). An $\mathcal{F}_t$-progressively measurable R-valued process X is called an g-martingale (resp. g-supermartingale, g-submartingale), if for each* $0 \le t \le T$, $E[|X_t|^2] < \infty$, *and* $\mathcal{E}^g_{s,t}[X_t] = X_s$ *a.s., (resp.* $\mathcal{E}^g_{s,t}[X_t] \le X_s$ *a.s.,* $\mathcal{E}^g_{s,t}[X_t] \ge X_s$ *a.s.) for all* $s \in [0,t]$.

The theory of BSDE has wildly used in many fields such as financial mathematics and stochastic control. Let us mention that reflected BSDE has been found useful to solve the optimal stopping problem and investigate Dynkin game. In the following, the notion of reflected BSDE is given.

Definition 2.3. *(Reflected BSDE, see [18]). Suppose that* $\xi \in L^2(\mathcal{F}_T)$, *and g satisfies the assumptions (A.0)-(A.2). Consider two processes* L, U *in* $S^2_T(R)$ *such that satisfy*

$$L(t) \le U(t) \quad a.s., \qquad \forall t \in [0,T) \quad and \quad L(T) \le \xi \le U(T) \quad a.s.$$

We say that a quadruple $(Y,Z,A,K) \in S^2_T(R) \times H^2_T(R^d) \times S^2_T(R) \times S^2_T(R)$ *is a solution of reflected BSDE (BSDE reflected between a lower obstacle L and an upper obstacle U) with parameters* (ξ, g), *if*
(i) *A, K are increasing:* $\mathrm{d}A \ge 0$, $\mathrm{d}K \ge 0$.
(i) (Y,Z) *solves the following BSDE on* $[0,T]$*:*

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s \cdot \mathrm{d}W_s; \tag{8}$$

(ii) *Y is dominated by L and U, i.e.,*

$$L(t) \le Y_t \le U(t) \quad a.s., \qquad \forall t \in [0,T]; \tag{9}$$

(iii) *The following Skorohod condition holds:*

$$\int_0^T (Y_{t-} - L(t-))\mathrm{d}A_t = \int_0^T (U(t-) - Y_{t-})\mathrm{d}K_t = 0 \quad a.s. \tag{10}$$

The processes L, U play the role of reflecting barriers; these are allowed to be random and time-varying, and the state-process Y is not allowed to cross them on its way to the prescribed terminal target condition ξ. Cvitanic and Karatzas [2] investigated Dynkin game by solving this reflected BSDE with double obstacles. Peng and Xu [18] proved that to obtain the solution of this reflected BSDE with double obstacles is equivalent to solving a problem of the smallest g-supermartingale and the largest g-submartingale.

In this paper, the constrained case of Dynkin game is also considered. So we introduce BSDE with a constraint. First, we present the notion of g-supersolution.

Definition 2.4. *(g-supersolution, see [17]) Suppose that* $\xi \in L^2(\mathcal{F}_T)$, *and g satisfies the assumptions (A.0)-(A.2). If a triple* (Y,Z,A) *belongs to* $S^2_T(R) \times H^2_T(R^d) \times S^2_{i;T}(R)$ *and* (Y,Z) *solves the following BSDE on* $[0,T]$*:*

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)\mathrm{d}s + A_T - A_t - \int_t^T Z_s \cdot \mathrm{d}W_s, \tag{11}$$

then we call Y a g-supersolution on $[0,T]$

Let $\phi : \Omega \times [0,T] \times R \times R^d \mapsto R^+$ be a given nonnegative, $\mathcal{F}_t$-progressively measurable function such that $\phi(t,0,0) \in H^2_T(R)$ and ϕ is Lipschitz with respect to (y,z), i.e., there exists a positive constant μ such that for all $y_1, y_2 \in R$, $z_1, z_2 \in R^d$,

$$|\phi(t, y_1, z_1) - \phi(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|).$$

Now we consider BSDE of the form (11) with a constraint

$$K_t(\omega) := \{(y, z) \in R \times R^d; \phi(\omega, t, y, z) = 0\}$$

imposed to the solution. That is, we want to find the solution of BSDE of the form (11) such that

$$\phi(t, Y_t, Z_t) = 0 \quad \text{a.s.} \tag{12}$$

In order to make the problem of BSDE with a constraint be meaningful, we need the following assumption: (H) There exists at least a g-supersolution $\hat{Y}$ with terminal value ξ and the decomposition $(\hat{Z}, \hat{A})$, such that the constraint (12) is satisfied.

Definition 2.5. *(The smallest g-supersolution subject to a constraint, g_Γ-expectation, see [17]) A g-supersolution Y with the decomposition (Z, A) is said to be the smallest g-supersolution, given terminal value ξ, subject to the constraint (12), if (12) is satisfied and $Y_t \leq \hat{Y}_t$ a.s., for any g-supersolution $\hat{Y}$ with terminal value ξ and the decomposition $(\hat{Z}, \hat{A})$ subject to $\phi(t, \hat{Y}_t, \hat{Z}_t) = 0$ a.s. When $g(\cdot, y, 0) = 0$ and $\phi(\cdot, y, 0) = 0$, $\forall y \in R$, the smallest g-supersolution Y_t subject to the constraint (12) is denoted by $\mathcal{E}_t^{g,\phi}[\xi]$. For convenience, we call $\mathcal{E}_t^{g,\phi}[\xi]$ the g_Γ-expectation of ξ.*

Remark 2.1. *In order to construct the smallest g-supersolution of BSDE (11) subject to the constraint (12), Peng [17] introduced the following BSDEs on $[0, T]$:*

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^T Z_s^n \cdot \mathrm{d}W_s, n = 1, 2, \cdots, \tag{13}$$

where $g_n := g + n\phi$. In [17], the author proved that Y^n increasingly converges to Y and Y is the smallest g-supersolution of BSDE (11) subject to the constraint (12).

Remark 2.2. *Suppose that $g(\cdot, y, 0) \equiv 0$ and $\phi(\cdot, y, 0) \equiv 0$, $\forall y \in R$. Then the smallest g-supersolution subject to the constraint (12) is well defined for terminal value $\xi \in L^\infty(\mathcal{F}_T)$, the space of all essentially bounded $\mathcal{F}_T$-measurable variables. In fact, for any $\xi \in L^\infty(\mathcal{F}_T)$ i.e., there exists a positive constant D such that $|\xi| \leq D$ a.s., then Y is g-supersolution of BSDE (11) with terminal value ξ and the decomposition (Z, A), such that the constraint (12) is satisfied, where*

$$A_t = \begin{cases} 0, & t \in [0, T) \\ D - \xi, & t = T \end{cases},$$

$$Y_t = \begin{cases} D, & t \in [0, T) \\ \xi, & t = T \end{cases}$$

and $Z_t = 0$. Thus, the smallest g-supersolution subject to the constraint (12) exists.

At last, we give a comparison theorem of the smallest g-supersolution subject to a constraint.

Proposition 2.2. *Suppose g and ϕ satisfy the assumptions (A.0), (A.2) and (A.3). If $\xi, \eta \in L^2(\mathcal{F}_T)$ and $\xi \leq \eta$ a.s., then*

$$\mathcal{E}_t^{g,\phi}[\xi] \leq \mathcal{E}_t^{g,\phi}[\eta] \quad a.s., \quad \forall t \in [0, T]. \tag{14}$$

proof. Let $\left(Y^{1,n}, Z^{1,n}\right)$ and $\left(Y^{2,n}, Z^{2,n}\right)$ be the solutions of the following BSDEs on $[0,T]$:

$$Y_t^{1,n} = \xi + \int_t^T \left[g\left(s, Y_s^{1,n}, Z_s^{1,n}\right) + n\phi\left(s, Y_s^{1,n}, Z_s^{1,n}\right)\right] \mathrm{d}s - \int_t^T Z_s^{1,n} \cdot \mathrm{d}W_s, n = 1, 2, \cdots$$

and

$$Y_t^{2,n} = \eta + \int_t^T \left[g\left(s, Y_s^{2,n}, Z_s^{2,n}\right) + n\phi\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s - \int_t^T Z_s^{2,n} \cdot \mathrm{d}W_s, n = 1, 2, \cdots,$$

respectively. By the comparison theorem of BSDE (see Peng [15]), we have for each n,

$$Y_t^{1,n} \leq Y_t^{2,n} \quad \text{a.s.,} \quad \forall t \in [0,T].$$

Noting the fact that $\lim_{n\to\infty} Y_t^{1,n} = \mathcal{E}_t^{g,\phi}[\xi]$ a.s. and $\lim_{n\to\infty} Y_t^{2,n} = \mathcal{E}_t^{g,\phi}[\eta]$ a.s., $\forall t \in [0,T]$, we have

$$\mathcal{E}_t^{g,\phi}[\xi] \leq \mathcal{E}_t^{g,\phi}[\eta] \quad \text{a.s.,} \quad \forall t \in [0,T].$$

□

3 Dynkin game under ambiguity

In this section we first study Dynkin game without constraints. Second, the more complicated case with a constraint will be considered.

In order to make Dynkin game that we study be meaningful, we need the following assumption: (B.1) $L(t), U(t)$ are two continuous processes belonging to $S_T^2(R)$ and satisfy that

$$L(t) < U(t) \quad \text{a.s.,} \qquad \forall t \in [0,T) \quad \text{and} \quad L(T) \leq \xi \leq U(T) \quad \text{a.s.;}$$

(B.2) Mokobdski's condition: there exist two non-negative supermartingales h and $h^{'}$ of $S_T^2(R)$ such that

$$L(t) \leq h_t - h_t^{'} \leq U(t), \quad \forall t \in [0,T].$$

From Cvitanic and Karatzas [2], we know that: Suppose the assumptions (A.0)-(A.2) and (B.1), (B.2) hold. For each $\xi \in L^2(\mathcal{F}_T)$, then there exists a unique solution $(Y, Z, A, K) \in S_T^2(R) \times H_T^2(R^d) \times S_{c,i;T}^2(R) \times S_{c,i;T}^2(R)$ of reflected BSDE (8) formulated in Definition 2.3. In particular, let (Y^*, Z^*, A^*, K^*) be the unique solution of reflected BSDE (8) with terminal value $L(T)$. $\mathcal{T}_t$ is the set of all stopping times taking values in $[t,T]$.

3.1 Dynkin game without constraints

In Cvitanic and Karatzas [2], the method for studying Dynkin game states as follows. We define the lower and upper value functions

$$\underline{V}(t) := ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[\hat{R}_t(\tau,\sigma)|\mathcal{F}_t]$$

and

$$\overline{V}(t) := ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[\hat{R}_t(\tau,\sigma)|\mathcal{F}_t],$$

respectively, where

$$R_t(\tau,\sigma) = \int_t^{\tau\wedge\sigma} g(s, Y_s^*, Z_s^*)\mathrm{d}s + L(\tau)1_{\{\tau\leq\sigma\}} + U(\sigma)1_{\{\sigma<\tau\}}. \tag{15}$$

They found a pair of saddle point (τ_t^*, σ_t^*) such that

$$E[R_t(\tau, \sigma_t^*)|\mathcal{F}_t] \le E[R_t(\tau_t^*, \sigma_t^*)|\mathcal{F}_t] \le E[R_t(\tau_t^*, \sigma)|\mathcal{F}_t]$$

holds, and then the game has a value function $V(t) := \overline{V}(t) = \underline{V}(t)$. Furthermore,

$$Y_t^* = V(t) \quad \text{a.s.}, \quad \forall t \in [0, T]. \tag{16}$$

In our framework, we also need the assumption (A.3) holds. The method for studying Dynkin game in our framework can be formulated as follows. Define the lower and upper value functions

$$\underline{V}_t := ess \sup_{\tau \in \mathcal{T}_t} ess \inf_{\sigma \in \mathcal{T}_t} \mathcal{E}_t^g[R(\tau, \sigma)] \tag{17}$$

and

$$\overline{V}_t = ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} \mathcal{E}_t^g[R(\tau, \sigma)], \tag{18}$$

respectively, where $R(\tau, \sigma) := L(\tau)1_{\{\tau \le \sigma\}} + U(\sigma)1_{\{\sigma < \tau\}}$. By Definition 2.1, we know that

$$\mathcal{E}_t^g[R(\tau, \sigma)] = E\left[\int_t^{\tau \wedge \sigma} g(s, Y_s^{\tau,\sigma}, Z_s^{\tau,\sigma})\mathrm{d}s + L(\tau)1_{\{\tau \le \sigma\}} + U(\sigma)1_{\{\sigma < \tau\}}|\mathcal{F}_t\right], \tag{19}$$

where $(Y^{\tau,\sigma}, Z^{\tau,\sigma})$ is the solution of BSDE (7) with terminal value $\eta := L(\tau)1_{\{\tau \le \sigma\}} + U(\sigma)1_{\{\sigma < \tau\}}$. We find a pair of saddle point (τ_t^*, σ_t^*) such that

$$\mathcal{E}_t^g[R(\tau, \sigma_t^*)] \le \mathcal{E}_t^g[R(\tau_t^*, \sigma_t^*)] \le \mathcal{E}_t^g[R(\tau_t^*, \sigma)] \tag{20}$$

holds, and then the game has a value function $V(t) := \overline{V}_t = \underline{V}_t$. Furthermore, we prove that Equation (16) also holds

From the expressions of (15) and (19), we can easily find the differences between our method and the method of Cvitanic and Karatzas [2]. In (15), (Y, Z) is the solution of reflected BSDE (8) formulated in Definition 2.3, with terminal value $L(T)$, and do not depend on the stopping times σ and τ. But in (19) $(Y^{\tau,\sigma}, Z^{\tau,\sigma})$ is the solution of BSDE (7) with terminal value $L(\tau)1_{\{\tau \le \sigma\}} + U(\sigma)1_{\{\sigma < \tau\}}$, and depend on the stopping times σ and τ.

With the help of the theory of g-expectation, we can find a pair of saddle point of Dynkin game. The main reason is that g-expectation enjoys almost all properties of classical expectation except for linearity. But as we can see from the proof of Theorem 3.1, linearity is not crucial to study Dynkin game without constraints. The following theorem (Theorem 3.1) is the main result of Dynkin game without constraints.

Theorem 3.1. *Suppose the assumptions (A.0), (A.2), (A.3) and (B.1), (B.2) hold. Let (Y^*, Z^*, A^*, K^*) be the unique solution of reflected BSDE (8) with terminal value $L(T)$. Then Dynkin game stated in (17) and (18) has a pair of saddle point (τ_t^*, σ_t^*), where*

$$\tau_t^* = \inf\{s \ge t : L(s) = Y_s^*\} \wedge T \tag{21}$$

and

$$\sigma_t^* = \inf\{s \ge t : U(s) = Y_s^*\} \wedge T, \tag{22}$$

such that

$$\mathcal{E}_t^g[R(\tau, \sigma_t^*)] \le \mathcal{E}_t^g\left[Y_{\tau \wedge \sigma_t^*}^*\right] \le \mathcal{E}_t^g\left[Y_{\tau_t^* \wedge \sigma_t^*}^*\right] \le \mathcal{E}_t^g\left[Y_{\tau_t^* \wedge \sigma}^*\right] \le \mathcal{E}_t^g\left[R(\tau_t^*, \sigma)\right] \tag{23}$$

holds for any $\tau, \sigma \in \mathcal{T}_t$, and hence

$$\underline{V}_t = \overline{V}_t = Y_t^* \quad a.s., \quad \forall t \in [0, T].$$

proof. For any $\tau \in \mathcal{T}_t$, we have

$$\begin{aligned}
\mathcal{E}_t^g[R(\tau,\sigma_t^*)] &= \mathcal{E}_t^g\left[L(\tau)1_{\{\tau\le\sigma_t^*\}} + U(\sigma_t^*)1_{\{\sigma_t^*<\tau\}}\right] \\
&\le \mathcal{E}_t^g\left[Y_\tau^* 1_{\{\tau\le\sigma_t^*\}} + U(\sigma_t^*)1_{\{\sigma_t^*<\tau\}}\right] \\
&= \mathcal{E}_t^g\left[Y_\tau^* 1_{\{\tau\le\sigma_t^*\}} + Y_{\sigma_t^*}^* 1_{\{\sigma_t^*<\tau\}}\right] \\
&= \mathcal{E}_t^g\left[Y_{\tau\wedge\sigma_t^*}^*\right].
\end{aligned}$$

Now we prove that

$$\mathcal{E}_t^g\left[Y_{\tau\wedge\sigma_t^*}^*\right] \le \mathcal{E}_t^g\left[Y_{\tau_t^*\wedge\sigma_t^*}^*\right] \tag{24}$$

holds for any $\tau \in \mathcal{T}_t$.
Case 1: On the set $\{\omega : \sigma_t^* < \tau \le \tau_t^*\}$, (24) obviously holds.
Case 2: On the set $\{\omega : \tau \le \tau_t^* < \sigma_t^*\}$, by (10) and (21), (22), we have

$$A_{\tau_t^*} - A_\tau = 0$$

and

$$K_{\tau_t^*} - K_\tau = 0.$$

So BSDE (8) can be rewritten as follows:

$$Y_\tau^* = Y_{\tau_t^*}^* + \int_\tau^{\tau_t^*} g(s, Y_s^*, Z_s^*)\mathrm{d}s - \int_\tau^{\tau_t^*} Y_s^* \cdot \mathrm{d}W(s).$$

This means

$$\mathcal{E}_t^g\left[Y_\tau^*\right] = \mathcal{E}_t^g\left[Y_{\tau_t^*}^*\right], \tag{25}$$

by Proposition 2.1.
Case 3: On the set $\{\omega : \tau \le \sigma_t^* \le \tau_t^*\}$, by (10) and (21), (22), we have

$$A_{\sigma_t^*} - A_\tau = 0$$

and

$$K_{\sigma_t^*} - K_\tau = 0.$$

So BSDE (8) can be rewritten as follows:

$$Y_\tau^* = Y_{\sigma_t^*}^* + \int_\tau^{\sigma_t^*} g(s, Y_s^*, Z_s^*)\mathrm{d}s - \int_\tau^{\sigma_t^*} Y_s^* \cdot \mathrm{d}W(s).$$

This means

$$\mathcal{E}_t^g\left[Y_\tau^*\right] = \mathcal{E}_t^g\left[Y_{\sigma_t^*}^*\right], \tag{26}$$

by Proposition 2.1. By (25) and (26), we have that on the set $\{\omega : \tau \le \tau_t^*\}$,

$$\mathcal{E}_t^g[Y_{\tau\wedge\sigma_t^*}^*] = \mathcal{E}_t^g[Y_{\tau_t^*\wedge\sigma_t^*}^*]. \tag{27}$$

Case 4: On the set $\{\omega : \sigma_t^* \le \tau_t^* < \tau\}$, (24) obviously holds.
Case 5: On the set $\{\omega : \tau_t^* < \tau \le \sigma_t^*\}$, by (10) and (21), (22), we have

$$K_\tau - K_{\tau_t^*} = 0.$$

So BSDE (8) can be rewritten as follows:

$$Y^*_{\tau^*_t} = Y^*_\tau + \int_{\tau^*_t}^{\tau} g(s, Y^*_s, Z^*_s)\mathrm{d}s + A^*_\tau - A^*_{\tau^*_t} - \int_{\tau^*_t}^{\tau} Y^*_s \cdot \mathrm{d}W(s).$$

This means

$$\mathcal{E}^g_t\left[Y^*_\tau\right] \le \mathcal{E}^g_t\left[Y^*_{\tau^*_t}\right], \tag{28}$$

by Proposition 2.1.

Case 6: On the set $\{\omega : \tau^*_t < \sigma^*_t < \tau\}$, by (10) and (21), (22), we have

$$K_{\sigma^*_t} - K_{\tau^*_t} = 0.$$

So BSDE (8) can be rewritten as follows:

$$Y^*_{\tau^*_t} = Y^*_{\sigma^*_t} + \int_{\tau^*_t}^{\sigma^*_t} g(s, Y^*_s, Z^*_s)\mathrm{d}s + A^*_{\sigma^*_t} - A^*_{\tau^*_t} - \int_{\tau^*_t}^{\sigma^*_t} Y^*_s \cdot \mathrm{d}W(s).$$

This means

$$\mathcal{E}^g_t\left[Y^*_{\sigma^*_t}\right] \le \mathcal{E}^g_t\left[Y^*_{\tau^*_t}\right], \tag{29}$$

by Proposition 2.1. By (28) and (29), we have that on the set $\{\omega : \tau^*_t < \tau\}$,

$$\mathcal{E}^g_t[Y^*_{\tau\wedge\sigma^*_t}] \le \mathcal{E}^g_t[Y^*_{\tau^*_t\wedge\sigma^*_t}]. \tag{30}$$

From (27) and (30), we know (24) holds.

Next we prove that

$$\mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma^*_t}\right] \le \mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma}\right] \tag{31}$$

holds for any $\sigma \in \mathcal{T}_t$. In a similar manner as above, we have that on the set $\{\omega : \sigma \le \sigma^*_t\}$,

$$\mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma^*_t}\right] = \mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma}\right] \tag{32}$$

and on the set $\{\omega : \sigma > \sigma^*_t\}$,

$$\mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma^*_t}\right] \le \mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma}\right]. \tag{33}$$

From (32) and (33), we know (31) holds.

At last, we prove that

$$\mathcal{E}^g_t\left[Y^*_{\tau^*_t\wedge\sigma}\right] \le \mathcal{E}^g_t\left[R(\tau^*_t, \sigma)\right] \tag{34}$$

holds for any $\sigma \in \mathcal{T}_t$. In fact,

$$R(\tau^*_t, \sigma) = L(\tau^*_t)1_{\{\tau^*_t \le \sigma\}} + U(\sigma)1_{\{\sigma < \tau^*_t\}} \ge Y^*_{\tau^*_t}1_{\{\tau^*_t \le \sigma\}} + Y^*_\sigma 1_{\{\sigma < \tau^*_t\}} = Y^*_{\tau^*_t\wedge\sigma}.$$

So we complete the proof of Theorem 3.1. □

Remark 3.1. *Comparing with the method used to prove Theorem 4.1 in Cvitanic and Karatzas [2], we find that our method is simpler. The main reason is that we use the theory of g-expectation to handle this problem. In Subsection 3.2, it can be found that the theory of g-expectation is very convenient for us to handle the constrained case.*

3.2 Dynkin game with a constraint

Dynkin game with a constraint that we study can be formulated as follows. Define the lower and upper value functions

$$\underline{V}_t := ess \sup_{\tau \in \mathcal{T}_t} ess \inf_{\sigma \in \mathcal{T}_t} \mathcal{E}_t^{g,\phi}[R(\tau,\sigma)] \tag{35}$$

and

$$\overline{V}_t = ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} \mathcal{E}_t^{g,\phi}[R(\tau,\sigma)], \tag{36}$$

respectively, where $R(\tau,\sigma) = L(\tau)1_{\{\tau\le\sigma\}} + U(\sigma)1_{\{\sigma<\tau\}}$. We want to find a pair of saddle point (τ_t^*, σ_t^*) such that

$$\mathcal{E}_t^{g,\phi}[R(\tau,\sigma_t^*)] \le \mathcal{E}_t^{g,\phi}[R(\tau_t^*,\sigma_t^*)] \le \mathcal{E}_t^{g,\phi}[R(\tau_t^*,\sigma)] \tag{37}$$

holds for any $\tau, \sigma \in \mathcal{T}_t$, and then the game has a value function $V(t) := \overline{V}_t = \underline{V}_t$.

Remark 3.2. *Suppose the assumptions (A.0), (A.2), (A.3) and (B.1), (B.2) hold. For each $m \in N$, we consider the following BSDE on $[0,T]$:*

$$Y_t = L(T) + \int_t^T g_m(s, Y_s^m, Z_s^m)\mathrm{d}s + A_T^m - A_t^m - (K_T^m - K_t^m) - \int_t^T Z_s^m \cdot \mathrm{d}W_s, \tag{38}$$

where $g_m =: g + m\phi$. Let $(Y^m, Z^m, A^m, K^m) \in S_T^2(R) \times H_T^2(R^d) \times S_{c,i;T}^2(R) \times S_{c,i;T}^2(R)$ be the unique solution of reflected BSDE (38) satisfying

$$L(t) \le Y_t^m \le U(t) \quad a.s., \qquad \forall t \in [0,T] \tag{39}$$

and

$$\int_0^T (Y_t^m - L(t))\mathrm{d}A_t^m = \int_0^T (U(t) - Y_t^m)\mathrm{d}K_t^m = 0 \quad a.s. \tag{40}$$

From Peng and Xu [18], we know that (Y^m, Z^m, A^m, K^m) can be obtain by the penalization method. For each $m, n \in N$, let $(Y^{n,m}, Z^{n,m})$ be the solution of the following BSDE on $[0,T]$:

$$\begin{aligned} Y_t^{n,m} &= L(T) + \int_t^T g_m(s, Y_s^{n,m}, Z_s^{n,m})\mathrm{d}s + n\int_t^T (Y_s^{n,m} - L(s))^-\mathrm{d}s \\ &- n\int_t^T (Y_s^{n,m} - U(s))^+\mathrm{d}s - \int_t^T Z_s^{n,m} \cdot \mathrm{d}W(s). \end{aligned}$$

Fixing n, by the comparison theorem of BSDE, we have $Y^{n,m}$ is increasing with respect to m. By Theorem 3.1 in Peng and Xu [18], we know that

$$\left(Y_t^{n,m}, Z_t^{n,m}, n\int_t^T (Y_s^{n,m} - L(s))^-\mathrm{d}s, n\int_t^T (Y_s^{n,m} - U(s))^+\mathrm{d}s\right) \to (Y_t^m, Z_t^m, A_t^m, K_t^m),$$

as $n \to \infty$, Y^m is increasing with respect to m and Y, the limit of Y^m is RCLL.

The following theorem (Theorem 3.2) is the main result of Dynkin game with a constraint.

Theorem 3.2. *Suppose the assumptions (A.0), (A.2), (A.3) and (B.1), (B.2) hold. We also assume that $L(t), U(t)$ satisfy the following conditions:*
(C) $L(t)$ is increasing and there exists some constant $B > 0$ such that

$$L(t) \le U(t) \le B \quad a.s., \qquad \forall t \in [0,T]$$

and
(D) $\phi(\cdot, y, 0) \equiv 0, \quad \forall y \in R$. *Then Dynkin game stated in (35) and (36) has a pair of saddle point* (τ_t^*, σ_t^*)*, where*

$$\tau_t^* = \inf\{s \geq t : L(s) = Y_s\} \wedge T \tag{41}$$

and

$$\sigma_t^* = \inf\{s \geq t : U(s) = Y_s\} \wedge T, \tag{42}$$

such that

$$\mathcal{E}_t^{g,\phi}[R(\tau, \sigma_t^*)] \leq \mathcal{E}_t^{g,\phi}\left[Y_{\tau \wedge \sigma_t^*}\right] \leq \mathcal{E}_t^{g,\phi}\left[Y_{\tau_t^* \wedge \sigma_t^*}\right] \leq \mathcal{E}_t^{g,\phi}\left[Y_{\tau_t^* \wedge \sigma}\right] \leq \mathcal{E}_t^{g,\phi}\left[R(\tau_t^*, \sigma)\right] \tag{43}$$

holds for any $\tau, \sigma \in \mathcal{T}_t$*, and hence*

$$\underline{V}_t = \overline{V}_t = Y_t \quad a.s., \quad \forall t \in [0, T].$$

proof. Define

$$\tau_t^*(m) := \inf\{s \geq t : L(s) = Y_s^m\} \wedge T$$

and

$$\sigma_t^*(m) := \inf\{s \geq t : U(s) = Y_s^m\} \wedge T.$$

It is easy to check that $\{\tau_t^*(m)\}$ is increasing and $\tau_t^* \geq \tau_t^*(m)$ for any m, and $\{\sigma_t^*(m)\}$ is decreasing and $\sigma_t^* \leq \sigma_t^*(m)$ for any m.

From the expressions of τ_t^* and σ_t^*, it is easy to obtain that

$$R(\tau, \sigma_t^*) = L(\tau)1_{\{\tau \leq \sigma_t^*\}} + U(\sigma_t^*)1_{\{\sigma_t^* < \tau\}} \leq Y_\tau 1_{\{\tau \leq \sigma_t^*\}} + Y_{\sigma_t^*}1_{\{\sigma_t^* < \tau\}} = Y_{\tau \wedge \sigma_t^*}$$

and

$$R(\tau_t^*, \sigma) = L(\tau_t^*)1_{\{\tau_t^* \leq \sigma\}} + U(\sigma)1_{\{\sigma < \tau_t^*\}} \geq Y_{\tau_t^*}1_{\{\tau_t^* \leq \sigma\}} + Y_\sigma 1_{\{\sigma < \tau_t^*\}} = Y_{\tau_t^* \wedge \sigma}$$

for any $\tau, \sigma \in \mathcal{T}_t$. Thus, by Proposition 2.2, we have

$$\mathcal{E}_t^{g,\phi}[R(\tau, \sigma_t^*)] \leq \mathcal{E}_t^{g,\phi}\left[Y_{\tau \wedge \sigma_t^*}\right] \tag{44}$$

and

$$\mathcal{E}_t^{g,\phi}\left[Y_{\tau_t^* \wedge \sigma}\right] \leq \mathcal{E}_t^{g,\phi}\left[R(\tau_t^*, \sigma)\right]. \tag{45}$$

Now we prove that

$$\mathcal{E}_t^{g,\phi}[Y_{\tau \wedge \sigma_t^*}] \leq Y_t \tag{46}$$

hold for any $\tau \in \mathcal{T}_t$. For any $n \leq m$, by the comparison theorem of BSDE and Theorem 3.1, we have

$$\mathcal{E}_t^{g_n}\left[Y_{\tau \wedge \sigma_t^*}^m\right] \leq \mathcal{E}_t^{g_m}\left[Y_{\tau \wedge \sigma_t^*}^m\right] = \mathcal{E}_t^{g_m}\left[Y_{\tau \wedge \sigma_t^*(m)}^m\right] \leq \mathcal{E}_t^{g_m}\left[Y_{\tau_t^*(m) \wedge \sigma_t^*(m)}^m\right] = Y_t^m. \tag{47}$$

By Proposition 2.1, fixing n and taking limit in (47) as $m \to \infty$, we have

$$\mathcal{E}_t^{g_n}\left[Y_{\tau \wedge \sigma_t^*}\right] \leq Y_t. \tag{48}$$

By Remark 2.1 and taking limit in (48) as $n \to \infty$, (46) holds. In particular,

$$\mathcal{E}_t^{g,\phi}\left[Y_{\tau_t^* \wedge \sigma_t^*}\right] \leq Y_t. \tag{49}$$

Next we prove that

$$Y_t \leq \mathcal{E}_t^{g,\phi}[Y_{\tau_t^* \wedge \sigma}] \tag{50}$$

hold for any $\sigma \in \mathcal{T}_t$. By Theorem 3.1 and Remark 2.1, we have

$$Y_t^m = \mathcal{E}_t^{g_m}\left[Y^m_{\tau_t^*(m)\wedge\sigma_t^*(m)}\right] \leq \mathcal{E}_t^{g_m}\left[Y^m_{\tau_t^*(m)\wedge\sigma}\right] \leq \mathcal{E}_t^{g,\phi}\left[Y^m_{\tau_t^*(m)\wedge\sigma}\right]. \tag{51}$$

Case 1: On the set $\{\omega : \sigma < \tau_t^*(m) \leq \tau_t^*\}$, by Remark 3.2, we have

$$Y^m_{\tau_t^*(m)\wedge\sigma} = Y^m_\sigma = Y^m_{\tau_t^*\wedge\sigma} \leq Y_{\tau_t^*\wedge\sigma}. \tag{52}$$

Case 2: On the set $\{\omega : \tau_t^*(m) \leq \sigma \leq \tau_t^*\}$, by the increasing property of $L(t)$, we have

$$Y^m_{\tau_t^*(m)\wedge\sigma} = Y^m_{\tau_t^*(m)} = L(\tau_t^*(m)) \leq L(\sigma) \leq Y_\sigma = Y_{\tau_t^*\wedge\sigma}. \tag{53}$$

Case 3: On the set $\{\omega : \tau_t^*(m) \leq \tau_t^* < \sigma\}$, by the increasing property of $L(t)$, we have

$$Y^m_{\tau_t^*(m)\wedge\sigma} = Y^m_{\tau_t^*(m)} = L(\tau_t^*(m)) \leq L(\tau_t^*) = Y_{\tau_t^*} = Y_{\tau_t^*\wedge\sigma}. \tag{54}$$

By Proposition 2.2, it follows from (52), (53) and (54) that

$$\mathcal{E}_t^{g,\phi}\left[Y^m_{\tau_t^*(m)\wedge\sigma}\right] \leq \mathcal{E}_t^{g,\phi}\left[Y_{\tau_t^*\wedge\sigma}\right]. \tag{55}$$

From (51), (55) and taking limit for Y_t^m as $m \to \infty$, (50) holds. In particular,

$$Y_t \leq \mathcal{E}_t^{g,\phi}[Y_{\tau_t^*\wedge\sigma_t^*}]. \tag{56}$$

Thus, it follows from (44), (45), (46), (49) (50) and (56) that (43) holds for any $\tau, \sigma \in \mathcal{T}_t$. So we complete the proof of Theorem 3.2. □

For applications of Dynkin game under ambiguity, a good example is the pricing and hedging of game options. Our results can be used to analyze game options when there is ambiguity in the incomplete market. The further study about this topic will appear in our future work.

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Competing interests

The authors declare that they have no competing interests.

Authors contributions

The authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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