Notes on q-Hermite based unified Apostol type polynomials

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Abstract. In this article, a new class of q-Hermite based unified Apostol type polynomials is introduced by means of generating function and series representation. Several important formulas and recurrence relations for these polynomials are derived via different generating methods. We also introduce q-analog of Stirling numbers of second kind of order ν by which we construct a relation including aforementioned polynomials.

Keywords: q-Hermite type polynomials, q-unified Apostol type polynomials, q-Hermite based unified Apostol type polynomials.

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1. Introduction and Preliminaries

With the development of q-calculus started appearing in the nineteenth century, Many authors made generalizations to special functions and polynomial families based on the q-analogs (cf. [1-12, 18, 20]). During the process, properties and relations have been demonstrated and contributed to solving different kinds of problems in other subjects (see [4, 5, 20]). The applications of q-calculus in various fields of mathematics, physics and engineering.

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a), n \in \mathbb{N}.$$

The q-analogue of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1-q^a}{1-q}, q \in \mathbb{C} - \{1\}; a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n}, q \neq 1; n \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$

The Gauss q-binomial coefficient $\left(\begin{array}{c} n \\ k \end{array}\right)_q$ is given by

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, k = 0, 1, \cdots, n.$$

The q-analogue of the function $(x+y)_q^n$ is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q q^{k(k-1)/2} x^{n-k} y^k, n \in \mathbb{N}_0. \tag{1.1}$$

The q-analogue of exponential function are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, 0 <\mid q \mid< 1; \mid x \mid<\mid 1-q \mid^{-1}, \tag{1.2}$$

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)x;q)_\infty, 0 <\mid q < 1; x \in \mathbb{C}. \tag{1.3}$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x), D_q E_q(x) = E_q(qx), \tag{1.4}$$

where the q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, 0 <\mid q \mid< 1.$$

For any two arbitrary functions $f(z)$ and $g(z)$, the q-derivative operator D_q satisfies the following product and quotient relations:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z), \tag{1.5}$$

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{q,z}f(z) - f(qz)D_{q,z}g(z)}{g(z)g(qz)}. \tag{1.6}$$

The Apostol type q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α, the Apostol type q-Euler polynomials $E_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α and the Apostol type q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α are defined by means of the following generating function (see [9-12, 18]):

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{n!}, (\mid t + \log\lambda \mid) < 2\pi, 1^{\alpha} = 1, \tag{1.7}$$

$$\left(\frac{2}{\lambda e_q(t) + 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{n!}, (\mid t + \log\lambda \mid) < \pi, 1^{\alpha} = 1, \tag{1.8}$$

$$\left(\frac{2t}{\lambda e_q(t) + 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{n!}, (\mid t + \log\lambda \mid) < \pi, 1^{\alpha} = 1. \tag{1.9}$$

Clearly, we have

$$B_{n,q}^{(\alpha)}(\lambda) = B_{n,q}^{(\alpha)}(0,0;\lambda), E_{n,q}^{(\alpha)}(\lambda) = E_{n,q}^{(\alpha)}(0,0;\lambda), G_{n,q}^{(\alpha)}(\lambda) = G_{n,q}^{(\alpha)}(0,0;\lambda).$$

Recently, Ozarslan [13] introduced the following unification of the Apostol Bernoulli, Apostol Euler and Apostol Genocchi polynomials. Explicitly Ozarslan sudied the following generating function:

$$f_{a,b}^{(\alpha)}(x;t,a,b) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{1.10}$$

$$\left(|t + b\ln(\frac{\beta}{\alpha})| < 2\pi, k \in \mathbb{N}_0; a,b \in \Re\backslash\{0\}; \alpha,\beta \in \mathbb{C}\right).$$

For $\alpha = 1$ in (1.10), we get

$$f_{a,b}(x;t,a,b) = \frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}(x;k,a,b)\frac{t^n}{n!}, \tag{1.11}$$

$$\left(|t + b\ln(\frac{\beta}{\alpha})| < 2\pi, k \in \mathbb{N}_0; a, b \in \Re^+; \beta \in \mathbb{C}\right).$$

From (1.10) and (1.11), we have

$$P^{(1)}_{n,\beta}(x;k,a,b) = P_{n,\beta}(x;k,a,b), (n \in \mathbb{N}),$$

which is defined by Ozden and Simsek [15] and Ozden et al. [14] introduced many properties of these polynomials.

Very recently, Riyasat and Khan [18] introduced a new type of q-Hermite based Appell polynomials as follows.

Definition 1.1. The 2D q-Hermite based Appell polynomials ${}_HA^{(s)}_{n,q}(q \in \mathbb{C}, 0 <| q |< 1)$ are defined by means of the following generating function

$$\frac{1}{g_q(t)} e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) = \sum_{n=0}^{\infty} {}_HA^{(s)}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{1.12}$$

$${}_HA^{(s)}_{n,q} = {}_HA^{(s)}_{n,q}(0,0).$$

Definition 1.2. The 2D q-Hermite polynomials $H^{(s)}_{n,q}(x,y)(0 <| q |< 1, 0 \neq s \in \mathbb{R})$ are defined by means of the following generating function

$$e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) = \sum_{n=0}^{\infty} H^{(s)}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{1.13}$$

where $H^{(s)}_{n,q} = H^{(s)}_{n,q}(0,0)$ are the q-Hermite numbers defined by

$$e_q\left(\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H^{(s)}_{n,q}(0)\frac{t^n}{[n]_q!}.$$

The generalized Stirling numbers of the second kinds $S(n,\nu,a,b,\beta)$ of order ν are defined in [21] as follows:

$$\sum_{n=0}^{\infty} S(n,\nu,a,b,\beta)\frac{t^n}{n!} = \frac{(\beta^b e^t - a^b)^\nu}{\nu!}. \tag{1.14}$$

For $\beta = \lambda$, $a = b = 1$, (1.14) reduces to

$$\sum_{n=0}^{\infty} S(n,\nu,\lambda)\frac{t^n}{n!} = \frac{(\lambda e^t - 1)^\nu}{\nu!}. \tag{1.15}$$

In this paper, we introduce unified q-Hermite based unified Apostol Bernoulli, Euler and Genocchi polynomials of order α and to investigate some properties of them. Moreover, we consider q analog of new generalization of Stirling numbers of the second kind of order ν by which we derive a relation including unified q-analog of Apostol type polynomials of order α.

2. q-Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$

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In this section, we introduce q-Hermite-based unified Apostol type polynomials (qHbAtp) ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ by means of the generating function and series representation. Certain relations for these polynomials are also derived by using various identities. Now we start at the following definition.

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Definition 2.1. Let $q\in\mathbb{C},k\in\mathbb{N}_0,a,b\in\mathbb{R}\setminus\{0\},\alpha,\beta\in\mathbb{N},0<\mid q\mid<1$. The q-Hermite-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α are defined by means of the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\sum_{n=0}^{\infty}{}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}. \tag{2.1}$$

When $x=y=s=0$ in (2.1), ${}_HP^{(\alpha,0)}_{n,\beta,q}(0,0;k,a,b)=P^{(\alpha)}_{n,\beta,q}(k,a,b)$ are called the n^{th} q-unified Apostol type numbers of order α.

Remark 2.1. For $x=y=0$ in (2.1), ${}_HP^{(\alpha,s)}_{n,\beta,q}(k,a,b)={}_HP^{(\alpha,s)}_{n,\beta,q}(0,0;k,a,b)$ are the q-Hermite based unified Apostol type numbers defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha e_q\left(\frac{st^2}{1+q}\right)=\sum_{n=0}^{\infty}{}_HP^{(\alpha,s)}_{n,\beta,q}(k,a,b)\frac{t^n}{[n]_q!}. \tag{2.2}$$

Remark 2.2. For $s=0$ in (2.1), the result reduces to known result of Kurt [6] as follows

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^\alpha e_q(xt)E_q(yt)=\sum_{n=0}^{\infty}P^{(\alpha)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}. \tag{2.3}$$

Remark 2.3. Taking $q\longrightarrow 1$ and $s=y=0$ in (2.1), we get the known result of Ozarslan [13] as follows

$$\lim_{q\to 1}\sum_{n=0}^{\infty}{}_HP^{(\alpha,0)}_{n,\beta,q}(x,0;k,a,b)\frac{t^n}{[n]_q!}=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{xt}. \tag{2.4}$$

Theorem 2.1. Unified q-Hermite-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α holds true:

$$\begin{aligned}{}_HP^{(\alpha,s)}_{n,\lambda,q}(x,y;1,1,1)&={}_HB^{(\alpha,s)}_{n,q}(x,y;\lambda),\\ {}_HP^{(\alpha,s)}_{n,\lambda,q}(x,y;0,-1,1)&={}_HE^{(\alpha,s)}_{n,q}(x,y;\lambda),\\ {}_HP^{(\alpha,s)}_{n,\frac{\lambda}{2},q}(x,y;1,-\frac{1}{2},1)&={}_HG^{(\alpha,s)}_{n,q}(x,y;\lambda).\end{aligned} \tag{2.5}$$

Theorem 2.2. The following series representation for the q-Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)=\sum_{m=0}^{n}\begin{pmatrix}n\\m\end{pmatrix}_q P^{(\alpha)}_{n-m,\beta,q}(o,y;k,a,b)H^{(s)}_{m,q}(x). \tag{2.6}$$

Proof. Using equation (1.10) and (1.12) in the l.h.s. of equation (2.1) and then applying the Cauchy product rule and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.6).

Theorem 2.3. The following summation formula for the q-Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)=\sum_{m=0}^{n}\begin{pmatrix}n\\m\end{pmatrix}_q {}_HP^{(\alpha,s)}_{m,\beta,q}(0,0;k,a,b)(x+y)^{n-m}_q. \tag{2.7}$$

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$$ {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right)_{q} {}_{H}P^{(\alpha,s)}_{m,\beta,q}(0,y;k,a,b)x^{n-m}. \tag{2.8} $$

$$ {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right)_{q} q^{(n-k)(n-k-1)/2} {}_{H}P^{(\alpha,s)}_{m,\beta,q}(x,0;k,a,b)y^{n-m}. \tag{2.9} $$

Proof. Suitably using equations (1.1)-(1.3) in generating function (2.1) to get three different form. Further making use of the Cauchy product rule in the resultant expressions and then comparing the like powers of t in the both sides of resultant equation, we find formulas (2.7)-(2.9).

Theorem 2.4. The following recursive formulas for the q-Hermite-based unified Apostol type polynomials ${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$$ D_{q,x\,H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = [n]_{q\,H}P^{(\alpha,s)}_{n-1,\beta,q}(x,y;k,a,b), \tag{2.10} $$

$$ D_{q,y\,H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = [n]_{q\,H}P^{(\alpha,s)}_{n-1,\beta,q}(x,qy;k,a,b). \tag{2.11} $$

Proof. Differentiating generating function (2.1) with respect to x and y with the help of equation (1.4) and then simplifying with the help of the Cauchy product rule formulas (2.10) and (2.11) are obtained.

Theorem 2.5. The following relation for the q-Hermite-based unified Apostol-type polynomials ${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$$ a^{b}\,{}_{H}P^{(s)}_{n,\beta,q}(x,y;k,a,b) = \beta^{b} \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_{q} {}_{H}P^{(s)}_{n-r,\beta,q}(x,y;k,a,b) - \frac{[n]_q!}{[n-k]_q!} 2^{1-k} H^{(s)}_{n-k,q}(x,y). \tag{2.12} $$

Proof. Consider the following identity

$$ \frac{a^b}{(\beta^b e_q(t) - a^b)e_q(t)} = \frac{\beta^b}{\beta^b e_q(t) - a^b} - \frac{1}{e_q(t)}. $$

Evaluating the following fraction using above identity, we find

$$ \frac{a^b 2^{1-k} t^k e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt)}{(\beta^b e_q(t) - a^b)e_q(t)} = \frac{\beta^b 2^{1-k} t^k e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt)}{\beta^b e_q(t) - a^b} $$

$$ - \frac{2^{1-k} t^k e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt)}{e_q(t)} $$

$$ a^b \sum_{n=0}^{\infty} {}_{H}P^{(s)}_{n,\beta,q}(x,y;k,a,b) \frac{t^n}{[n]_q!} $$

$$ = \beta^b \sum_{n=0}^{\infty} {}_{H}P^{(s)}_{n,\beta,q}(x,y;k,a,b) \frac{t^n}{n!} \sum_{k=0}^{\infty} t^r \frac{t^r}{[r]_q!} - 2^{1-k} \sum_{n=0}^{\infty} H^{(s)}_{n-k,q}(x,y) \frac{t^n}{[n-k]_q!}. $$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.12) follows.

Theorem 2.6. The following relation for the q-Hermite-based unified Apostol-type polynomials ${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$$ {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) $$

$$= 2^{k-1}\sum_{k=0}^{n}\left(\begin{array}{c} n+k \\ m \end{array}\right)_q \Big[\beta^b P_{n-m+k,\beta,q}(1,0;k,a,b)\,{}_HP^{(\alpha,s)}_{m,\beta,q}(x,y;k,a,b)$$
$$-a^b P_{n-m+k,\beta,q}(0,0;k,a,b)\,{}_HP^{(\alpha,s)}_{m,\beta,q}(x,y;k,a,b)\Big]. \qquad (2.13)$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$
$$= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)\left(\frac{\beta^b e_q(t)-a^b}{2^{1-k}t^k}\right)\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$= \frac{2^{k-1}}{t^k}\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)e_q(t)\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$-a^b\frac{2^{k-1}}{t^k}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$= \frac{2^{k-1}}{t^k}\beta^b\sum_{n=0}^{\infty}P_{n\beta,q}(1,0;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty}{}_HP^{(\alpha,s)}_{m,\beta,q}(x,y;k,a,b)\frac{t^m}{[m]_q!}$$
$$-\frac{2^{k-1}}{t^k}a^b\sum_{n=0}^{\infty}P_{n,\beta,q}(0,0;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty}{}_HP^{(\alpha,s)}_{m,\beta,q}(x,y;k,a,b)\frac{t^m}{[m]_q!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.13) follows.

Theorem 2.7. The following recurrence relation for the q-Hermite-based unified Apostol-type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$${}_HA^{(s)}_{n+1,q}(x,y;\lambda) = -\left(\frac{2s}{1+q}\right)[n]_q\,{}_HA^{(s)}_{n-1,q}(qx,qy;\lambda) + x\,{}_HA^{(s)}_{n,q}(x,y;\lambda)$$
$$+y\,{}_HA^{(s)}_{n,q}(qx,qy;\lambda) + \frac{1}{(\lambda-1)^2}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q {}_HA^{(s)}_{n-k,q}(x,y;\lambda)q^{n-k}A_{k,q}(1,\lambda-1;\lambda). \qquad (2.14)$$

Proof. Taking $\alpha = 1$ and then applying q-derivative on both sides of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty}{}_HP^{(s)}_{n+1,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!} = 2^{1-k}D_{q,t}\left[\frac{t^k e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)}{\beta^b e_q(t)-a^b}\right],$$

which on performing differentiation in l.h.s. using formula (1.6) yields

$$\sum_{n=0}^{\infty}{}_HP^{(s)}_{n+1,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!} = 2^{1-k}\left[\frac{(\beta^b e_q(qt)-a^b)D_{q,t}\left(t^k e_q(xt)e_q\left(-\frac{st^2}{1+q}\right)E_q(yt)\right)}{(\beta^b e_q(t)-a^b)(\beta^b e_q(qt)-a^b)}\right.$$
$$\left.-\frac{e_q(qxt)E_q(yqt)e_q\left(-\frac{sq^2t^2}{1+q}\right)D_{q,t}(e_q^{t(\lambda-1)}-\lambda)}{(e_q^{t(\lambda-1)}-\lambda)(e_q^{qt(\lambda-1)}-\lambda)}\right]$$

$$= -\left(\frac{2s}{1+q}\right)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)}-\lambda}\right)e_q\left(qxt-\frac{st^2}{1+q}\right)E_q(qyt)t$$
$$+x\left(\frac{1-\lambda}{e_q^{t(\lambda-1)}-\lambda}\right)e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$+y\left(\frac{1-\lambda}{e_q^{t(\lambda-1)}-\lambda}\right)e_q\left(qxt-\frac{st^2}{1+q}\right)E_q(qyt)$$
$$+\frac{1}{(\lambda-1)^2}\left(\frac{1-\lambda}{e_q^{t(\lambda-1)}-\lambda}\right)e_q\left(qxt-\frac{st^2}{1+q}\right)E_q(qyt)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)}-\lambda}\right)e_q^{(\lambda-1)t},$$

which on making use of the Cauchy product rule in the r.h.s. and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the resultant equation gives recurrence relation (2.13).

3. Summation formulae for q-Hermite based unified Apostol type polynomials

In this section, we provide implicit and explicit formulae, Stirling numbers of the second kind and some relationships for q-Hermite based Apostol type polynomials of order α related to Apostol type Bernoulli polynomials, Apostol type Euler polynomials and Apostol type Genocchi polynomials. We now begin with the following theorem.

Theorem 3.1. The following summation formulae for q-Hermite based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$$ {}_HP^{(\alpha,s)}_{r+l,\beta,q}(z,y;k,a,b)$$
$$=\sum_{n,m=0}^{r,l}\binom{l}{m}_q\binom{r}{n}_q(z-x)^{n+m}\,{}_HP^{(\alpha,s)}_{r+l-n-m,\beta,q}(x,y;k,a,b). \tag{3.1}$$

Proof. We replace t by $t+w$ and rewrite the generating function (2.1) as

$$\left(\frac{2^{1-k}t^k}{\beta^b e_q(t+w)-a^b)}\right)^{\alpha}E_q(y(t+w))e_q\left(\frac{-s(t+u)^2}{1+q}\right)$$
$$=e_q(-x(t+w))\sum_{r,l=0}^{\infty}{}_HP^{(\alpha,s)}_{r+l,\beta,q}(x,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!},\quad \text{(see [16, 17])}. \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e_q\left((z-x)(t+w)\right)\sum_{r,l=0}^{\infty}{}_HP^{(\alpha,s)}_{r+l,\beta,q}(x,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}$$
$$=\sum_{k,l=0}^{\infty}{}_HP^{(\alpha,s)}_{r+l,\beta,q}(z,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+w)]^N}{[N]_q!}\sum_{r,l=0}^{\infty}{}_HP^{(\alpha,s)}_{r+l,q}(x,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}$$
$$=\sum_{r,l=0}^{\infty}{}_HP^{(\alpha,s)}_{k+l,\beta,q}(z,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}, \tag{3.4}$$

which on using formula [19,p.52(2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

in the left hand side becomes

$$\sum_{n,m=0}^{\infty} \frac{(z-x)^{n+m}t^n w^m}{[n]_q![m]_q!} \sum_{k,l=0}^{\infty} {}_HP^{(\alpha,s)}_{r+l,\beta,q}(x,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}$$

$$= \sum_{r,l=0}^{\infty} {}_HP^{(\alpha,s)}_{r+l,\beta,q}(z,y;\lambda)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}. \tag{3.6}$$

Now replacing r by $r-n$, and l by $l-m$ in the left hand side of (3.6), we get

$$\sum_{r,l=0}^{\infty} \sum_{n,m=0}^{r,l} \frac{(z-x)^{n+m}}{[n]_q![m]_q!} {}_HP^{(\alpha,s)}_{r+l-n-m,q}(x,y;k,a,b)\frac{t^r}{(r-n)_q!}\frac{w^l}{(l-m)_q!}$$

$$= \sum_{r,l=0}^{\infty} {}_HP^{(\alpha,s)}_{r+l,\beta,q}(z,y;k,a,b)\frac{t^r}{[r]_q!}\frac{w^l}{[l]_q!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and w in the above equation, we get the required result.

Remark 3.1. By taking l = 0 in Eq. (3.1), we immediately deduce the following result.

Corollary 3.1. The following summation formula for q-Hermite based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$${}_HP^{(\alpha,s)}_{k+l,\beta,q}(z,y;k,a,b) = \sum_{n=0}^{r} \left(\begin{array}{c} r \\ n \end{array}\right)_q (z-x)^n {}_HP^{(\alpha,s)}_{r-n,\beta,q}(x,y;k,a,b). \tag{3.8}$$

Remark 3.2. On replacing z by $z+x$ and setting $y=0$ in Theorem (3.1), we get the following result involving q-Hermite based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of one variable

$${}_HP^{(\alpha,s)}_{r+l,\beta,q}(z+x;k,a,b)$$

$$= \sum_{n,m=0}^{r,l} \left(\begin{array}{c} l \\ m \end{array}\right)_q \left(\begin{array}{c} r \\ n \end{array}\right)_q z^{n+m} {}_HP^{(\alpha,s)}_{r+l-n-m,\beta,q}(x;k,a,b), \tag{3.9}$$

whereas by setting $z=0$ in Theorem 3.1, we get another result involving q-Hermite based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of one and two variables

$${}_HP^{(\alpha,s)}_{r+l,\beta,q}(y;k,a,b)$$

$$= \sum_{n,m=0}^{r,l} \left(\begin{array}{c} l \\ m \end{array}\right)_q \left(\begin{array}{c} r \\ n \end{array}\right)_q (-x)^{n+m} {}_HP^{(\alpha,s)}_{r+l-n-m,\beta,q}(x,y;k,a,b). \tag{3.10}$$

Theorem 3.2. The following summation formulae for q-Hermite based unified Apostol type polynomials ${}_HP^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ of order α holds true:

$${}_HP^{(\alpha+1,s)}_{n,\beta,q}(x,y;k,a,b) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right)_q P_{n-m,\beta,q}(k,a,b) {}_HP^{(\alpha,s)}_{m,\beta,q}(x,y;k,a,b). \tag{3.11}$$

Proof. From (2.1), we have

$$\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$=\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\sum_{m=0}^{\infty} {}_HP_{m,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^m}{[m]_q!}$$
$$\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha+1,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty} P_{n,\beta,q}(k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} {}_HP_{m,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^m}{[m]_q!}.$$

Now replacing n by $n-m$ and equating the coefficients of t^n leads to formula (3.11).

Theorem 3.3. The following summation formulae for q-Hermite based unified Apostol type polynomials ${}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)$ of order α holds true:

$${}_HP_{n,\beta,q}^{(\alpha,s)}(x+1,y;k,a,b)=\sum_{m=0}^{n}\binom{n}{m}_q {}_HP_{m,\beta,q}^{(\alpha,s)}(x,y;k,a,b). \tag{3.12}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x+1,y;k,a,b)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$
$$=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)(e_q(t)-1)$$
$$=\left(\sum_{m=0}^{\infty} {}_HP_{m,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^m}{[m]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\right)-\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HP_{m,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{(n-m)_q!}-\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.12).

Theorem 3.4. The following relationship holds true:

$$a^{b\alpha}\alpha!\sum_{r=0}^{n}\binom{n}{r}_q {}_HP_{n-r,\beta,q}^{(\alpha,s)}(x,y;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)=2^{(1-k)\alpha}H_{n-k\alpha,q}^{(s)}(x,y)\frac{[n]_q!}{[n-k\alpha]_q!}. \tag{3.13}$$

Proof. By using equation (1.13) and (1.14), we have

$$\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$

$$=\frac{2^{(1-k)\alpha}t^{k\alpha}}{a^{b\alpha}\left(\left(\frac{\beta}{a}\right)^b e_q(t)-1\right)}e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)=\frac{2^{(1-k)\alpha}t^{k\alpha}e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)}{a^{b\alpha}\alpha!\sum\limits_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{[r]_q!}}$$

$$\sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(x,y;k,a,b)\frac{t^n}{[n]_q!}a^{b\alpha}\alpha!\sum_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^r}{[r]_q!}=2^{(1-k)\alpha}t^{k\alpha}\sum_{n=0}^{\infty}H_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!}$$

$$\sum_{n=0}^{\infty}a^{b\alpha}\alpha!\sum_{r=0}^{n}\binom{n}{r}_q {}_HP_{n-r,\beta,q}^{(\alpha,s)}(x,y;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^n}{[n]_q!}$$

$$= 2^{(1-k)\alpha}\sum_{n=0}^{\infty} H^{(s)}_{n-k\alpha,q}(x,y)\frac{t^n}{(n-k\alpha)_q!}$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result (3.13).

Theorem 3.5. The following relationship holds true:

$$_{H}P^{(\alpha,s)}_{n-\nu k,\beta,q}(x,y;k,a,b) = \frac{[\nu]_q!2^{(k-1)\nu}[n-\nu k]_q!}{[n]_q!}$$
$$\times\sum_{l=0}^{n}\binom{n}{l}_q S(n-l,\nu,a,b,\beta)\,_{H}P^{(\alpha-\nu,s)}_{l,\beta,q}(x,y;k,a,b). \tag{3.14}$$

Proof. From (2.1) and (1.14), we have

$$\sum_{n=0}^{\infty}{}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)\frac{\left(\beta^b e_q(t)-a^b\right)^{\nu}}{[\nu]_q!}$$
$$\times\frac{[\nu]_q!}{(\beta^b e_q(t)-a^b)^{\nu}}$$
$$=\frac{[\nu]_q!}{(2^{1-k}t^k)^{\nu}}\sum_{l=0}^{\infty}{}_{H}P^{(\alpha-\nu,s)}_{l,\beta,q}(x,y;k,a,b)\frac{t^l}{(l)_q!}\sum_{n=0}^{\infty}S(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!}$$
$$\sum_{n=0}^{\infty}{}_{H}P^{(\alpha,s)}_{n-\nu k,\beta,q}(x,y;k,a,b)\frac{t^n}{[n-\nu k]_q!}$$
$$=[\nu]_q!2^{(k-1)\nu}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}_q S(n-l,\nu,a,b,\beta)\,_{H}P^{(\alpha-\nu,s)}_{l,\beta,q}(x,y;k,a,b)\right)\frac{t^n}{[n]_q!}.$$

On comparing the coefficients of t^n in both sides, we get (3.14).

Theorem 3.6. The following relationship holds true:

$$_{H}P^{(\alpha-m,s)}_{n,\beta,q}(x,y;k,a,b) = \sum_{r=0}^{n}\binom{n}{r}_q\,{}_{H}P^{(\alpha,s)}_{n-r,\beta,q}(x,y;k,a,b)P^{(-m)}_{r,\beta,q}(0,0;k,a,b). \tag{3.15}$$

Proof. By using generating function (2.1), we have

$$\sum_{n=0}^{\infty}{}_{H}P^{(\alpha-m,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha-m} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$
$$= e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)\left(1-\frac{e_q((\lambda-1)t)-1}{1-\lambda}\right)^{-\alpha}$$
$$=\left(\sum_{n=0}^{\infty}{}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}\right)\left(\sum_{r=0}^{\infty}P^{(-m)}_{r,\beta,q}(0,0;k,a,b)\frac{t^r}{[r]_q!}\right).$$

Using Cauchy product and comparing the coefficients of t^n in both sides, we arrive at the required result (3.15).

Theorem 3.7. The following relation between the q-Hermite based unified Apostol type polynomials $_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ and q-Apostol type Bernoulli polynomials $B_{n,q}(x;\lambda)$ holds true:

$$_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = \sum_{m=0}^{n+1}\binom{n+1}{m}_q\left(\lambda\sum_{r=0}^{m}\binom{m}{r}_q B_{m-r,q}(x;\lambda)-B_{m,q}(x;\lambda)\right)$$

$$\times {}_{H}P^{(\alpha,s)}_{n-m+1,q}(0,y;k,a,b). \tag{3.16}$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$
$$= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right) E_q(yt)\left(\frac{t}{\lambda e_q(t)-1}\right)\left(\frac{\lambda e_q(t)-1}{t}\right)$$
$$= \frac{1}{t}\left(\lambda \sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} B_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.$$
$$\left. -\sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} B_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\right). \tag{3.17}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.17), assertion (3.16) follows.

Theorem 3.8. The following relation between the q-Hermite based unified Apostol type polynomials ${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ and q-Apostol type Euler polynomials $E_{n,q}(x;\lambda)$ holds true:

$${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}_q\left(\lambda\sum_{r=0}^{m}\binom{m}{r}_q E_{m-r,q}(x;\lambda)+E_{m,q}(x;\lambda)\right)$$
$$\times {}_{H}P^{(\alpha,s)}_{n-m,\beta,q}(0,y;k,a,b). \tag{3.18}$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$
$$= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right) E_q(yt)\left(\frac{2}{\lambda e_q(t)+1}\right)\left(\frac{\lambda e_q(t)+1}{2}\right)$$
$$= \frac{1}{2}\left(\lambda \sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} E_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.$$
$$\left. +\sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} E_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\right). \tag{3.19}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.19), assertion (3.18) follows.

Theorem 3.9. The following relation between the q-Hermite based unified Apostol type polynomials ${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)$ and q-Apostol type Genocchi polynomials $G_{n,q}(x;\lambda)$ holds true:

$${}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b) = \frac{1}{2}\sum_{m=0}^{n+1}\binom{n+1}{m}_q\left(\lambda\sum_{r=0}^{m}\binom{m}{r}_q G_{m-r,q}(x;\lambda)+G_{m,q}(x;\lambda)\right)$$
$$\times {}_{H}P^{(\alpha,s)}_{n-m+1,\beta,q}(0,y;k,a,b). \tag{3.20}$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_{H}P^{(\alpha,s)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}$$

$$= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^{\alpha} e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) \left(\frac{2t}{\lambda e_q(t)+1}\right)\left(\frac{\lambda e_q(t)+1}{2t}\right)$$

$$= \frac{1}{2t}\left(\lambda \sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} G_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.$$

$$\left. + \sum_{n=0}^{\infty} {}_HP_{n,\beta,q}^{(\alpha,s)}(0,y;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} G_{m,q}(x;\lambda)\frac{t^m}{[m]_q!}\right). \tag{3.21}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.21), assertion (3.20) follows.

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